Exact spatial soliton solutions of the two-dimensional generalized nonlinear Schrödinger equation with distributed coefficients

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An improved homogeneous balance principle and an F-expansion technique are used to construct exact periodic wave solutions to the generalized two-dimensional nonlinear Schrödinger equation with distributed dispersion, nonlinearity, and gain coefficients. For limiting parameters, these periodic wave solutions acquire the form of localized spatial solitons. Such solutions exist under certain conditions, and impose constraints on the functions describing dispersion, nonlinearity, and gain (or loss). We establish a simple procedure to select different classes of solutions, using the dispersion and the gain coefficient in one case, or the chirp function and the gain coefficient in the other case, as independent parameter functions. We present a few characteristic examples of periodic wave and soliton solutions with physical relevance.

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I. INTRODUCTION

Exact solutions of nonlinear partial differential equations (PDEs) have been investigated in recent years by many workers interested in nonlinear physical phenomena [1,2]. Various sophisticated methods for solving PDEs have been created to find soliton solutions, for example the inverse scattering transform [3], Hirota's bilinear operators [4], Backlund transformation [5], and the truncated Painlevé expansion [6]. With the advent of symbolic computation systems, the methods of direct algebraic manipulation have become feasible. New powerful methods of solution have been invented, such as the homogeneous balance principle [7], the F-expansion technique [8], the hyperbolic tangent expansion method [9], the generalized Riccati equation method [10], and the Jacobi elliptic function (JEF) expansion method [11], among others. We combine the first two of these methods, to obtain exact periodic wave and soliton solutions of the two-dimensional nonlinear Schrödinger equation (NLSE) with distributed coefficients.

The NLSE is one of the most important universal nonlinear models that naturally arises in many physical systems. It appears in numerous branches of physics, such as nonlinear optics, condensed matter physics, Bose-Einstein condensates, plasma physics, and hydrodynamics [12]. It is integrable in $1+1$ dimensions [13,14], with many stable soliton solutions reported thus far. The one-dimensional NLSE with distributed coefficients contains exact sech- and tanh-shaped stable solution pulses as solutions [15,16]. Additional periodic solutions in terms of JEFs are reported in Ref. [17]. These soliton pulses exhibit linear chirp and possess huge applicative potential in optical communication technology [18]. No comparable stable solutions are known for the two-dimensional NLSE.

The problem is that the inverse scattering method, instrumental in finding stable soliton solutions in one dimension (1D), is not applicable to the 2D NLSE. All localized solutions of the 2D NLSE with constant coefficients and self-focusing nonlinearity either spread out with propagation (for input powers less than a critical value) or collapse at a finite distance (for powers above the critical value) [19]. Even the radially symmetric solution at the critical power (or the critical exponent), known as the Townes soliton, is unstable. Small perturbation of this solution below the critical power leads to expansion, whereas perturbation above the critical power leads to blowup. Great interest has been generated recently when it was suggested that the 2D NLSE with distributed coefficients may lead to stable 2D solitons [20]. The stabilizing mechanism was the sign-alternating Kerr nonlinearity in a layered medium. A vigorous search for stabilized periodic solutions of the 2D NLSE with distributed coefficients has been launched [21]; however, out of necessity, it has been numerical. In this paper we present a method for finding analytical periodic wave solutions to the 2D NLSE with distributed coefficients.

We also present additional classes of exact spatial soliton solutions to the 2D NLSE with varying dispersion, nonlinearity, and gain or loss. The question of stability of these solitons will be addressed elsewhere. We use the homogeneous balance principle and the F-expansion technique, which are applicable for finding analytical solutions to a class of nonlinear of PDEs.

Section II of the paper introduces the model equation, and Sec. III the solution procedure. Section IV presents the periodic wave solutions and Sec. V the solitary wave solutions. Section VI gives the conclusions.

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II. GENERALIZED NONLINEAR SCHRÖDINGER EQUATION

In Cartesian coordinates, the generalized NLSE describing the propagation of 2D spatial solitons in bulk optical media with varying coefficients is expressed as [16,17]

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\beta(z)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u+\chi(z)|u|^2u=i\gamma(z)u, \tag{1}$$

where $u(z,x,y)$ is the complex envelope of the electrical field in the moving frame, z is the normalized distance of propagation, and x and y are the normalized coordinates in the transverse plane. The function $\beta(z)$ represents the dispersion coefficient, $\chi(z)$ the nonlinearity coefficient, and $\gamma(z)$ the gain ($\gamma>0$) or the loss ($\gamma<0$) coefficient. Thus, Eq. (1) describes the propagation behavior of an optical beam in a Kerr-like medium with varying dispersion, nonlinearity, and gain or loss. We define the complex u field as [22,23]

$$u(z,x,y)=A(z,x,y)e^{iB(z,x,y)}, \tag{2}$$

where $A(z,x,y)$ and $B(z,x,y)$ are real functions. Substituting $u(z,x,y)$ into Eq. (1), we find the following coupled equations for the phase $B(z,x,y)$ and the amplitude $A(z,x,y)$:

$$\frac{\partial A}{\partial z}+\frac{1}{2}\beta(z)\left(2\frac{\partial A}{\partial x}\frac{\partial B}{\partial x}+2\frac{\partial A}{\partial y}\frac{\partial B}{\partial y}+A\frac{\partial^2 B}{\partial x^2}+A\frac{\partial^2 B}{\partial y^2}\right)=\gamma(z)A, \tag{3}$$

$$-A\frac{\partial B}{\partial z}+\frac{1}{2}\beta(z)\left[\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}-A\left(\frac{\partial B}{\partial x}\right)^2-A\left(\frac{\partial B}{\partial y}\right)^2\right]+\chi(z)A^3=0. \tag{4}$$

These equations are treated by the homogeneous balance principle and the F-expansion technique, mentioned in the previous section.

III. SOLUTION PROCEDURE

According to the balance principle and the F-expansion technique [7,8], the solution of Eqs. (3) and (4) can be expressed in the following form:

$$A(z,x,y)=f_0(z)+f_1(z)F(\theta)+f_{-1}(z)F^{-1}(\theta), \tag{5a}$$

$$\theta=k(z)x+l(z)y+\omega(z), \tag{5b}$$

$$B(z,\rho)=a(z)\rho^2+b(z)(x+y)+e(z), \tag{5c}$$

where $\rho^2=x^2+y^2$ and f_{-1}, f_0, f_1, k, l, ω, a, b, and e are the parameters to be determined. In this quadratic form of Eq. (5c) the parameter $a(z)$ is related to the wave front curvature; it is also a measure of the phase chirp imposed on the solitary wave. The function $F(\theta)$ is one of JEFs, which in general satisfy the following general first- and second-order nonlinear ordinary differential equations:

$$\left(\frac{dF}{d\theta}\right)^2=c_0+c_2F^2+c_4F^4, \tag{6a}$$

TABLE I. Jacobi elliptic functions.

	c_0	c_2	c_4	$F(\theta)$
1	1	$-(1+m)$	m	sn
2	$1-m$	$2m-1$	$-m$	cn
3	$m-1$	$2-m$	-1	dn
4	m	$-(1+m)$	1	ns
5	$-m$	$2m-1$	$1-m$	nc
6	-1	$2-m$	$m-1$	nd
7	1	$2-m$	$1-m$	sc
8	$1-m$	$2-m$	1	cs
9	1	$-(1+m)$	m	cd
10	m	$-(1+m)$	1	dc
11	1	$2m-1$	$-m(1-m)$	sd
12	$-m(1-m)$	$2m-1$	1	ds

$$\frac{d^2F}{d\theta^2}=c_2F+2c_4F^3, \tag{6b}$$

where c_0, c_2, and c_4 are real constants related to the elliptic modulus of the JEFs (see Table I). Substituting Eqs. (5) into Eqs. (3) and (4) and requiring that x^qF^n, y^qF^n ($q=0,1,2$; $n=0,1,2,3$), and $\sqrt{c_0+c_2F^2+c_4F^4}$ of each term be separately equal to zero, we obtain a system of algebraic or first-order ordinary differential equations for f_p, k, l, ω, a, b, and e:

$$\frac{df_p}{dz}+2a\beta f_p-\gamma(z)f_p=0, \tag{7a}$$

$$f_j\left(\frac{dk}{dz}+2ka\beta(z)\right)=0, \tag{7b}$$

$$f_j\left(\frac{dl}{dz}+2la\beta(z)\right)=0, \tag{7c}$$

$$f_j\left(\frac{d\omega}{dz}+\beta(z)(k+l)b\right)=0, \tag{7d}$$

$$-f_p\left(\frac{da}{dz}+2\beta(z)a^2\right)=0, \tag{7e}$$

$$-f_p\left(\frac{db}{dz}+2\beta(z)ab\right)=0, \tag{7f}$$

$$-f_0\left(\frac{de}{dz}+\beta(z)b^2-\chi(z)f_0^2-6\chi(z)f_1f_{-1}\right)=0, \tag{7g}$$

$$-f_j\left(\frac{de}{dz}-\frac{1}{2}\beta(z)k^2c_2-\frac{1}{2}\beta(z)l^2c_2+\beta(z)b^2-3\chi(z)f_0^2-3\chi(z)f_1f_{-1}\right)=0, \tag{7h}$$

$$f_1[\beta(z)(k^2+l^2)c_4+\chi(z)f_1^2]=0, \tag{7i}$$

$$f_{-1}[\beta(z)(k^2+l^2)c_0+\chi(z)f_{-1}^2]=0, \quad (7j)$$

$$3\chi(z)f_0f_j^2=0, \quad (7k)$$

where $p=0,-1,1$ and $j=-1,1$. By solving Eqs. (7) self-consistently, one obtains a set of conditions on the coefficients and parameters, necessary for Eq. (1) to have exact periodic wave solutions.

IV. PERIODIC WAVE SOLUTIONS

After finding the consistent solutions of Eqs. (7), we obtain exact periodic solutions of Eq. (1) in the following cases.

Case (1). $\beta(z)$ and $\gamma(z)$ are arbitrary functions of the propagation distance. $f_0=f_{-1}=a=0$, $b(z)=b_0$, $k(z)=k_0$, $l(z)=l_0$, $f_1=f_{10}\exp[\int_0^z\gamma(z)dz]$, $\omega=-(k_0+l_0)b_0\int_0^z\beta(z)dz+\omega_0$, $e=[\frac{1}{2}c_2(k_0^2+l_0^2)-b_0^2]\int_0^z\beta(z)dz+e_0$, $\theta=k_0x+l_0y-(k_0+l_0)b_0\int_0^z\beta\times(z)dz+\omega_0$, and

$$u(z,x,y)=f_{10}\exp\left(\int_0^z\gamma(z)dz\right)F(\theta)e^{i[b_0(x+y)+e]}. \quad (8a)$$

Case (2). $\beta(z)$ and $\gamma(z)$ are arbitrary functions of the propagation distance. $f_0=a=0$, $b=b_0$, $k=k_0$, $l=l_0$, $f_1=f_{10}\exp[\int_0^z\gamma(z)dz]$, $f_{-1}=\pm\sqrt{c_0/c_4}f_{10}\exp[\int_0^z\gamma(z)dz]$, $e=[\frac{1}{2}c_2(k_0^2+l_0^2)-b_0^2\mp3\sqrt{c_0c_4}(k_0^2+l_0^2)]\int_0^z\beta(z)dz+e_0$, $\omega=-(k_0+l_0)b_0\int_0^z\beta(z)dz+\omega_0$, $\theta=k_0x+l_0y-(k_0+l_0)b_0\int_0^z\beta(z)dz+\omega_0$, and

$$u(z,x,y)=f_{10}\exp\left(\int_0^z\gamma(z)dz\right)\times\left(F(\theta)\pm\sqrt{\frac{c_0}{c_4}}\frac{1}{F(\theta)}\right)e^{i[b_0(x+y)+e]}. \quad (8b)$$

Case (3). $a(z)$ and $\gamma(z)$ are arbitrary functions of the propagation distance. $f_0=f_{-1}=0$, $b=b_0a$, $k=k_0a$, $l=l_0a$, $f_1=f_{10}a\exp[\int_0^z\gamma(z)dz]$, $\omega=\frac{1}{2}(k_0+l_0)b_0a+\omega_0$, $e=-\frac{1}{4}(k_0^2+l_0^2)c_2a+(b_0^2/2)a+e_0$, $\theta=k_0ax+l_0ay+\frac{1}{2}(k_0+l_0)b_0a+\omega_0$, and

$$u(z,x,y)=af_{10}\exp\left(\int_0^z\gamma(z)dz\right)F(\theta)e^{i[a\rho^2+b_0a(x+y)+e]}. \quad (8c)$$

Case (4). $a(z)$ and $\gamma(z)$ are arbitrary functions of the propagation distance. $f_0=0$, $b=b_0a$, $k=k_0a$, $l=l_0a$, $f_1=f_{10}a\exp[\int_0^z\gamma(z)dz]$, $e=-\frac{1}{4}(k_0^2+l_0^2)c_2a+(b_0^2/2)a\pm3(k_0^2+l_0^2)\times\sqrt{c_0c_4}a/2+e_0$, $\omega=\frac{1}{2}(k_0+l_0)b_0a+\omega_0$, $f_{-1}=\pm\sqrt{c_0/c_4}af_{10}\times\exp[\int_0^z\gamma(z)dz]$, $\theta=k_0ax+l_0ay+\frac{1}{2}(k_0+l_0)b_0a+\omega_0$, and

$$u(z,x,y)=af_{10}\exp\left(\int_0^z\gamma(z)dz\right)\times\left(F(\theta)\pm\sqrt{\frac{c_0}{c_4}}\frac{1}{F(\theta)}\right)e^{i[a\rho^2+b_0a(x+y)+e]}. \quad (8d)$$

Here and in what follows the symbols with the subscript 0 are used to represent the initial values of the corresponding parameters at the initial distance $z=0$, with all of them being constant.

In all four cases the nonlinearity coefficient $\chi(z)$ is expressed in terms of the other coefficients $\beta(z)$ and $\gamma(z)$:

$$\chi=-\frac{\beta(z)(k_0^2+l_0^2)c_4}{f_{10}^2}\exp\left(-2\int_0^z\gamma(z)dz\right). \quad (9a)$$

This equation can be conveniently understood as an integrability condition on Eq. (1):

$$\frac{1}{\chi}\frac{d\chi}{dz}-\frac{1}{\beta}\frac{d\beta}{dz}+2\gamma(z)=0. \quad (9b)$$

Hence, the solutions found can exist only under certain conditions and the system parameter functions $\beta(z)$, $\gamma(z)$, and $\chi(z)$ cannot be all chosen independently. In cases (1) and (2) we chose $\beta(z)$ and $\gamma(z)$ as independent, and the nonlinearity coefficient $\chi(z)$ is determined by Eq. (9a). In these two cases the beams are unchirped pulses. In cases (3) and (4), the chirp function $a(z)$ and the gain coefficient $\gamma(z)$ are chosen as independent, and the dispersion coefficient $\beta(z)$ is determined from the equation

$$\beta(z)=-\frac{1}{2a^2}\frac{da}{dz}. \quad (10)$$

Hence, for chirped pulses, $\beta(z)$ and $a(z)$ cannot be arbitrary simultaneously. As long as we want to have beams with chirp, the dispersion coefficient $\beta(z)$ will be determined by Eq. (10). When the chirp function $a(z)$ and the gain $\gamma(z)$ are chosen as independent parameters, then the other parameters follow from Eqs. (9a), (9b), and (10). In this way different periodic and solitary solutions are obtained, depending in each case on two real independent functions, $\beta(z)$ and $\gamma(z)$, or $a(z)$ and $\gamma(z)$.

In Table I we list JEFs that may appear in the solutions. As long as we choose the constants according to the relationships listed in the table, and substitute the appropriate $F(\theta)$ into Eqs. (8a)–(8d), we obtain exact periodic traveling wave solutions to the generalized 2D NLSE. The parameter m ($0\leqslant m\leqslant1$) in the table is the square of the elliptic modulus of JEFs. When $m\to0$, JEFs degenerate into trigonometric functions, i.e., $\mathrm{sn}(\theta)\to\sin(\theta)$, $\mathrm{cn}(\theta)\to\cos(\theta)$, $\mathrm{dn}(\theta)\to1$, and the periodic traveling wave solutions become the periodic trigonometric solutions. When $m\to1$, JEFs degenerate into the hyperbolic functions, i.e., $\mathrm{sn}(\theta)\to\tanh(\theta)$, $\mathrm{cn}(\theta)\to\mathrm{sech}(\theta)$, $\mathrm{dn}(\theta)\to\mathrm{sech}(\theta)$, etc., and the periodic traveling wave solutions become the spatial soliton solutions of the NLSE.

V. PROPAGATION CHARACTERISTICS OF SPATIAL SOLITONS

As mentioned, the solutions to the generalized NLSE can be categorized in terms of two sets of arbitrary functions, $\beta(z)$ and $\gamma(z)$ or $a(z)$ and $\gamma(z)$. We have freedom in selecting these functions appropriately, according to some actual physical requirements, so as to improve the soliton propagation characteristics. For example, having in mind inherent instability of multidimensional spatial solitons, one can select suitable soliton-supporting optical media in which long-distance stable propagation can be achieved.

TABLE II. Some single-JEF soliton solutions.

Soliton type	Case	Soliton intensity	θ expression	Existence condition	Independent functions
Bright soliton	Case (1): Unchirped soliton	$\lvert u_1(z,x,y)\rvert^2=f_{10}^2\exp(2\int_0^z\gamma(z)dz)\operatorname{sech}^2(\theta)$	$\theta=k_0x+l_0y-(k_0+l_0)b_0\int_0^z\beta(z)dz+\omega_0$	$\frac{c_4\beta}{\chi}<0$	$\beta(z),\gamma(z)$
	Case (2): Chirped soliton	$\lvert u_2(z,x,y)\rvert^2=a^2f_{10}^2\exp(2\int_0^z\gamma(z)dz)\operatorname{sech}^2(\theta)$	$\theta=a(k_0x+l_0y+\frac{1}{2}(k_0+l_0)b_0)+\omega_0$	$\frac{c_4}{\chi}\frac{da}{dz}>0$	$a(z),\gamma(z)$
Dark soliton	Case (3): Unchirped soliton	$\lvert u_3(z,x,y)\rvert^2=f_{10}^2\exp(2\int_0^z\gamma(z)dz)\tanh^2(\theta)$	$\theta=k_0x+l_0y-(k_0+l_0)b_0\int_0^z\beta(z)dz+\omega_0$	$\frac{c_4\beta}{\chi}<0$	$\beta(z),\gamma(z)$
	Case (4): Chirped soliton	$\lvert u_4(z,x,y)\rvert^2=a^2f_{10}^2\exp(2\int_0^z\gamma(z)dz)\tanh^2(\theta)$	$\theta=a(k_0x+l_0y+\frac{1}{2}(k_0+l_0)b_0)+\omega_0$	$\frac{c_4}{\chi}\frac{da}{dz}>0$	$a(z),\gamma(z)$

Substituting JEFs from Table I into Eqs. (8a)–(8d), with the condition $m=1$, we obtain the solitary wave solutions. The form of the solitons is controlled by the parameter functions $\beta(z)$ and $\gamma(z)$ or $a(z)$ and $\gamma(z)$. In Table II we exhibit some interesting single-JEF soliton solutions. To display unique behavior of these exact soliton solutions, we choose the dispersion coefficient $\beta(z)$ and the phase chirp parameter $a(z)$ in terms of trigonometric and hyperbolic functions, in addition to the standard constant or linear functions. Trigonometric functions are physically relevant because they provide for alternating regions of positive and negative dispersion and nonlinearity, which is indicated in improved stability of 2D spatial solitons [20,21]. Hyperbolic functions are convenient for displaying abrupt one-time changes in the propagation of spatial solitons. Thus, we present some examples using the linear, trigonometric and hyperbolic distributed control systems.

We see from Table II that the speed of the unchirped dark soliton, u_3, is related to $-(k_0+l_0)b_0[\beta(z)-\beta_0]$, while its phase shift is determined by $[\frac{1}{2}c_2(k_0^2+l_0^2)-b_0^2]\int_0^z\beta(z)dz$ [case (3)]. In addition, from Eq. (9a) the wave amplitude is given by $f_{10}\exp\int_0^z\gamma(z)dz=\sqrt{-(k_0^2+l_0^2)c_4\beta/\chi}$, where $c_4\beta/\chi<0$. Therefore, one can choose suitable initial conditions and the dispersion coefficient of the bulk optical material to control the speed and the phase shift of the soliton. For convenience, we fix $f_{10}=k_0=l_0=b_0=1$ and $\omega_0=e_0=0$ throughout. The amplitude and solitary characteristics can solely be controlled by the dispersion $\beta(z)$ and the gain $\gamma(z)$. Likewise for the unchirped bright solitary wave [case (1)]. For solitary waves with chirp from Table II [cases (2) and (4)], the chirp function $a(z)$ and the gain coefficient $\gamma(z)$ will determine the wave propagation characteristics. An interesting property of the chirped soliton is that the soliton intensity $|u|^2$ is periodic along the propagation distance.

In Fig. 1 we present a comparison between a periodic wave solution and a solitary wave solution. As one can see, the spatial soliton is reminiscent of a single period of the JEF periodic wave solution. In Figs. 2 and 3 we present contour plots of the solitary wave intensity $|u|^2$ distributions of the four cases from Table II, for specific choices of control functions. As is evident, when $\beta(z)$ is linear or hyperbolic tan function (not shown), there occurs a turning point in the steady pulse propagation. When $\beta(z)$ is a trigonometric function, it produces a sign-changing nonlinearity, and there is a snakelike appearance in the intensity distribution, as in Fig. 4.

Finally, we discuss propagation properties of an unchirped solitary wave possessing constant gain (or loss) in the bulk optical medium with a specific dispersion and nonlinearity [24]. We assume the dispersion coefficient $\beta(z)=[\chi(z)/\sigma_0]e^{\sigma z}$, and the Kerr nonlinearity coefficient $\chi(z)=\chi_0+\chi_1\cos(gz)$, where σ_0 is linked to the initial peak power and χ_0, χ_1, and g characterize the Kerr effect. For appropriate values of χ_0 and χ_1 this nonlinearity consists of positive/negative layers along the propagation direction. The gain is $\gamma(z)=\sigma/2$, corresponding to an exponentially decreasing or increasing dispersion in the medium. From Table II we know the dark solitary wave solution:

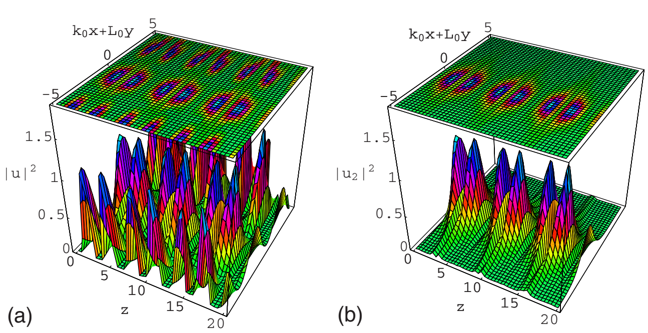

FIG. 1. (Color online) Comparison of intensity distributions of a cn wave with a chirped bright solitary wave, for the same parameter functions: $a(z)=\sin(z)$, $\gamma(z)=\sin(z)$. (a) cn wave, $m=0.5$: (b) chirped bright solitary wave, $m=1$.

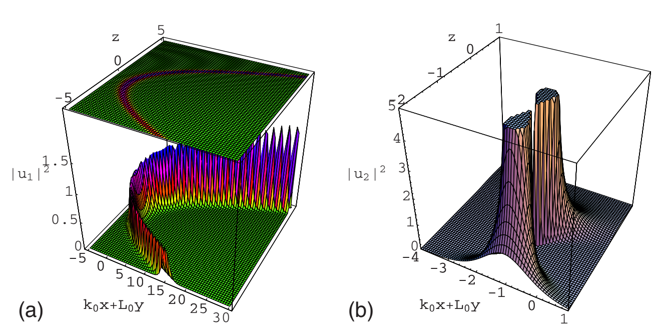

FIG. 2. (Color online) Comparison of intensity distributions of an unchirped bright solitary wave with a chirped bright solitary wave. Parameters: (a) Unchirped wave, $a(z)=0$, $\beta(z)=1+z$, $\gamma(z)=0.05$: (b) chirped wave, $a(z)=1/(1+z)^2$, $\beta(z)=1+z$, $\gamma(z)=-0.05$.

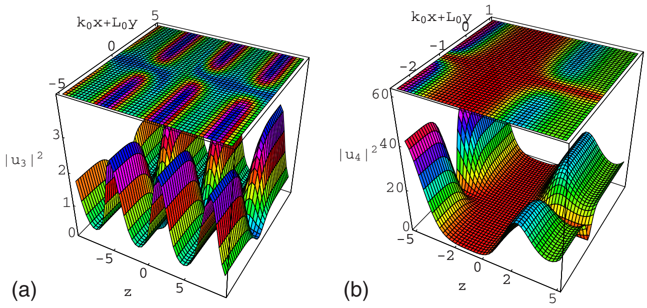

FIG. 3. (Color online) Comparison of intensity distributions of an unchirped dark solitary wave with a chirped dark solitary wave. Parameters: (a) Unchirped wave, $a(z)=0$, $\beta(z)=-1/2(1+z)^2$, $\gamma(z)=\cos(z)/2$; (b) chirped wave, $a(z)=1+z$, $\beta(z)=-1/2(1+z)^2$, $\gamma(z)=\cos(z)/2$;

$$u_5=f_{10}e^{\sigma z/2}\tanh(\theta)\exp\left\{i\left[b_0(x+y)+\frac{e^{\sigma z}}{\sigma_0}\left(\frac{1}{2}(k_0^2+l_0^2)-b_0^2\right)\times\left(\frac{\chi_0}{\sigma}+\frac{\sigma\chi_1\cos(gz)+g\chi_1\sin(gz)}{\sigma^2+g^2}\right)\right]\right\}, \tag{11}$$

where

$$\theta=k_0x+l_0y-\frac{(k_0+l_0)b_0e^{\sigma z}}{\sigma_0}\times\left(\frac{\chi_0}{\sigma}+\frac{\sigma\chi_1\cos(gz)+g\chi_1\sin(gz)}{\sigma^2+g^2}\right).$$

In Fig. 4 we present the evolution of the unchirped solitary wave u_5. We see that the phase shift and the speed of this solitary wave evolve according to the changes in the dispersion parameter only, and the solitary wave width remains unchanged. This is an important feature of an unchirped dark solitary wave. It is also remarkable that the pulse position varies periodically for $\sigma=0$; however, for $\sigma<0$ (or $\sigma>0$) the amplitude of this variation, as well as the amplitude of the wave itself, is decreasing (or increasing) with propagation.

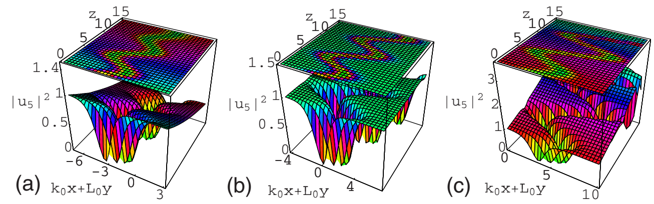

FIG. 4. (Color online) Intensity plots of the solution u_5, for three values of the gain: $\sigma=$(a)-0.06, (b) 0, and (c) 0.06. Other parameters are: $\chi_0=0.1$, $\chi_1=\sigma_0=g=1$.

VI. CONCLUSIONS

An improved homogeneous balance principle and an F-expansion technique are applied to the two-dimensional generalized nonlinear Schrödinger equation with distributed dispersion, nonlinearity, and gain. Abundant exact periodic wave solutions are obtained. Unusual soliton solutions are found. A procedure is presented for control of spatial solitons, in which one may select the dispersion and the gain coefficient, or the chirp function and the gain coefficient, to control propagation behavior of solitons. The present solution method provides a reliable technique that is more transparent and less tedious than the Jacobi elliptic function ansatz, or other expansion and variational methods. The technique is also applicable to other multidimensional nonlinear partial differential equation systems.

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