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Generation and discrimination of a type of four-partite entangled state

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Recently, a genuine four-qubit entangled state $|\chi^{00}\rangle_{3214}$ has been proposed [Y. Yeo and W. K. Chua, Phys. Rev. Lett. **96**, 060502 (2006)]. This state has many interesting entanglement properties and possible applications in quantum information processing and fundamental tests of quantum physics. Here we present a simple scheme for generating such a state in an ion-trap system. We also demonstrate how to discriminate between the 16 basis states $\{|\chi^{ij}\rangle_{3214}=\sigma_3^i\sigma_1^j|\chi^{00}\rangle_{3214}, i,j=0,1,2,3\}$.

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Entanglement, a consequence of the quantum-state superposition principle applied to composite quantum systems, is one of the most striking features of quantum mechanics. It not only helps quantum mechanics win over the local hidden-variable theory [1], but also lies at the heart of quantum information science [2]. It is therefore of importance to explore and prepare entangled states. Bipartite entanglement is well understood [3]. Multipartite entanglement, however, is still under intensive research [4]. It is well known that revealing the properties of a type of entangled states can help us to identify what quantum information processing (QIP) tasks can be achieved by them. Conversely, studying how a given QIP task can be accomplished with entanglement is helpful for exploring different types of entangled states. Teleportation is a typical example of a QIP task which provides a theoretical framework to study entanglement [5], especially multipartite entanglement [6], in addition to its practical importance [7]. Lately, in researching for faithful teleportation of an arbitrary two-qubit state with multipartite entanglement, Yeo and Chua [8] introduced a genuine four-qubit entangled state

$$|\chi^{00}\rangle_{3214}=\frac{1}{2\sqrt{2}}(|0000\rangle-|0011\rangle-|0101\rangle+|0110\rangle+|1001\rangle+|1010\rangle+|1100\rangle+|1111\rangle)_{3214}. \quad (1)$$

This state does not belong to the well-known three types of multipartite entangled states, i.e., Greenberger-Horne-Zeilinger (GHZ) state [9], W state [10], and linear cluster state [11], in terms of stochastic local operations and classical communication. The state $|\chi^{00}\rangle_{3214}$ has many interesting entanglement properties. It has been shown that a new Bell inequality is optimally violated by $|\chi^{00}\rangle$ but not by the above three types of four-qubit entangled states [12]. Another important property is that it has the maximum entanglement between qubits (3,2) and (1,4), and between (3,1) and (2,4). More importantly, it has many applications in QIP, such as teleportation and dense coding [8]. In this Brief Report, we propose a simple method to create such a type of multipartite entangled states in an ion-trap system. We also demonstrate how to discriminate between the 16 basis states $\{|\chi^{ij}\rangle=\sigma_3^i\sigma_1^j|\chi^{00}\rangle_{3214}, i,j=0,1,2,3\}$ ($\sigma^0=I$ is the two-dimensional identity operator, $\sigma^1,\sigma^2,\sigma^3$ are the conventional Pauli matrices) [8]. Our ideas may be helpful for in-depth study on these states and their applications. For convenience, they will be called χ-type entangled states in the following text.

We consider that four identical ions are confined in a linear Paul trap. Each of them has the ground state $|0\rangle$ and the excited state $|1\rangle$. We drive the neighboring two ions, i.e., the lth and $(l+1)$th ($l=1,2,3$) ions, by two classical homogeneous lasers with frequencies $\omega_0+\nu+\delta$ and $\omega_0-\nu-\delta$, respectively. Here ω_0 is the frequency of the transition $|1\rangle\leftrightarrow|0\rangle$, ν is the frequency of the center-of-mass mode of the collective motion of the ions, and δ is the detuning. Assuming $\delta\ll\nu$, then the excitation of the stretch modes is far off-resonant and is negligible. We consider the resolved sideband regime, where the vibrational frequency ν is much larger than other characteristic frequencies. In the Lamb-Dicke regime, i.e., the spatial extension of the atomic wave function is much smaller than the wavelength of the lasers, the Hamiltonian in the interaction picture is

$$H_{l,l+1}^{i}=i\eta\Omega e^{-i\varphi}\sum_{j=l}^{l+1}\sigma_j^{+}(a^{\dagger}e^{-i\delta t}+ae^{i\delta t})+\text{H.c.}, \quad (2)$$

where $a^{\dagger}$ (a) denotes the creation (annihilation) operator for the vibrational mode, $\sigma_j^{+}=|1_j\rangle\langle 0_j|$ and $\sigma_j^{-}=|0_j\rangle\langle 1_j|$ are the spin flip operators, Ω and φ are the Rabi frequency and phase of the laser fields, and $\eta=\sqrt{(k)^2/(4\nu m)}$ is the Lamb-Dicke parameter with k being the effective wave vector of the laser fields along the direction of the string of the ions and m being the mass of an ion. The Hamiltonian is mathematically identical to that describing the interaction of two-level atoms with a single-mode cavity field [13,14], with the cavity mode replaced by the vibrational mode.

When $\delta\gg\eta\Omega$, the energy conserving transitions are $|1_l1_{l+1}n\rangle\leftrightarrow|0_l0_{l+1}n\rangle$ and $|0_l1_{l+1}n\rangle\leftrightarrow|1_l0_{l+1}n\rangle$. The transition $|1_l1_{l+1}n\rangle\leftrightarrow|0_l0_{l+1}n\rangle$ is mediated by $|0_l1_{l+1}n\pm1\rangle$ and $|1_l0_{l+1}n\pm1\rangle$. The contributions of $|0_l1_{l+1}n\pm1\rangle$ are equal to those of $|1_l0_{l+1}n\pm1\rangle$. The corresponding Rabi frequency is given by $\lambda=2(\Omega\eta)^2/\delta$. Since the transition paths interfere destructively, there is no transfer of population in states with different vibrational excitation and thus the Rabi frequency is independent of the vibrational quantum number [15]. The

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Rabi frequency for $|0_l 1_{l+1} n\rangle \leftrightarrow |1_l 0_{l+1} n\rangle$, mediated by $|1_l 1_{l+1} n\pm 1\rangle$ and $|0_l 0_{l+1} n \pm 1\rangle$, is also $2(\Omega\eta)^2/\delta$. Then the effective Hamiltonian can be described by [14,15]

$$H^e_{l,l+1} = \frac{\lambda}{2}\sum_{j=l}^{l+1}(|1_j\rangle\langle 1_j| + |0_j\rangle\langle 0_j|) + \lambda(\sigma_l^+\sigma_{l+1}^+ + \sigma_l^+\sigma_{l+1}^- + \text{H.c.}). \tag{3}$$

The effective Hamiltonian is independent of the vibrational quantum number. Thus the vibrational state of the ions can be a thermal state in principle.

Assume that the initial state of the ionic system is $|0_1 0_2 0_3 0_4\rangle$. After a coupling time τ between every pair of neighboring ions, the state of the system is

$$\begin{aligned}|\chi(\tau)\rangle = e^{-i3\lambda\tau}[&\cos^3(\lambda\tau)|0_1 0_2 0_3 0_4\rangle \\ &- i\cos^2(\lambda\tau)\sin(\lambda\tau)|0_1 1_2 1_3 0_4\rangle - i\cos^2(\lambda\tau)\sin(\lambda\tau) \\ &\times|0_1 0_2 1_3 1_4\rangle - \cos(\lambda\tau)\sin^2(\lambda\tau)|0_1 1_2 0_3 1_4\rangle \\ &- i\cos^2(\lambda\tau)\sin(\lambda\tau)|1_1 1_2 0_3 0_4\rangle - \cos(\lambda\tau)\sin^2(\lambda\tau) \\ &\times|1_1 0_2 1_3 0_4\rangle - \cos(\lambda\tau)\sin^2(\lambda\tau)|1_1 1_2 1_3 1_4\rangle + i\sin^3(\lambda\tau) \\ &\times|1_1 0_2 0_3 1_4\rangle].\end{aligned} \tag{4}$$

By choosing $\tau=\pi/(4\lambda)$, we obtain a χ-type entangled state

$$\begin{aligned}|\chi^{00}\rangle'_{3214} = \frac{1}{2\sqrt{2}}(&|0_3 0_2 0_1 0_4\rangle - i|1_3 1_2 0_1 0_4\rangle - i|1_3 0_2 0_1 1_4\rangle \\ &- |0_3 1_2 0_1 1_4\rangle - i|0_3 1_2 1_1 0_4\rangle - |1_3 0_2 1_1 0_4\rangle \\ &- |1_3 1_2 1_1 1_4\rangle + i|0_3 0_2 1_1 1_4\rangle),\end{aligned} \tag{5}$$

where a common phase factor $e^{-i3\pi/4}$ is discarded. The state $|\chi^{00}\rangle'_{3214}$=l.u.$|\chi^{00}\rangle_{3214}$, where "l.u." indicates that the equality holds up to a local unitary transformation on one or more of the qubits [16]. If the operation time on the pair of ions 2 and 3 is not $\pi/(4\lambda)$ but at any time T, we can obtain

$$\begin{aligned}|\bar{\chi}\rangle_{3214} = \frac{1}{2}[&\cos(\lambda T)(|0_3 0_2 0_1 0_4\rangle - i|1_3 0_2 0_1 1_4\rangle - i|0_3 1_2 1_1 0_4\rangle \\ &- |1_3 1_2 1_1 1_4\rangle) + \sin(\lambda T)(i|0_3 0_2 1_1 1_4\rangle - |1_3 0_2 1_1 0_4\rangle \\ &- |0_3 1_2 0_1 1_4\rangle - i|1_3 1_2 0_1 0_4\rangle)],\end{aligned} \tag{6}$$

where a common phase factor $e^{-i(\lambda T+\pi/2)}$ is discarded. This state is another genuine four-qubit "maximally" entangled state, i.e., it has the maximum entanglement between ions (3,2) and (1,4), and between (3,1) and (2,4), and can also be used to realize perfect teleportation and dense coding. But the amount of entanglement with von Neumann measure between ions (1,2) and (3,4) is less than that of $|\chi^{00}\rangle_{3214}$ [8].

Next, we demonstrate how to implement χ-type-basis measurement, i.e., deterministically distinguish between the 16 basis states $\{|\chi^{ij}\rangle_{3214}=\sigma_3^i\sigma_1^j|\chi^{00}\rangle'_{3214}, i,j=0,1,2,3\}$, with the setup above. First we assume that the trapped four ions are initially in the state $|\chi^{00}\rangle'_{3214}$ of Eq. (5). We, respectively, drive the neighboring two ions by the abovementioned lasers with duration τ. Then the state of the ionic system evolves into

$$\begin{aligned}|\chi^{00}\rangle'_{3214} \to \frac{1}{2\sqrt{2}}\{&[\cos(\lambda\tau) - \sin(\lambda\tau)]\times[1-\cos(4\lambda\tau)]|0_1 0_2 0_3 0_4\rangle - i[\cos(\lambda\tau)-\sin(\lambda\tau)]\cos(4\lambda\tau)|0_1 0_2 1_3 1_4\rangle - i[\cos(\lambda\tau) \\ &- \sin(\lambda\tau)]\cos(4\lambda\tau)|1_1 1_2 0_3 0_4\rangle - [\cos(\lambda\tau)-\sin(\lambda\tau)][1+\sin(4\lambda\tau)]|1_1 1_2 1_3 1_4\rangle - i[\cos(\lambda\tau)+\sin(\lambda\tau)][1-\sin(4\lambda\tau)] \\ &\times|0_1 1_2 1_3 0_4\rangle - [\cos(\lambda\tau)+\sin(\lambda\tau)]\cos(4\lambda\tau)|0_1 1_2 0_3 1_4\rangle - [\cos(\lambda\tau)+\sin(\lambda\tau)]\cos(4\lambda\tau)|1_1 0_2 1_3 0_4\rangle + i[\cos(\lambda\tau) \\ &+ \sin(\lambda\tau)]\times[1+\sin(4\lambda\tau)]|1_1 0_2 0_3 1_4\rangle\}.\end{aligned} \tag{7}$$

Choosing $\lambda\tau=\pi/4$, we obtain

$$|\chi^{00}\rangle'_{3214} \to i|1_1\rangle|0_2\rangle|0_3\rangle|1_4\rangle. \tag{8}$$

Equation (8) indicates that the state $|\chi^{00}\rangle'_{3214}$ is transformed into a product state of ions 1, 2, 3, and 4. Similarly, the other 15 basis states can also be converted into corresponding product states. For simplicity, we rewrite the 16 basis states $\{|\chi^{ij}\rangle_{3214}, i,j=0,1,2,3\}$ as the following form:

$$\begin{aligned}|\chi^{\pm+}\rangle = \frac{1}{2\sqrt{2}}\{&|m_1\rangle[|m_3\rangle(|0_2 0_4\rangle - |1_2 1_4\rangle) + i|\bar{m}_3\rangle(|1_2 0_4\rangle \\ &+ |0_2 1_4\rangle)] \pm i|\bar{m}_1\rangle[|m_3\rangle(|1_2 0_4\rangle - |0_2 1_4\rangle) + i|\bar{m}_3\rangle \\ &\times(|0_2 0_4\rangle + |1_2 1_4\rangle)]\},\end{aligned}$$

$$\begin{aligned}|\chi^{\pm-}\rangle = \frac{1}{2\sqrt{2}}\{&|m_1\rangle[|m_3\rangle(|0_2 0_4\rangle - |1_2 1_4\rangle) - i|\bar{m}_3\rangle(|1_2 0_4\rangle \\ &+ |0_2 1_4\rangle)] \pm i|\bar{m}_1\rangle[|m_3\rangle(|1_2 0_4\rangle - |0_2 1_4\rangle) - i|\bar{m}_3\rangle \\ &\times(|0_2 0_4\rangle + |1_2 1_4\rangle)]\},\end{aligned} \tag{9}$$

where $m\in\{0,1\}$ and $\bar{m}$ is the counterpart of the binary number m. After the abovementioned operations, we have

$$|\chi^{++}\rangle \to |m_1\rangle|0_2\rangle|m_3\rangle|0_4\rangle,$$

$$|\chi^{+-}\rangle \to |m_1\rangle|1_2\rangle|m_3\rangle|1_4\rangle,$$

$$|\chi^{-+}\rangle \rightarrow |\bar{m}_1\rangle|1_2\rangle|m_3\rangle|0_4\rangle,$$

$$|\chi^{--}\rangle \rightarrow |\bar{m}_1\rangle|0_2\rangle|m_3\rangle|1_4\rangle, \quad (10)$$

where the nonsense global phases are discarded. It can be verified that another group of basis states $\{\sigma_1^i\sigma_4^j|\chi^{00}\rangle'_{3214}, i,j=0,1,2,3\}$ can also be transformed into corresponding product states with the method above. So a deterministic χ-type-basis measurement can be achieved by individual detection of the related qubits.

Now we give a brief discussion on the experimental matters. Based on the current ion-trap techniques [17], we can use one Zeeman level of the $S_{1/2}$ ground state of $^{40}Ca^+$ ions as the state $|0\rangle$ and one Zeeman level of the metastable $D_{5/2}$ state as the state $|1\rangle$. The lifetime of the metastable $D_{5/2}$ state is $\tau_l \approx 1.16s$. With the accessible center-of-mass mode frequency $\nu=1.2$ MHz [18], we can choose $\delta=0.1\nu$, $\eta=0.1$, $\Omega=0.1\nu$, thus the condition $\Omega\eta \ll \delta \ll \nu$ can be satisfied. Then the mean phonon number $\bar{n}$ of the center-of-mass mode should be less than 3 in order to satisfy $\eta\sqrt{\bar{n}+1} \ll 1$, i.e., the Lamb-Dicke criterion. According to recent experiments [17–20], these parametric values are available, with which the duration of our schemes is only $3\tau=\pi\delta/(8\eta^2\Omega^2) \approx 1$ μs much shorter than τ_l. In addition, it is experimentally achievable to precisely control the laser-illuminating time and locate the ions in a trap [17–19]. Thus our schemes are realizable with the present ion-trap techniques.

The main factor that decreases the quality (fidelity) of the generated state may be the intensity fluctuation of the laser fields [20]. Now we estimate such an error. Suppose that the fluctuation of the Rabi frequency of the laser fields is δ_Ω ($\delta_\Omega \ll \Omega$). Then the infidelity caused by the fluctuation is

$$f_i \approx \frac{3(\pi\delta_\Omega)^2}{4\Omega^2}. \quad (11)$$

It can be seen that the infidelity is proportional to $(\delta_\Omega)^2$ for a given Ω. Setting $\delta_\Omega=0.05$ Ω, we have $f_i=0.0185$. This result implies that our scheme is only slightly affected by the small intensity fluctuation of the laser fields. The erroneous rate of χ-type-basis measurement caused by the fluctuation of the Rabi frequency of the laser fields is also described by the formula of Eq. (11).

In summary, the χ-type entangled states [8] have many interesting properties and possible applications in QIP and fundamental tests of quantum physics. Thus it is meaningful to study them in concrete systems. In this paper, we presented a simple scheme for generating such a type of multipartite entangled states in an ion-trap system. The idea can also be used in other systems, such as cavity quantum electrodynamics [14]. Moreover, we demonstrated how to realize the χ-type-basis measurement. It has been shown that the 16 χ-type basis states can be deterministically distinguished only by individual detection of related qubits. Our work indicates that the schemes for teleportation and dense coding with this type of states [8] can be directly demonstrated in the ion-trap system.

We anticipate that the χ-type entangled states may have other potential applications, such as error prevention in QIP [21]. In addition, it may be an interesting work to generalize the χ-type entangled states to the case where more than four particles are involved, and to multilevel and continuous-variable systems like the generalization of the cluster states [22].

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