Classical states versus separable states

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A bipartite state is called classical (with regard to correlations) if it is left undisturbed by a certain local von Neumann measurement, and is called separable if it can be represented as a convex combination of product states. Due to the perfect distinguishability of orthogonal vectors, a classical state can essentially be identified with a convenient bivariate probability distribution, and moreover, it is separable, but not vice versa. The notion of separability plays a key role in quantum information theory because entanglement is defined via separability. However, the definition of separability is *ad hoc* and formal. In this paper, we present an intrinsic characterization of separable states via classical states from the measurement perspective: Separable states are precisely those states that are reductions of classical states in higher dimensions with the natural partitions. Consequently, entangled states are precisely those states that cannot be represented as such reductions of classical states. This observation highlights the hidden mutually exclusive and complementary relations between classicality and entanglement.

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I. INTRODUCTION

Consider a bipartite quantum state ρ shared by parties a and b and the fundamental issue of qualifying and quantifying the correlations therein. A standard approach is based on the entanglement versus separability dichotomy [1]. In this framework, a bipartite state ρ is termed separable if it can be represented as

$$\rho = \sum_i p_i \rho_i^a \otimes \rho_i^b \tag{1}$$

for some probability distribution $\{p_i\}$ and local density operators $\{\rho_i^a\}$ for party a and $\{\rho_i^b\}$ for party b. Otherwise it is defined as entangled. It should be emphasized that the above representation is not unique and usually there are infinitely many representations of a separable state. The intuitive meaning of this formulation is clear: Any product state $\rho_i^a \otimes \rho_i^b$ evidently does not possess any correlations (classical or quantum), and the procedure of taking a convex combination can be realized as a classical mixing process, which should not generate any entanglement which is understood as a truly quantum phenomenon arising from superposition, thus the name separable state. Indeed, there is a classical mechanism generating separable states [1]. In spite of the simplicity and naivete of Eq. (1), it is notoriously difficult to determine whether a given state is separable or not because the representation is not unique, though several ingenious separability criteria useful for some particular classes of states have been devised [2].

An essential and intuitive difference between classicality and quantumness lies in the distinguished role played by measurements: While in the classical realm, we can always (at least in principle) make measurements without disturbing a classical system; this is not so in the quantum world. A quantum measurement usually disturbs the system, except when the system happens to be classical.

Due to the fundamental importance of the notion of separability and its plausible relations to classicality, one may inquire whether there is any intrinsic characterization of separability. This paper is devoted to present such a characterization from the measurement perspective. The key idea is to augment both parties a and b separately, consider classical correlations between these two augmented parties, and then trace out the ancillas. The main result is that separable states are exactly those states that are natural nontrivial reductions (partial states) of classical states. This characterization puts the entanglement in a concrete and direct contrast to classicality: Entangled states are exactly those states that *cannot* be regarded as such reduced states of classical ones. Here classicality (more precisely, classicality of correlations) is defined via nondisturbance by local quantum measurements [3].

The paper is organized as follows. In Sec. II, we review the notion of classical states and their canonical representations from the quantum measurement perspective. Intuitively, a bipartite state is classical if it is not disturbed by a local von Neumann measurement. This definition exploits the fundamental difference of measurements in classical and quantum systems. In Sec. III, we express separable states as reductions (partial states) of classical states in higher dimensions. Section IV is devoted to some discussion.

II. CLASSICAL STATES AND LOCAL VON NEUMANN MEASUREMENTS

In order to put our discussion in a precise setting, let us first review some basic notions. Consider a quantum system

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whose states are described by vectors in a Hilbert space $\mathcal{H}$. A (complete) von Neumann measurement on this system is a family of orthogonal one-dimensional projections $\{\Pi_i\}$ such that $\Sigma_i \Pi_i = \mathbf{1}$ (orthogonal resolution of identity on $\mathcal{H}$). In contrast to this standard and traditional formulation of quantum measurement, a more general notion of quantum measurement is a positive-operator-valued measure (POVM), also called a generalized measurement: Any family of nonnegative operators $\{E_j\}$ on $\mathcal{H}$ is called a POVM if $\Sigma_j E_j = \mathbf{1}$ (generalized resolution of identity) [4]. A von Neumann measurement is a particular case of a POVM, but not vice versa. The importance of POVM lies in that many tasks, such as optimal extraction of information, state discrimination, and entanglement transformation, require generalized measurements, rather than only von Neumann measurements [5].

Following Ref. [3], a bipartite state ρ is called classical (with regard to correlations) if there are a local von Neumann measurement $\{\Pi_i^a\}$ for party a and a local von Neumann measurement $\{\Pi_j^b\}$ for party b such that ρ is left undisturbed by the joint measurement $\{\Pi_i^a \otimes \Pi_j^b\}$; that is,

$$\rho = \sum_{ij} \Pi_i^a \otimes \Pi_j^b \rho \Pi_i^a \otimes \Pi_j^b.$$

In such a case, there is a bivariate probability distribution $\{p_{ij}\}$ such that

$$\rho = \sum_{ij} p_{ij} \Pi_i^a \otimes \Pi_j^b, \tag{2}$$

and ρ can actually be identified as the classical bivariate probability distribution $\{p_{ij}\}$ due to the orthogonality of the measurement operators (which are thus perfectly distinguishable) [3]. In particular, we see that classical states are separable, but not vice versa. In general, a linear combination of the maximally mixed state with any nonclassical state (in particular, entangled state) is not classical. For example, in a two-qubit system, let $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and consider the Werner state

$$\rho_c \coloneqq (1-c)\frac{1}{4} + c|\Psi^-\rangle\langle\Psi^-|, \quad c \in [-1/3, 1],$$

which is separable when $c \leq 1/3$ and entangled when $c > 1/3$. ρ_c cannot be classical except for the trivial case $c = 0$ because ρ_c is the linear combination of the maximally mixed state (which is classical and left invariant by any quantum measurement) and a Bell state, which is always disturbed by any jointly local von Neumann measurement. Consequently, we have the following strict inclusion relation:

$$\{\text{classical states}\} \subset \{\text{separable states}\}.$$

However, any product state $\rho^a \otimes \rho^b$ is classical because we can always perform spectral decompositions

$$\rho^a = \sum_i \lambda_i^a \Pi_i^a, \quad \rho^b = \sum_j \lambda_j^b \Pi_j^b,$$

and rewrite $\rho^a \otimes \rho^b$ as

$$\rho^a \otimes \rho^b = \sum_{ij} \lambda_i^a \lambda_j^b \Pi_i^a \otimes \Pi_j^b,$$

which is a classical state.

Though it is usually very difficult to detect separability [2], it is straightforward to determine whether a state is classical or not: Just perform spectral decomposition of the operator representing the state, and check whether each eigenvector is a product state or not, and check furthermore whether the marginal states are mutually orthogonal or not.

While the set of separable states (in a fixed bipartite system) is apparently convex, this is not the case for the set of classical states. One may argue that this is a serious drawback of the very definition of classical states. But in certain circumstances, convexity is not intrinsic. An interesting example is the cleanness of POVMs, which is, quite unexpectedly, largely not related to the convex structure of POVMs [6].

A nontrivial combination of two classical states is classical if they commute. But when the two classical states do not commute, their linear combination may still be classical (of course, in general cases, not classical). For example, consider two product states $\rho^a \otimes \rho_1^b$ and $\rho^a \otimes \rho_2^b$. If ρ_1^b and ρ_2^b do not commute, then $\rho^a \otimes \rho_1^b$ and $\rho^a \otimes \rho_2^b$ do not commute either, but the linear combination

$$\frac{1}{2}\rho^a \otimes \rho_1^b + \frac{1}{2}\rho^a \otimes \rho_2^b = \rho^a \otimes (\rho_1^b + \rho_2^b)/2$$

is still a product state, and thus is classical.

Another sharp contrast between a classical state and a separable state is that the representation [i.e., Eq. (2)] of a nondegenerate classical state is always unique, while that [i.e., Eq. (1)] of a separable state is not.

The key idea of our characterization of separable states is that although separable states are not necessarily classical, they can be regarded as very natural shadows (reductions) of classical states in larger systems.

III. SEPARABLE STATES AS SHADOWS OF CLASSICAL STATES

Our main result is the following characterization of separable states.

Theorem. A bipartite state ρ on $\mathcal{H}^a \otimes \mathcal{H}^b$ shared by two parties a and b is separable if and only if there exists a classical state σ on $(\mathcal{K}^a \otimes \mathcal{H}^a) \otimes (\mathcal{H}^b \otimes \mathcal{K}^b)$ such that

$$\rho = \mathrm{tr}_{\mathcal{K}^a \otimes \mathcal{K}^b} \sigma.$$

Here $\mathcal{K}^a$ is an auxiliary Hilbert space for party a and $\mathcal{K}^b$ an auxiliary Hilbert space for party b, and $\mathrm{tr}_{\mathcal{K}^a \otimes \mathcal{K}^b}$ denotes the partial trace over $\mathcal{K}^a \otimes \mathcal{K}^b$.

Before presenting a proof, we should mention that there is always a trivial way to represent any bipartite state ρ (not necessarily separable) as the partial trace of a classical state in higher dimensions if we do not employ the natural partition of separating the two parties: Just put $\rho' \coloneqq \rho \otimes \tau$ with τ any state in an auxiliary Hilbert space $\mathcal{H}^c$, then clearly $\rho = \mathrm{tr}_{\mathcal{H}^c} \rho'$. But this representation does not shed any light on the correlations in ρ and is useless for our purpose of characterizing correlations in separable states.

Now we proceed to establish the above theorem. First note that if σ is a classical state on $(\mathcal{K}^a\otimes\mathcal{H}^a)\otimes(\mathcal{H}^b\otimes\mathcal{K}^b)$ (with the division of party a in possession of $\mathcal{K}^a\otimes\mathcal{H}^a$ and party b in possession of $\mathcal{H}^b\otimes\mathcal{K}^b$), then by Theorem 2 in Ref. [3], there are von Neumann measurements $\{\Pi_i^a\}$ on $\mathcal{K}^a\otimes\mathcal{H}^a$ and $\{\Pi_j^b\}$ on $\mathcal{H}^b\otimes\mathcal{K}^b$, and a classical bivariate probability distribution $\{p_{ij}\}$ such that

$$\sigma=\sum_{ij} p_{ij}\Pi_i^a\otimes\Pi_j^b.$$

Taking partial trace over $\mathcal{K}^a\otimes\mathcal{K}^b$, and putting

$$p_i:=\sum_j p_{ij},\quad \rho_i^a:=\mathrm{tr}_{\mathcal{K}^a}\,\Pi_i^a,\quad \rho_i^b:=\sum_j \frac{p_{ij}}{p_i}\mathrm{tr}_{\mathcal{K}^b}\,\Pi_j^b,$$

we obtain

$$\mathrm{tr}_{\mathcal{K}^a\otimes\mathcal{K}^b}\,\sigma=\sum_{ij}p_{ij}\,\mathrm{tr}_{\mathcal{K}^a}\,\Pi_i^a\otimes\mathrm{tr}_{\mathcal{K}^b}\,\Pi_j^b=\sum_i p_i\rho_i^a\otimes\rho_i^b,$$

which is clearly a separable state on $\mathcal{H}^a\otimes\mathcal{H}^b$.

Conversely, suppose that $\rho=\sum_{i=1}^N p_i\rho_i^a\otimes\rho_i^b$ is a separable state on $\mathcal{H}^a\otimes\mathcal{H}^b$. Let the spectral decompositions of ρ_i^a and ρ_i^b be

$$\rho_i^a=\sum_j\lambda_{ij}^a|\psi_{ij}^a\rangle\langle\psi_{ij}^a|,\quad \rho_i^b=\sum_k\lambda_{ik}^b|\psi_{ik}^b\rangle\langle\psi_{ik}^b|,$$

respectively. Let $\{|i\rangle\}$ be an orthonormal base for C^N (n-dimensional complex Hilbert space), and C_a^N and C_b^N be two copies of C^N. We take $\mathcal{K}^a=C_a^N$ as an ancilla for party a and $\mathcal{K}^b=C_b^N$ as an ancilla for party b. Let

$$|\Psi_{ij}^a\rangle:=|i\rangle\otimes|\psi_{ij}^a\rangle,\quad |\Psi_{ik}^b\rangle:=|\psi_{ik}^b\rangle\otimes|i\rangle,$$

then $\{\Psi_{ij}^a\}$ and $\{\Psi_{ik}^b\}$ are orthogonal sets (of course, not necessarily complete) for $\mathcal{K}^a\otimes\mathcal{H}^a$ and $\mathcal{H}^b\otimes\mathcal{K}^b$, respectively, and consequently,

$$\{\Pi_{ij}^a:=|\Psi_{ij}^a\rangle\langle\Psi_{ij}^a|\},\quad \{\Pi_{ik}^b:=|\Psi_{ik}^b\rangle\langle\Psi_{ik}^b|\}$$

can be extended to von Neumann measurements on $\mathcal{K}^a\otimes\mathcal{H}^a$ and $\mathcal{K}^b\otimes\mathcal{H}^b$, respectively. Now constructing a state

$$\sigma:=\sum_{ijk}p_i\lambda_{ij}^a\lambda_{ik}^b\Pi_{ij}^a\otimes\Pi_{ik}^b$$

on $(\mathcal{K}^a\otimes\mathcal{H}^a)\otimes(\mathcal{H}^b\otimes\mathcal{K}^b)$, then clearly this σ is a classical state. Moreover, if we put

$$\rho_{ij}^a:=\mathrm{tr}_{\mathcal{K}^a}\,\Pi_{ij}^a,\quad \rho_{ik}^b:=\mathrm{tr}_{\mathcal{K}^b}\,\Pi_{ik}^b,$$

and

$$\rho_i^a:=\sum_j\lambda_{ij}^a\rho_{ij}^a,\quad \rho_i^b:=\sum_k\lambda_{ik}^b\rho_{ik}^b,$$

then we have

$$\begin{aligned}\mathrm{tr}_{\mathcal{K}^a\otimes\mathcal{K}^b}\,\sigma&=\sum_{ijk}p_i\lambda_{ij}^a\lambda_{ik}^b\,\mathrm{tr}_{\mathcal{K}^a}\,\Pi_{ij}^a\otimes\mathrm{tr}_{\mathcal{K}^b}\,\Pi_{ik}^b\\&=\sum_{ijk}p_i\lambda_{ij}^a\lambda_{ik}^b\rho_{ij}^a\otimes\rho_{ik}^b\\&=\sum_i p_i\Big(\sum_j\lambda_{ij}^a\rho_{ij}^a\Big)\otimes\Big(\sum_k\lambda_{ik}^b\rho_{ik}^b\Big)\\&=\sum_i p_i\rho_i^a\otimes\rho_i^b=\rho,\end{aligned}$$

which demonstrates that ρ is a reduced state of the classical state σ.

IV. DISCUSSION

According to the Neumark theorem [4], any rank-one POVM can be reduced to a von Neumann measurement in a larger space (system plus ancilla). Our main theorem may be formally regarded as a correlation analog of this result. While it is true that any separable states can be formally reduced to classical states on a larger system, this does not diminish the need and significance of the notion of separable states in describing correlations. Although the correlations in a separable state are often understood as classical, these correlations, unlike that in a classical state [as defined by Eq. (2)], cannot be really identified as generated by a classical bivariate probability distribution, and there are some subtle and delicate issues here with significant consequences [3]. For instance, the quantum discord of a separable state may not vanish [7], separable states can be used to distribute entanglement [8], and there are certain tasks with quantum advantage based on separable states rather than on entanglement [9].

Because usually there are many ways of representing a separable state as convex combinations of product states, there are also many classical states whose reductions are the same separable state. On the other hand, our result illustrates an alternative method of reproducing correlations in a separable state, and in turn highlights the nonclassical nature of entanglement.

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