PHYSICAL REVIEW A **78**, 023616 (2008)

# Spin-orbit coupled Bose-Einstein condensates

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(Received 18 December 2007; revised manuscript received 15 July 2008; published 11 August 2008)

We consider a many-body system of pseudo-spin-1/2 bosons with spin-orbit interactions, which couple the momentum and the internal pseudo-spin degree of freedom created by spatially varying laser fields. The corresponding single-particle spectrum is generally anisotropic and contains two degenerate minima at finite momenta. At low temperatures, the many-body system condenses into these minima generating a type of entangled Bose-Einstein condensate. We show that in the presence of weak density-density interactions the many-body ground state is characterized by a twofold degeneracy. The corresponding many-body wave function describes a condensate of "left-" and "right-moving" bosons. By fine-tuning the parameters of the laser field, one can obtain a bosonic version of the spin-orbit coupled Rashba model. In this symmetric case, the degeneracy of the ground state is very large, which may lead to phases with nontrivial topological properties. We argue that the predicted type of Bose-Einstein condensates can be observed experimentally via time-of-flight imaging, which will show characteristic multipeak structures in momentum distribution.

DOI: 10.1103/PhysRevA.78.023616 PACS number(s): 03.75.Ss, 05.30.Jp, 71.70.Ej, 72.25.Rb

## I. INTRODUCTION

Bose-Einstein condensation is an old and thoroughly studied quantum phenomenon, where a many-body system of bosons undergoes a phase transition in which a single-particle state becomes macroscopically occupied. This phenomenon has been observed in condensed matter systems and more recently in experiments on cold atomic gases [1,2], which provided a unique avenue to visualize the formation of the Bose-Einstein condensate (BEC). Bose-Einstein condensation is a phase transition driven mostly by the statistics of the underlying bosonic excitations and not by interactions. The statistics of basic particles are determined by the particle spin via the fundamental Pauli spin-statistics theorem [3]: The spin must be an integer for bosons and a half-integer for fermions.

In this paper, we discuss a cold atomic system of multilevel bosons moving in the presence of spatially varying laser fields, which give rise to an emergent pseudo-spin-1/2 degree of freedom for the bosons. We emphasize from the outset that the symmetry operations in the pseudo-spin space are not related to real-space rotations and thus there is no contradiction between the existence of the pseudo-spin-1/2 bosons and the fundamental Pauli theorem. To "create" the pseudo-spin-1/2 bosons, one can use the experimental setup, proposed in Refs. [4–10], in which three degenerate hyperfine ground states $|1\rangle$, $|2\rangle$, $|3\rangle$ are coupled to an excited state $|0\rangle$ by spatially varying laser fields. This "tripod scheme" leads to the appearance of a pair of degenerate dark states, spanning a subspace which is well-separated in energy from two nondegenerate bright states. The coupling between the dark and the bright states is very weak and will be neglected (adiabatic approximation). The parameter which labels the dark states plays the role of a pseudo-spin index. This emergent pseudo-spin degree of freedom is similar to that studied recently in the context of spinor condensates [11,12]. In particular, various aspects of the pseudo spin-1/2 boson physics were addressed [13–22] using two hyperfine states to support the internal degree of freedom associated with the pseudo spin. The key distinctive feature of the systems studied in this paper is that the single-particle Hamiltonian projected onto the subspace of the degenerate dark states generally possesses a non-Abelian gauge structure. That is, the kinetic term of the effective Hamiltonian has the form $\check{H}_{\rm kin} = (\mathbf{p}\check{1} - \check{\mathbf{A}})^2/2m$, where $\check{\mathbf{A}}(\mathbf{r})$ is a matrix in the pseudo-spin space. In a recent Letter [10], we pointed out that under certain conditions this non-Abelian gauge structure is equivalent to a spin-orbit interaction. To understand the nature of this interaction, we note that the dark states are eigenstates of an atom at rest. Once the atom moves in the spatially modulated laser field, the dark state label, i.e., the pseudo-spin index, is not a good quantum number and the pseudo-spin starts to precess about the direction of the momentum. This coupling between the internal degree of freedom associated with the dark state subspace and the orbital movement of the particle represents the spin-orbit interaction. The spin-orbit coupling parameters can be adjusted by changing the properties of the spatially modulated light beams. These conclusions are based entirely on single-particle physics; the particle statistics play no role. Below, we consider a many-particle system of bosons within this tripod scheme. Due to spin-orbit coupling, the degenerate ground states of the system correspond to nonzero momenta, leading to a type of BEC, the spin-orbit coupled Bose-Einstein condensate (SOBEC).

The paper is organized as follows: In Sec. II we introduce our model and discuss the properties of a noninteracting many body system of bosons with spin-orbit interactions. We find that, in general, the single-particle spectrum is characterized by two degenerate minima at finite momenta and we determine the transition temperature for the bosons condensing into these minima. In Sec. III we study the effects of density-density interactions using a generalized Bogoliubov transformation (Sec. III A). We show that the quasiparticle excitation spectrum contains an anisotropic free particle component and an anisotropic sound similar to the conventional Bogoliubov phonon. By calculating the energy of the condensate, we find that for a system of $N$ bosons the ($N$

 

+1)-fold degeneracy of the noninteracting ground state is reduced by the interactions to a twofold degeneracy corresponding to "left-" or "right-moving" particles. The corresponding many-body wave function describes a ($N00N$)—type state [23] [i.e., an entangled quantum state of the form $(|N,0\rangle+|0,N\rangle)$], suggesting that future studies of the SOBEC state in the context of quantum entanglement and quantum interference are highly relevant. For completeness, we also derive the Gross-Pitaevskii equations for the spin-orbit coupled condensate (Sec. III B). Linearizing the coupled nonlinear equations in the vicinity of a stationary solution leads to a spectrum of excitations that reproduces the generalized Bogoliubov result. A possible experimental signature of the SOBEC is described in Sec. IV. We argue that a SOBEC can be observed via time-of-flight imaging, which will show a characteristic multipeak structure of the density profiles. We demonstrate that such a measurement generates distinct outputs for left- and right-moving condensates and thus can be viewed as a measurement of a qubit. A summary of the paper along with our conclusions are presented in Sec. V.

## II. SPIN-ORBIT INTERACTING HAMILTONIAN AND SINGLE-PARTICLE SPECTRUM

We start with the following many-body Hamiltonian describing spin-orbit coupled bosons,

$$\hat{\mathcal{H}} = \sum_{\mathbf{p};\alpha,\beta} \hat{b}^\dagger_{\alpha\mathbf{p}} \left\{ \frac{\mathbf{p}^2}{2m}\check{1} - v p_x \check{\sigma}_2 - v' p_y \check{\sigma}_3 \right\}_{\alpha\beta} \hat{b}_{\beta\mathbf{p}}, \tag{1}$$

where $\hat{b}^\dagger_{\alpha\mathbf{p}}$ and $\hat{b}_{\alpha\mathbf{p}}$ are the creation and annihilation operators for bosons in the state with momentum $\mathbf{p}$ and pseudo-spin $\alpha=\uparrow,\downarrow$, $\check{\sigma}_i$ are the Pauli matrices in the pseudo-spin space, and the parameters $v$ and $v'$ characterize the strength and anisotropy of the spin-orbit coupling. We reiterate that this type of spin-orbit-coupled Hamiltonian (1) will appear within the recently proposed tripod scheme [4–10] in which three hyperfine ground states of an atom $|1\rangle$, $|2\rangle$, and $|3\rangle$ are coupled to an excited state $|0\rangle$ via spatially modulated laser fields. The underlying laser-atom Hamiltonian is

$$\mathcal{H}_{\text{a-l}} = \Omega_0|0\rangle\langle 0| + \sum_{\mu=1}^{3}[\Omega_\mu(\mathbf{r})|0\rangle\langle\mu| + \text{H.c.}], \tag{2}$$

where $\Omega_0$ is a constant detuning to the excited state and the Rabi frequencies consistent with the realization of an effective spin-orbit interaction can be taken as $\Omega_1(\mathbf{r}) = \Omega \sin\theta \cos(mv_a x)e^{imv_b y}$, $\Omega_2(\mathbf{r})=\Omega \sin\theta \sin(mv_a x)e^{imv_b y}$, and $\Omega_3(\mathbf{r})=\Omega\cos\theta$, with $\Omega$, $\theta$, $v_a$, and $v_b$ being constants (see, e.g., Refs. [6,10] for details). Diagonalizing the atom-laser Hamiltonian (2) via a position-dependent rotation $R_{\mu\alpha}(\mathbf{r})$, with $\alpha\in\{\uparrow,\downarrow,b_1,b_2\}$ and $\mu\in\{0,1,2,3\}$, generates a pair of degenerate dark states

$$|\uparrow\rangle = \sin\Phi_x e^{-iS_y}|1\rangle - \cos\Phi_x e^{-iS_y}|2\rangle, \tag{3}$$

$$|\downarrow\rangle = \cos\theta\cos\Phi_x e^{-iS_y}|1\rangle + \cos\theta\sin\Phi_x e^{-iS_y}|2\rangle - \sin\theta|3\rangle,$$

with $\Phi_x=mv_a x$ and $S_y=mv_b y$, and two nondegenerate bright states $|b_{1(2)}\rangle$. The pseudo-spin-1/2 structure emerges when the problem is projected onto the subspace spanned by the pair of degenerate dark states [10]. Applying the position-dependent rotation $R_{\mu\alpha}(\mathbf{r})$ to the kinetic energy term in the Hamiltonian generates a coupling of the pseudo-spin to momentum [see Eq. (1)] with $v=v_a\cos\theta$ and $v'=v_b\sin^2(\theta/2)$, in the given parametrization. These coupling constants can be easily adjusted by changing the parameters $v_a$, $v_b$, and $\theta$ of the laser fields, which provides a knob to tune the strength and form of the spin-orbit interaction.

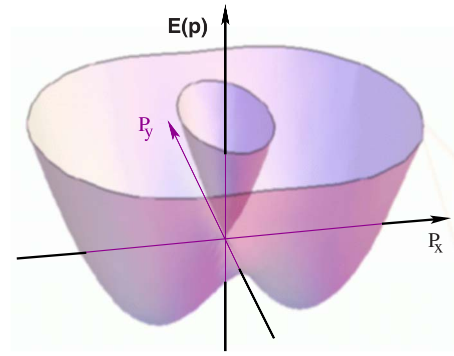


FIG. 1. (Color online) Schematic picture of the band structure described by Eq. (4) with $v/v'=2.5$ for a constant value of $p_z$. The inside sheet represents the $\lambda=+1$ band, while the outside sheet corresponds to $\lambda=-1$ and has a double-well structure with minima at $p_x=\pm mv$ and $p_y=0$.

Now, we concentrate on the generic case characterized by anisotropic spin-orbit interactions and assume for concreteness that $v>v'>0$. The trap potential and the interparticle interaction are initially disregarded and discussed in the following sections. The single-particle spectrum of Hamiltonian (1) is (see Fig. 1)

$$E_\lambda(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + \lambda\sqrt{v^2p_x^2 + v'^2p_y^2}, \tag{4}$$

where $\lambda=\pm1$ labels the bands. The corresponding eigenfunctions $\vec{\phi}_{\lambda\mathbf{p}}(\mathbf{r})=e^{i\mathbf{pr}}\vec{U}_\lambda(\chi_\mathbf{p})$ are spinors with components

$$U_{\uparrow\lambda}(\chi_\mathbf{p}) = \frac{[\sqrt{\cos^2\chi_\mathbf{p}+\Delta^2\sin^2\chi_\mathbf{p}} - \Delta\lambda\sin\chi_\mathbf{p}]^{1/2}}{\sqrt{2}[\cos^2\chi_\mathbf{p}+\Delta^2\sin^2\chi_\mathbf{p}]^{1/4}} \quad \text{and} \tag{5}$$

$$U_{\downarrow\lambda}(\chi_\mathbf{p}) = -i\lambda\,\text{sgn}[\cos\chi_\mathbf{p}]U_{\uparrow-\lambda}(\chi_\mathbf{p}), \tag{6}$$

where $\chi_\mathbf{p}$ is the azimuthal angle in the $(p_x,p_y)$ plane and $\Delta=v'/v<1$. The unitary matrix $U_{\alpha\lambda}(\chi_\mathbf{p})$ diagonalizes the Hamiltonian (1) (where $\alpha=\uparrow,\downarrow$ corresponds to the pseudo-spin index and $\lambda=\pm1$ labels the eigenstates). It is obvious from Eq. (4) that the spectrum of the single particle problem contains two minima at $\lambda=-1$ and momenta $p_y=p_z=0$ and $p_x=\pm mv\neq0$ (see Fig. 1). Consequently, the single particle ground state is double-degenerate and the most general expression for the corresponding wave function is

$$\Psi_{\mathrm{dw}}(\mathbf{r}) = \sqrt{w_{\mathrm{L}}}\begin{pmatrix}1\\ -i\end{pmatrix}e^{-imvx+i\phi_{\mathrm{L}}} + \sqrt{w_{\mathrm{R}}}\begin{pmatrix}1\\ i\end{pmatrix}e^{imvx+i\phi_{\mathrm{R}}}, \quad (7)$$

where $w_{\mathrm{L}} \geqslant 0$ and $w_{\mathrm{R}} \geqslant 0$ are the fractions of left- and right-moving states subjected to the constraint $w_{\mathrm{L}}+w_{\mathrm{R}}=1$, while $\phi_{\mathrm{L}}$ and $\phi_{\mathrm{R}}$ are arbitrary phases. Note that by left -or right-moving states we mean states with nonzero momentum average, $\langle \mathbf{p}\rangle = \mp mv\mathbf{e}_x$. However, the corresponding average velocity vanishes $\langle \nabla_{\mathbf{p}}\check{\mathcal{H}}(\mathbf{p})\rangle = \mathbf{0}$, so that quasiparticles characterized by these nonzero momentum single-particle states are not actually "moving," as long as the laser fields generating the spin-orbit coupling are maintained. Note that rotations in the manifold of the double-well ground states are distinct from rotations in the pseudo-spin Hilbert space, as real-space and pseudo-spin coordinates are mixed up by the spin-orbit interaction. The twofold degeneracy of the single-particle ground state is preserved if the system is placed in a harmonic trap. For a potential $V_{\mathrm{trap}}=m\omega^2\mathbf{r}^2/2$, we can write the Schödinger equation in momentum representation: The trap potential plays the role of "the kinetic energy" and the real kinetic term produces a double-well potential in momentum space, see Fig. 1. The tunneling processes connect the degenerate vacua in momentum space [24]. However, they do not eliminate the double-degeneracy of the single-particle states, which is protected by the Kramers-like symmetry (see Sec. III B).

At low temperatures, the many-body Bose system (1) condenses into the single-particle states corresponding to the double-well minima. The transition temperature of this double-well SOBEC can be calculated using standard textbook procedures [25]. Let us assume that near and below the transition the band with $\lambda=+1$ does not contribute and that we can expand the spectrum near the minima of the band (4). We define the momentum $\mathbf{q}$ in the vicinity of the left or right minima as follows: $\mathbf{p}=\pm mv\mathbf{e}_x+\mathbf{q}$, with $q \ll mv$. Equation (4) leads to the anisotropic spectrum:

$$\delta E(\mathbf{q}) = \frac{q_x^2+q_z^2}{2m} + \left[1-\left(\frac{v'}{v}\right)^2\right]\frac{q_y^2}{2m}. \quad (8)$$

The transition temperature is

$$T_{\mathrm{c}} = \frac{\pi}{2}\left[\frac{4}{\zeta(3/2)}\right]^{3/2}\left[1-\left(\frac{v'}{v}\right)^2\right]^{1/3}\frac{n^{2/3}}{m}. \quad (9)$$

We see that if $n^{1/3}[1-(v'/v)^2]^{1/6} \ll mv$, our approximation is justified and, in particular, the density of particles in the upper band $\lambda=+1$ is exponentially small.

In the isotropic limit $\Delta=v'/v\to 1$, the transition temperature formally vanishes. Note that in the isotropic case $v=v'$ the spin-orbit term of the Hamiltonian (1) is equivalent to the Rashba model [26] and can be reduced to the latter via the rotation $\exp(i\pi\check{\sigma}_2/4)$ in the pseudo-spin space. In this case, the spectrum (4) has minima on a one-dimensional circle $\sqrt{p_x^2+p_y^2}=mv$ (see Fig. 2). The single-particle ground state is infinitely degenerate and the most general expression for the corresponding wave function is

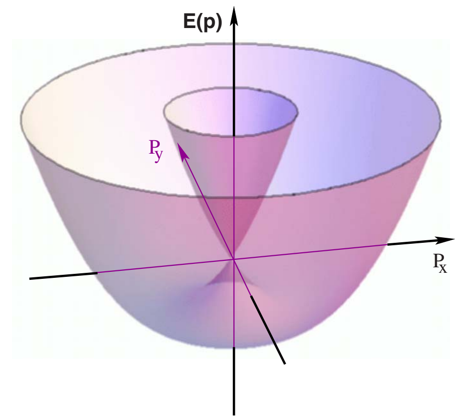


FIG. 2. (Color online) Schematic picture of the band structure described by Eq. (4) for the isotropic Rashba-type case with $v=v'$ for $p_z=0$. The inside sheet represents the $\lambda=+1$ band, while the outside sheet corresponds to $\lambda=-1$ and has minima a one-dimensional circle $\sqrt{p_x^2+p_y^2}=mv$.

$$\Psi_{\mathrm{ring}}(\mathbf{r}) = \int_0^{2\pi}\frac{d\chi}{2\pi}\sqrt{w(\chi)}\vec{U}_-(\chi)e^{i\phi(\chi)}e^{[imv(x\cos\chi+y\sin\chi)]}, \quad (10)$$

where $w(\chi)>0$ is the angle-dependent weight of the Bose condensate on a circle $[\int d\chi/(2\pi)w(\chi)=1]$ and $\phi(\chi)$ is the angle-dependent phase. An especially interesting class of ground states corresponds to $w(\chi)$ not vanishing anywhere on the circle. In this case, the phase $\phi(\chi)$ must satisfy the constraint $\phi(\chi+2\pi)-\phi(\chi)=2\pi n$, with $n\in\mathbb{Z}=\pi_1(S^1)$ being an integer winding number. Therefore there may exist a number of topologically distinct ground states (characterized by the winding number), which cannot be deformed into one another via any continuous transformation. We note here that a transition into the ring SOBEC is similar to a "weak-crystallization transition" discussed by Brazovsky [27] (see also Refs. [28]). In this case, the phase volume of fluctuations is very large, which drives the (classical) transition first order. Even though the transition temperature into the ring SOBEC vanishes in the thermodynamic limit, in a finite trapped system, the energy scale for the crossover into this state will be nonzero [29].

## III. EFFECTS OF DENSITY-DENSITY INTERACTION

The most general ground-state many-body wave function of a noninteracting "double well BEC" is

$$\|\Psi_N\rangle = \sum_{n=0}^{N}\frac{c_n}{\sqrt{n!(N-n)!}}(\hat{B}_L^\dagger)^n(\hat{B}_R^\dagger)^{N-n}\|\mathrm{vac}\rangle, \quad (11)$$

where $n$ and $N-n$ are the numbers of left and right movers, $\hat{B}_{L/R}^\dagger$ are the corresponding creation operators, and $c_n$ are arbitrary coefficients satisfying $\Sigma_n|c_n|^2=1$. In the absence of spin-orbit interaction, a two-component bosonic system has a ferromagnetic ground state with fully polarized pseudo-spin [12,30]. We emphasize that this is not the case for the double-well many-body ground state (11) that describes the spin-orbit interacting BEC. All the arguments used for proving the ferromagnetic nature of the ground state for a two-component system [12] are now irrelevant, as the real-space and spin components of the wave function cannot be factorized due to the spin-orbit coupling. The noninteracting ground state (11) has an $(N+1)$-fold degeneracy. We show below that this large degeneracy is partially lifted by interactions and reduced to a twofold degeneracy. We assume a density-density interaction $\hat{\mathcal{H}}_{\rm int}=\frac{1}{2}\int d^3\mathbf{r}d^3\mathbf{r}'\hat{n}(\mathbf{r})V_{\rm int}(\mathbf{r}-\mathbf{r}')\hat{n}(\mathbf{r}')$, where $\hat{n}(\mathbf{r})=\sum_\mu\hat{\psi}^\dagger_\mu(\mathbf{r})\hat{\psi}_\mu(\mathbf{r})$ and $\hat{\psi}_\mu(\mathbf{r})$ is the field operator, which is initially defined in terms of the creation or annihilation operators for the original hyperfine states. First, we perform the position-dependent rotation $R_{\mu\alpha}(\mathbf{r})$ to obtain the effective interaction term, which has the standard form

$$\hat{\mathcal{H}}_{\rm int}=\frac{1}{2V}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}V_{\rm int}(\mathbf{q})\hat{b}^\dagger_{\alpha\mathbf{p}}\hat{b}_{\alpha\mathbf{p}+\mathbf{q}}\hat{b}^\dagger_{\beta\mathbf{p}'}\hat{b}_{\beta\mathbf{p}'-\mathbf{q}}, \tag{12}$$

where $\hat{b}^\dagger_{\alpha\mathbf{p}}$ is the creation operator for a state with momentum $\mathbf{p}$ and pseudo-spin $\alpha$ in the dark state subspace. We need to perform a second momentum-dependent transformation defined by Eqs. (5) and (6), which introduces new bosonic operators labeled by the band index $\lambda=\pm1$ (4): $\hat{B}_{\lambda\mathbf{p}}=U^\dagger_{\lambda\alpha}(\mathbf{p})\hat{b}_{\alpha\mathbf{p}}$ and $\hat{B}^\dagger_{\lambda\mathbf{p}}=\hat{b}^\dagger_{\alpha\mathbf{p}}U_{\alpha\lambda}(\mathbf{p})$, where $U_{\alpha\lambda}(\mathbf{p})=U_{\alpha\lambda}(\chi_\mathbf{p})$ and the summation over the spin index $\alpha$ is implied. In the limit of relatively weak interactions, $V_{\rm int}\ll mv^2/2$ (we emphasize that the spin-orbit coupling strength can be tuned to be arbitrarily strong by adjusting the properties of laser fields), the upper band with $\lambda=+1$ is irrelevant for the low-energy physics. Thus it is convenient to express the Hamiltonian in terms of left or right-moving operators, defined as $\hat{B}_{L/R\mathbf{q}}=\hat{B}_{-1\mp(\mathbf{q}+m\mathbf{v})}$. Correspondingly, we have $U_{L/R\alpha}(\mathbf{q})=U_{-1\alpha}[\mp(\mathbf{q}+m\mathbf{v})]$. This leads to the following interaction Hamiltonian:

$$\hat{\mathcal{H}}_{\rm int}=\frac{1}{2V}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\sum_{\{\sigma_i\}}{}'V_{\rm int}(\mathbf{q}_\sigma)\hat{B}^\dagger_{\sigma_1\mathbf{p}}\hat{B}_{\sigma_2\mathbf{p}+\mathbf{q}}\hat{B}^\dagger_{\sigma_3\mathbf{p}'}\hat{B}_{\sigma_4\mathbf{p}'-\mathbf{q}}U^\dagger_{\sigma_1\alpha}(\mathbf{p})\times U_{\alpha\sigma_2}(\mathbf{p}+\mathbf{q})U^\dagger_{\sigma_3\alpha'}(\mathbf{p}')U_{\alpha'\sigma_4}(\mathbf{p}'-\mathbf{q}), \tag{13}$$

where the prime sign in the sum over the left and right indices $\sigma_i=L/R=\mp$ is restricted by the condition $\sigma_1+\sigma_3=\sigma_2+\sigma_4$, i.e., the numbers of left and right movers are conserved, and $\mathbf{q}_\sigma=\mathbf{q}-(\sigma_1-\sigma_2)mv\mathbf{e}_x$. We stress that Eq. (13) is valid in the limit of weak interactions (relative to the spin-orbit coupling) and low temperatures, when only single particle states with momenta in the vicinity of the two minima are occupied.

### A. Generalized Bogoliubov transformation

Next, we introduce the projection operators $\hat{\mathcal{P}}_{N,n}=\hat{\mathcal{P}}^2_{N,n}$ that select the subspace characterized by $n$ left-moving and $(N-n)$ right-moving quasiparticles. The Hamiltonian can be expressed as $\hat{\mathcal{H}}=\sum_{n=0}^N\hat{\mathcal{P}}_{N,n}\hat{\mathcal{H}}\hat{\mathcal{P}}_{N,n}=\sum_{n=0}^N\hat{\mathcal{H}}_n$. An important observation is that the Hamiltonian containing the interaction term (13) preserves the number of left and right movers and thus we can consider different "sectors," $\hat{\mathcal{H}}_n$, independently. Our goal is to diagonalize each term $\hat{\mathcal{H}}_n$ using a mean-field scheme and reduce the many-body Hamiltonian to the form

$$\hat{\mathcal{H}}=\sum_{n=0}^N\hat{\mathcal{P}}_{N,n}\left[\mathcal{E}_0(n)+\sum_{\mathbf{q},\sigma}\Omega_\sigma(n,\mathbf{q})\hat{\beta}^\dagger_{n,\sigma,\mathbf{q}}\hat{\beta}_{n,\sigma,\mathbf{q}}\right]\hat{\mathcal{P}}_{N,n}, \tag{14}$$

where $\mathcal{E}_0(n)$ is the contribution of the $(n,N-n)$ sector to the condensate energy, while $\Omega_\sigma(n,\mathbf{q})$ represents the spectrum of quasiparticle excitations. To obtain the mean-field result, we use a Bogoliubov-type approximation in which the operators corresponding to $\mathbf{q}=0$ are replaced within each sector $(n,N-n)$ by $c$ numbers, $\hat{B}_{L0}\to\sqrt{n_0}e^{i\phi/2}$ and $\hat{B}_{R0}\to\sqrt{N_0-n_0}e^{-i\phi/2}$. Next, we notice that at low temperatures, the momenta of uncondensed bosons are $q\ll mv$. Thus we can expand the products of $U$ vectors in Eq. (13) in terms of the deviations $\mathbf{q}$ from the minima of the energy bands

$$\vec{U}^\dagger_{\rm L}(\mathbf{q}_1)\vec{U}_{\rm L}(\mathbf{q}_2)=\vec{U}^\dagger_{\rm R}(\mathbf{q}_1)\vec{U}_{\rm R}(\mathbf{q}_2)\approx1-\frac{\Delta^2}{8}\frac{(q_{1y}-q_{2y})^2}{(mv)^2}+O(q^3_{1,2}),$$

$$\vec{U}^\dagger_{\rm R}(\mathbf{q}_1)\vec{U}_{\rm L}(\mathbf{q}_2)=\vec{U}^\dagger_{\rm L}(\mathbf{q}_1)\vec{U}_{\rm R}(\mathbf{q}_2)\approx\frac{\Delta}{2}\frac{q_{1y}+q_{2y}}{mv}+O(q^2_{1,2}), \tag{15}$$

with $\Delta=v'/v<1$. Consequently, contributions to the mean-field Hamiltonian can be expanded in the small parameter $x_\mathbf{q}=\Delta^2q_y^2/(mv)^2$. In the zero-order approximation, i.e., neglecting contributions of order $x_\mathbf{q}$ and higher, the mean-field Hamiltonian for the $(n,N-n)$ sector is

$$\hat{\mathcal{H}}^{(0)}_n=\frac{N}{2V}\sum_\mathbf{q}V_{\rm int}(\mathbf{q})\left[\hat{\vec{B}}^\dagger_\mathbf{q}\begin{pmatrix}s(\mathbf{q})+1+\delta & \sqrt{1-\delta^2}e^{-i\phi}\\ \sqrt{1-\delta^2}e^{i\phi} & s(\mathbf{q})+1-\delta\end{pmatrix}\hat{\vec{B}}_\mathbf{q}+\hat{\vec{B}}^{\rm T}_\mathbf{q}\begin{pmatrix}(1+\delta)e^{i\phi} & \sqrt{1-\delta^2}\\ \sqrt{1-\delta^2} & (1-\delta)e^{-i\phi}\end{pmatrix}\hat{\vec{B}}_{-\mathbf{q}}+\text{H.c.}\right], \tag{16}$$

where $\delta=2n/N-1$, $\hat{\vec{B}}^T_\mathbf{q}=(\hat{B}_{L\mathbf{q}},\hat{B}_{R\mathbf{q}})$ is the annihilation operator in a spinor notation, $s(\mathbf{q})=2\delta E(\mathbf{q})/[n_0V_{\rm int}(\mathbf{q})]$, and $\delta E(\mathbf{q})$ is the anisotropic spectrum (8) near the minima. We now introduce bosonic operators $\hat{B}_{-,\mathbf{q}}=\sqrt{1-n/N}\hat{B}_{L,\mathbf{q}}e^{-i\phi/2}-\sqrt{n/N}\hat{B}_{R,\mathbf{q}}e^{i\phi/2}$ and $\hat{B}_{+,\mathbf{q}}=\sqrt{n/N}\hat{B}_{L,\mathbf{q}}e^{-i\phi/2}+\sqrt{1-n/N}\hat{B}_{R,\mathbf{q}}e^{i\phi/2}$. The Hamiltonian becomes diagonal for the $\hat{B}_-$ particles, which have the "free" spectrum $\delta E(\mathbf{q})$, and has the standard Bogoliubov form [25] for the $\hat{B}_+$ particles. Introducing the operators $\hat{\beta}_{-,\mathbf{q}}\equiv\hat{B}_{-,\mathbf{q}}$ and $\hat{\beta}_{+,\mathbf{q}}\equiv(1-A^2_\mathbf{q})^{-1/2}(\hat{B}_{+,\mathbf{q}}+A_\mathbf{q}\hat{B}^\dagger_{+,-\mathbf{q}})$, with $A_\mathbf{q}=-s(\mathbf{q})-1+\sqrt{[s(\mathbf{q})+1]^2-1}$, we get

$$\hat{\mathcal{H}}^{(0)}_n=\mathcal{E}^{(0)}_0+\sum_\mathbf{q}\{\Omega_-(\mathbf{q})\hat{\beta}^\dagger_{-,\mathbf{q}}\hat{\beta}_{-,\mathbf{q}}+\Omega_+(\mathbf{q})\hat{\beta}^\dagger_{+,\mathbf{q}}\hat{\beta}_{+,\mathbf{q}}\}, \tag{17}$$

where $\mathcal{E}^{(0)}_0$ is the condensate energy [25] in the zero-order approximation, $\Omega_-(\mathbf{q})=\{q_x^2+q_z^2+q_y^2[1-(v'/v)^2]\}/(2m)$ is the anisotropic free particle quadratic spectrum and $\Omega_+(\mathbf{q})=\sqrt{[\Omega_-(\mathbf{q})+nV_{\rm int}(\mathbf{q})]^2-n^2V^2_{\rm int}(\mathbf{q})}$ is an anisotropic sound similar to the conventional Bogoliubov phonon mode in a

BEC. At this level of approximation the condensate energy is $n$-independent [i.e., it is the same for any particular sector characterized by $n$ left movers and $(N-n)$ right movers] and, consequently, the degeneracy of the noninteracting ground state (11) is preserved. In the first order approximation, the mean-field Hamiltonian (16) acquires sector-dependent corrections of order $x_{\mathbf{q}} \ll 1$. Following the above recipe, we introduce a set of new operators $\hat{B}_{\pm,\mathbf{q}}$ that diagonalize the $\hat{B}^{\dagger}_{\mathbf{q}}\hat{B}_{\mathbf{q}}$ term in the Hamiltonian (16) but not the other terms. Next, we diagonalize the full Hamiltonian [up to terms of order $O(x_{\mathbf{q}}^2)$] via a generalized Bogoliubov-type transformation

$$\hat{\beta}_{-,\mathbf{q}} = \hat{B}_{-,\mathbf{q}} + x_{\mathbf{q}} D_{\mathbf{q}} \hat{B}^{\dagger}_{-,-\mathbf{q}} + x_{\mathbf{q}} F_{1\mathbf{q}} \hat{B}_{+,\mathbf{q}} + x_{\mathbf{q}} F_{2\mathbf{q}} \hat{B}^{\dagger}_{+,-\mathbf{q}}, \quad (18)$$

$$\hat{\beta}_{+,\mathbf{q}} = (1 - A_{\mathbf{q}}^2)^{-1/2} (\hat{B}_{+,\mathbf{q}} + A_{\mathbf{q}} \hat{B}^{\dagger}_{+,-\mathbf{q}} + x_{\mathbf{q}} C_{1\mathbf{q}} \hat{B}_{-,\mathbf{q}} + x_{\mathbf{q}} C_{2\mathbf{q}} \hat{B}^{\dagger}_{-,-\mathbf{q}}). \quad (19)$$

In Eqs. (18) and (19) we already anticipated that some of the terms are corrections of order $x_{\mathbf{q}}$. The coefficients are determined by requiring that the $\beta$ operators obey standard commutation relations [to order $O(x_{\mathbf{q}})$] and that the off-diagonal contributions to the Hamiltonian vanish. Assuming for simplicity that we have a pointlike interaction, i.e., $V_{\text{int}}(\mathbf{q}) = V_{\text{int}}$ is momentum-independent for momenta in a range that is relevant for the problem, the ground-state energy in the $(n, N\text{-}n)$ sector is

$$\mathcal{E}_0(n) = \frac{V_{\text{int}} N^2}{2V} + \frac{V_{\text{int}} N}{2V} \sum_{\mathbf{q} \neq 0} \frac{1}{1 - A_{\mathbf{q}}^2} \Bigg( [2 + s(\mathbf{q})] A_{\mathbf{q}}^2 + 2 A_{\mathbf{q}} - \frac{x_{\mathbf{q}}}{8} \{ A_{\mathbf{q}}^2 [\cos(4\xi) + 3] - A_{\mathbf{q}} [\cos(4\xi) - 5] \} \Bigg) + O(x_{\mathbf{q}}^2), \quad (20)$$

where $\cos^2(\xi) = n/N$. The relevant coefficient of the generalized Bogoliubov transformation (18) and (19) has the form

$$A_{\mathbf{q}} = -1 - \frac{s}{2} + \frac{1}{2}\sqrt{s(4+s)} - \frac{x_{\mathbf{q}}}{32\sqrt{s(4+s)}} (2 + s\sqrt{s(4+s)}) \times [-4 - 5s + (4+s)\cos(4\xi)] + O(x_{\mathbf{q}}^2). \quad (21)$$

Explicitly evaluating Eq. (20) with $A_{\mathbf{q}}$ given by Eq. (21) shows that, at this level of approximation, the energy of the condensate becomes sector-dependent, $\mathcal{E}_0(n) \approx \mathcal{E}_0^{(0)} + \mathcal{E}_0^{(1)}(n)$, and is minimal for $n=0$ and $n=N$. Thus the density-density interaction reduces the large $(N+1)$-fold degeneracy of the ground state to a twofold degeneracy. Consequently, in the limit of vanishing interactions $V_{\text{int}} \to +0$, the most general expression for the many-body wave function is

$$\|\Psi_N\rangle = \frac{1}{\sqrt{N!}} [\sqrt{w_{\text{L}}} e^{i\phi_{\text{L}}} (\hat{B}^{\dagger}_{\text{L}})^N + \sqrt{w_{\text{R}}} e^{i\phi_{\text{R}}} (\hat{B}^{\dagger}_{\text{R}})^N] \|\text{vac}\rangle, \quad (22)$$

where $w_{\text{L/R}}$ represents the fraction of the left or right movers and $\phi_{\text{L/R}}$ are arbitrary phases. Notice that Eq. (22) describes a fragmented or entangled BEC, unless $w_{\text{L}} w_{\text{R}} = 0$. That is, the many-body state (22) does not correspond to the condensation into one single-particle state. We reiterate that the left and right movers in the condensate have nonzero momentum, but zero velocity and do not actually move while the laser fields responsible for the spin-orbit coupling are present. We also note that Eq. (22) describes a so-called $N00N$ state, [23,31] which is a quantum correlated state with properties that can be exploited in applications such as quantum sensing and quantum metrology. This suggests that the possibility of using spin-orbit coupled condensates as qubits deserves to be further investigated.

### B. Gross-Pitaevskii equations

Let us consider the density-density interaction potential as a contact pseudopotential, $V_{\text{int}}(\mathbf{r} - \mathbf{r}') = V_{\text{int}} \delta(\mathbf{r} - \mathbf{r}')$, where $V_{\text{int}} = \frac{4\pi\hbar^2}{m} a$ and $a$ is the interatomic scattering length. The full many body Hamiltonian can be written as

$$\hat{\mathcal{H}} = \sum_{\mu,\nu} \int d^3r \hat{\psi}^{\dagger}_{\mu}(\mathbf{r}) h_{\mu\nu} \hat{\psi}_{\nu}(\mathbf{r}) + \frac{V_{\text{int}}}{2} \sum_{\mu,\nu} \int d^3r \hat{\psi}^{\dagger}_{\mu}(\mathbf{r}) \hat{\psi}^{\dagger}_{\nu}(\mathbf{r}) \hat{\psi}_{\nu}(\mathbf{r}) \hat{\psi}_{\mu}(\mathbf{r}), \quad (23)$$

in terms of field operators $\hat{\psi}_{\mu}(\mathbf{r})$ for the original hyperfine states, $\mu \in \{0,1,2,3\}$. In Eq. (23) we used the notation $h_{\mu\nu} = \{\frac{\mathbf{p}^2}{2m} + V_{\text{trap}} + H_{a-l}\}_{\mu\nu}$ for the single particle Hamiltonian in the presence of a trap potential $V_{\text{trap}}$, in addition to the spatially varying laser fields that interact with the atom, $H_{a-l}$. In the adiabatic approximation, after projecting onto the dark state subspace, the first term in Eq. (23) becomes $\sum_{\mathbf{p};\alpha,\beta} \hat{b}^{\dagger}_{\alpha\mathbf{p}} \{[\mathbf{p}^2/2m + V_{\text{trap}}]1 - v p_x \sigma_2 - v' p_y \sigma_3\}_{\alpha\beta} \hat{b}_{\beta\mathbf{p}}$, where $\hat{b}^{\dagger}_{\alpha\mathbf{p}}$ and $\hat{b}_{\alpha\mathbf{p}}$ are the creation and annihilation operators for bosons with pseudo-spin $\alpha = \uparrow, \downarrow$. The interaction term is given by Eq. (12). Before writing down the Gross-Pitaevskii equations, let us summarize the three different representations used for describing the system of bosons interacting with the spatially modulated laser fields.

(i) *Hyperfine states representation*. This is the most straightforward way to describe the motion of the bosons and their interaction with the trap potential ($V_{\text{trap}}$) and the laser fields ($H_{a-l}$), as well as the density-density interaction [second term in Eq. (23)]. The field operator that creates a particle in the hyperfine state $\mu \in \{0,1,2,3\}$ at point $\mathbf{r}$ is $\hat{\psi}^{\dagger}_{\mu}(\mathbf{r})$, while the creation of a free-moving particle with momentum $\mathbf{p}$ is described by $\hat{c}^{\dagger}_{\mu\mathbf{p}} = \int d^3r e^{i\mathbf{p}\mathbf{r}} \hat{\psi}^{\dagger}_{\mu}(\mathbf{r})$. By performing the position-dependent rotation $R_{\mu\alpha}$ which diagonalizes the atom-laser Hamiltonian *and* projecting onto the dark states subspace we switch to the pseudo-spin representation.

(ii) *Pseudo-spin representation (dark states representation)*. This is the natural framework for describing the low-energy physics of the atomic system interacting with the laser field. The creation operator for free-moving particles with spin $\alpha \in \{\uparrow, \downarrow\}$ and momentum $\mathbf{p}$ is $\hat{b}^{\dagger}_{\alpha\mathbf{p}}$. We can define the corresponding field operator as $\hat{\tilde{\psi}}^{\dagger}_{\alpha}(\mathbf{r}) = \sum_{\mathbf{p}} e^{-i\mathbf{p}r} \hat{b}^{\dagger}_{\alpha\mathbf{p}}$. Note that the field operators in the hyperfine and pseudo-spin representations are related via the position-dependent rotation, $\hat{\psi}^{\dagger}_{\mu}(\mathbf{r}) = \sum_{\alpha} R_{\mu\alpha}(\mathbf{r}) \hat{\tilde{\psi}}^{\dagger}_{\alpha}(\mathbf{r})$. Diagonalizing the single-particle spin-orbit coupled Hamiltonian, $H = [\mathbf{p}^2/2m + V_{\text{trap}}]1 - v p_x \sigma_2$

$-v' p_y \sigma_3$, generates a set of eigenstates described by the spinor eigenfunctions $\vec{\phi}_{\sigma n}(\mathbf{r})$. The quantum number $\sigma=\pm$ can be viewed as labeling right (left) moving states.

(iii) *Right-or left-moving states representation*. This is the representation corresponding to the eigenstates of the spin-orbit coupled single particle Hamiltonian. In Sec. I we have shown that in the absence of a trap potential the single particle spectrum for the generic case $v \neq v'$ is characterized by two minima at nonzero momenta. Here we show explicitly that the double-degeneracy of the single-particle states is a general property of the spin-orbit interacting Hamiltonian, protected by a Kramers-like symmetry. Let us use the following parametrization for the eigenfunctions:

$$\vec{\phi}_{\sigma n}(\mathbf{r}) = e^{i\sigma m v x}\begin{pmatrix} u_{\sigma n}^{\uparrow}(\mathbf{r}) \\ i\sigma u_{\sigma n}^{\downarrow}(\mathbf{r}) \end{pmatrix}, \tag{24}$$

where $\sigma=\pm$ and $n$ is a set of quantum numbers. The components $u_{\sigma n}^{\alpha}(\mathbf{r})$ are the solutions of the following eigenproblem:

$$\begin{pmatrix} h_0 - v' p_y & i v p_x \\ -i v p_x & h_0 + v' p_y \end{pmatrix}\begin{pmatrix} u_{\sigma n}^{\uparrow}(\mathbf{r}) \\ i\sigma u_{\sigma n}^{\downarrow}(\mathbf{r}) \end{pmatrix} e^{i\sigma m v x} = E_{\sigma n}\begin{pmatrix} u_{\sigma n}^{\uparrow}(\mathbf{r}) \\ i\sigma u_{\sigma n}^{\downarrow}(\mathbf{r}) \end{pmatrix} e^{i\sigma m v x}, \tag{25}$$

where $h_0 = p^2/2m + V_{\text{trap}}$ is the Hamiltonian in the absence of spin-orbit interaction. More explicitly, $u_{\sigma n}^{\alpha}(\mathbf{r})$ satisfy the following system of coupled differential equations:

$$\left[-\frac{\nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) + i v' \frac{\partial}{\partial y} - E - \frac{m v^2}{2}\right] u_{\sigma}^{\uparrow}(\mathbf{r}) + \left[-i\sigma v \frac{\partial}{\partial x} + m v^2\right][u_{\sigma}^{\uparrow}(\mathbf{r}) - u_{\sigma}^{\downarrow}(\mathbf{r})] = 0,$$

$$\left[-\frac{\nabla^2}{2m} + V_{\text{trap}}(\mathbf{r}) - i v' \frac{\partial}{\partial y} - E - \frac{m v^2}{2}\right] u_{\sigma}^{\downarrow}(\mathbf{r}) - \left[-i\sigma v \frac{\partial}{\partial x} + m v^2\right][u_{\sigma}^{\uparrow}(\mathbf{r}) - u_{\sigma}^{\downarrow}(\mathbf{r})] = 0. \tag{26}$$

Taking the complex conjugate of Eq. (26) with $\sigma \to -\sigma$ we obtain an identical set of equations. Consequently we have

$$u_{-\sigma n}^{\uparrow}(\mathbf{r}) = [u_{\sigma n}^{\downarrow}(\mathbf{r})]^*,$$

$$u_{-\sigma n}^{\downarrow}(\mathbf{r}) = [u_{\sigma n}^{\uparrow}(\mathbf{r})]^*, \tag{27}$$

and the corresponding energies are degenerate, $E_{-\sigma n} = E_{\sigma n} = E_n$. Because $\langle \phi_{-\sigma n} | \phi_{\sigma n} \rangle = 0$, the two states are linearly independent. We conclude that the single-particle eigenstates of the spin-orbit coupled Hamiltonian are (at least) double degenerate independent of the symmetries (or lack of symmetry) of the trap potential. Note that this double degeneracy is a consequence of a Kramers-like symmetry of the spin-orbit interacting Hamiltonian, which contains terms that are either quadratic in momentum, or linear in both momentum and spin. The creation operator for a left-or right-moving particle described by the eigenstate $\vec{\phi}_{\sigma n}$ is $\hat{B}_{\sigma n}^{\dagger}$. The field operators in the pseudo-spin representation can be expressed in terms of $\hat{B}_{\sigma n}$ operators as

$$\hat{\tilde{\psi}}_{\uparrow}(\mathbf{r}) = \sum_n [e^{imvx} u_{+n}^{\uparrow}(\mathbf{r}) \hat{B}_{+n} + e^{-imvx} u_{-n}^{\uparrow}(\mathbf{r}) \hat{B}_{-n}],$$

$$\hat{\tilde{\psi}}_{\downarrow}(\mathbf{r}) = \sum_n [i e^{imvx} u_{+n}^{\downarrow}(\mathbf{r}) \hat{B}_{+n} - i e^{-imvx} u_{-n}^{\downarrow}(\mathbf{r}) \hat{B}_{-n}], \tag{28}$$

where the terms with $\sigma=+$ and $\sigma=-$ correspond to the right- and left-moving modes, respectively. Finally, note that in the translation invariant case, $V_{\text{trap}}=0$, we introduced the eigenfunctions $\vec{\phi}_{\lambda \mathbf{p}}(\mathbf{r}) = e^{i\mathbf{p}r}\vec{U}_{\lambda}(\chi_{\mathbf{p}})$, with $U_{\alpha\lambda}(\chi_{\mathbf{p}})$ given by Eqs. (5) and (6), and the corresponding creation operators, $\hat{B}_{\lambda \mathbf{p}}$. We then defined the left and right movers for the low energy band $\lambda=-1$ and small momenta $q < mv$ as $\hat{B}_{L/R\mathbf{q}} = \hat{B}_{(-1)\mp(\mathbf{q}+m\mathbf{v})}$. Alternatively, we can directly define the eigenfunctions $\vec{\phi}_{\sigma \mathbf{q}}(\mathbf{r})$ in the left and right moving representation using the parametrization (24), with no restriction for $\mathbf{q}$. The correspondence between the two representations is given by $\mathbf{p} = \sigma(\mathbf{q} + m\mathbf{v})$ and $\lambda = -\text{sgn}(q_x + mv)$. This generalizes our definition of the left-or right-moving modes to arbitrary energy. Notice, however, that a left (right) moving state from the high energy band $\lambda=+1$ has in fact a positive (negative) momentum.

To write the Gross-Pitaevskii equation in the pseudo-spin representation we use the standard procedure and calculate the commutator $[\hat{\tilde{\psi}}_{\alpha}(\mathbf{r}), \hat{\mathcal{H}}]$, where $\hat{\mathcal{H}}$ is the many-body Hamiltonian expressed in terms of pseudo-spin field operators. Using Eq. (23) and the relations between representations summarized above we obtain

$$i\frac{\partial}{\partial t}\tilde{\psi}_{\alpha}(\mathbf{r},t) = \sum_{\beta}\left\{\left[\frac{-\nabla^2}{2m} + V_{\text{trap}}(\mathbf{r})\right]\check{1} + iv\frac{\partial}{\partial x}\check{\sigma}_2 + iv'\frac{\partial}{\partial y}\check{\sigma}_3\right\}_{\alpha\beta}\tilde{\psi}_{\beta}(\mathbf{r},t) + V_{\text{int}}(|\tilde{\psi}_{\uparrow}|^2 + |\tilde{\psi}_{\downarrow}|^2)\tilde{\psi}_{\alpha}(\mathbf{r},t). \tag{29}$$

Relation (29), which is a system of two coupled nonlinear differential equations, represents the time-dependent Gross-Pitaevskii equation for a spin-orbit coupled Bose-Einstein condensate wave function. Similar equations can be written in the left or right-moving states representation. For simplicity, we will address here only the translation invariant case $V_{\text{trap}}=0$. The field operator for the left-or right-moving modes can be written in terms of the corresponding $\hat{B}_{\sigma\mathbf{q}}$ operators as

$$\hat{\tilde{\tilde{\psi}}}_{\sigma}(\mathbf{r}) = \sum_{\alpha,\mathbf{q}} \phi_{\sigma\mathbf{q}}^{\alpha}(\mathbf{r}) \hat{B}_{\sigma\mathbf{q}}. \tag{30}$$

The noninteracting part of the Hamiltonian is diagonal in terms of left-or right-moving operators, with eigenvalues that depend on the momentum $\mathbf{q}$ only. At low energies, these eigenvalues are given by the anisotropic spectrum $\delta E(\mathbf{q}) = (q_x^2 + q_z^2)/(2m) + q_y^2/(2m_y)$ with $m_y = m[1 - (v'/v)^2]^{-1}$. In general, the interacting Hamiltonian is given by Eq. (13), but in the low-energy limit we neglect all corrections of order $x_{\mathbf{q}}=\Delta^2 q_y^2/(mv)^2$ and higher coming from the momentum-dependent matrices $U_{\alpha\sigma}(\mathbf{q})$. In this limit we obtain

$$i\frac{\partial}{\partial t}\tilde{\tilde{\psi}}_\sigma(\mathbf{r},t)=\left(\frac{(-i\partial_x-\sigma mv)^2}{2m}-\frac{\partial_y^2}{2m_y}-\frac{\partial_z^2}{2m}\right)\tilde{\tilde{\psi}}_\sigma(\mathbf{r},t) + V_{\text{int}}(|\tilde{\tilde{\psi}}_L|^2+|\tilde{\tilde{\psi}}_R|^2)\tilde{\tilde{\psi}}_\sigma(\mathbf{r},t), \tag{31}$$

where $\partial_j=\partial/\partial x_j$, $j\in\{x,y,z\}$. Note that making the substitution $\tilde{\tilde{\psi}}_\sigma(\mathbf{r},t)=\tilde{\tilde{\psi}}'_\sigma(\mathbf{r},t)e^{i\sigma mvx}$, together with the rescaling $y\rightarrow y\sqrt{1-(v'/v)^2}$, reduces Eq. (31) to a standard Gross-Pitaevskii equaiton for spinor condensates. The time-independent Gross-Pitaevskii equations can be obtained by looking for a stationary solution of the form $\tilde{\tilde{\psi}}_\sigma(\mathbf{r},t)=\tilde{\tilde{\psi}}_{0\sigma}(\mathbf{r})e^{-i\mu t}$, where $\mu$ is the chemical potential which is determined by the condition $N=\int d^3r(|\tilde{\tilde{\psi}}_L|^2+|\tilde{\tilde{\psi}}_R|^2)$, with $N$ being the total number of bosons. We note that by linearizing $\tilde{\tilde{\psi}}_\sigma(\mathbf{r},t)$ with respect to the deviations from the stationary solution we obtain an excitation spectrum consisting of two modes, $\Omega_\pm(\mathbf{q})$, identical with those found using the generalized Bogoliubov treatment.

## IV. EXPERIMENTAL SIGNATURE OF SPIN-ORBIT COUPLED BEC: MEASURING A SOBEC QUBIT

A straightforward way to detect experimentally the BEC would be to probe the momentum distribution of the density of the particles via time-of-flight (TOF) expansion. After removing the trap *and* the laser fields, the boson gas represents a system of free particles, each characterized by a certain momentum and a hyperfine state index. In a TOF experiment one determines the momentum distribution by measuring the particle density at various times after the release of the boson cloud. The operator associated with a density measurement is $\hat{\rho}(\mathbf{r})=\Sigma_\mu\hat{\psi}^\dagger_\mu(\mathbf{r})\hat{\psi}_\mu(\mathbf{r})$, where $\hat{\psi}^\dagger_\mu(\mathbf{r})$ is the creation operator for a particle in the hyperfine state $\mu$ positioned at point $\mathbf{r}$. Determining the density profile involves a simultaneous measurement of $\hat{\rho}(\mathbf{r})$ for all the values of $\mathbf{r}\in\mathcal{V}$ corresponding to a certain region in space where the boson cloud is located. To insure formal simplicity, we consider a coarse-grained space, i.e., we treat $\mathbf{r}$ as a discrete variable. This is simply a technical trick and does not influence the final result. Our goal is to find the most likely spatial distributions of the particles at a given moment $t$ after the release of the atoms. In the limit of large particle numbers, the actual measured density profiles will involve only small fluctuations away from these "most likely" distributions.

For a system of $N$ bosons, the result of the measurement is a set of eigenvalues $\{\Sigma_\mu n_{\mathbf{r}\mu}\}_{(\mathbf{r}\in\mathcal{V})}$ that label an eigenvector of the density operator

$$\|\Phi_{\{n_{\mathbf{r}\mu}\}}\rangle=\prod_{\mu,\mathbf{r}\in\mathcal{V}}\frac{1}{\sqrt{(n_{\mathbf{r}\mu})!}}[\hat{\psi}^\dagger_\mu(\mathbf{r})]^{n_{\mathbf{r}\mu}}\|\text{vac}\rangle, \tag{32}$$

where the occupation numbers satisfy the constraint $\Sigma_{\mu,\mathbf{r}}n_{\mathbf{r}\mu}=N$, and the factors $1/\sqrt{(n_{\mathbf{r}\mu})!}$ insure the normalization to unity. Note that $n_{\mathbf{r}\mu}$ is an integer representing the number of particles located in a certain "cell" $\mathbf{r}$ of the coarse-grained space. At time $t$ after the release, the many-body state of $N$ bosons that were initially in a BEC ground state described by Eq. (22) is

$$\|\tilde{\Psi}_N(t)\rangle=\mathcal{N}\sum_\sigma\sqrt{w_\sigma}e^{i\phi_\sigma}\times\sum_{\{n_{\mathbf{r}\mu}\}_\mathcal{V}}\left\{\prod_{\mu,\mathbf{r}\in\mathcal{V}}\frac{1}{(n_{\mathbf{r}\mu})!}[Q^\sigma_\mu(\mathbf{r},t)\hat{\psi}^\dagger_\mu(\mathbf{r})]^{n_{\mathbf{r}\mu}}\|\text{vac}\rangle\right\}, \tag{33}$$

where $\mathcal{N}$ is a normalization factor, $\sigma$ labels the left ($\sigma=L\equiv-1$) and right ($\sigma=R\equiv+1$) modes and $\|\tilde{\Psi}_N(0)\rangle=\|\Psi_N\rangle$. The coefficients $Q^\sigma_\mu$ are normalized so that $\Sigma_{\mathbf{r},\mu}|Q^\sigma_\mu(\mathbf{r},t)|^2=1$. The second summation in Eq. (33) is over all possible spatial distributions of $N$ particles and, in the continuous limit, it becomes a path integral. Equation (33) represents the expansion of the many-body wave function in terms of eigenstates (32) of the density operator. The probability $\mathcal{P}[\{n_{\mathbf{r}\mu}\}]$ of measuring a certain density profile $n_{\mathbf{r}\mu}$ is determined by the coefficient of the corresponding term. If we focus, for simplicity, on the case when there are only left (right) movers in Eq. (22), this probability is proportional to $\Pi_{\mathbf{r},\mu}|Q^\sigma_\mu(\mathbf{r},t)|^{2n_{\mathbf{r}\mu}}/(n_{\mathbf{r}\mu})!$, with $\sigma=L(R)$. The probability $\mathcal{P}[\{n_{\mathbf{r}\mu}\}]$ has a maximum for $n^0_{\mathbf{r}\mu}=N|Q^\sigma_\mu(\mathbf{r},t)|^2$ corresponding, in the continuous limit, to a stationary point of the path integral in Eq. (33). For large particle number, $\mathcal{P}[\{n_{\mathbf{r}\mu}\}]$ becomes sharply peaked at $n^0_{\mathbf{r}\mu}$ and the actually measured density profiles will exhibit only relatively small deviations from the stationary profile. Therefore at time $t$ after release, the density of the boson cloud is

$$\rho(\mathbf{r},t)=N\sum_\mu|Q^\sigma_\mu(\mathbf{r},t)|^2. \tag{34}$$

If both $w_R$ and $w_L$ are nonzero, the result of a measurement will be either a right-moving density profile [$\sigma=R$ in Eq. (34)] with a probability $w_R$, or a left-moving profile [$\sigma=L$ in Eq. (34)] with a probability $w_L$, assuming that the two profiles are spatially well-separated. We are not addressing here the interesting effects of the interference between left- and right-moving condensates. These effects are negligible if the left- and right-moving density profiles are spatially separated, but become important otherwise, e.g., at small times after the release.

Next we determine explicitly the coefficients $Q^\sigma_\mu(\mathbf{r},t)$ for the exactly solvable model of bosons with "Ising-type" spin-orbit coupling, $v\neq v'=0$, placed in a harmonic trap, $V_{\text{trap}}(\mathbf{r})=m\omega^2r^2/2$ [10]. In this case, the operators $\hat{B}^\dagger_\sigma$ from Eq. (22) are creation operators for the single particle ground states

$$\vec{\phi}_{\sigma0}(\mathbf{r})=\varphi_0(\mathbf{r})e^{i\sigma mvx}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\sigma\end{pmatrix}, \tag{35}$$

where $\varphi_0(\mathbf{r})$ represents the ground-state wave function of the harmonic oscillator. The spinor (35) is written in the dark state basis. Performing the position-dependent rotation $R_{\mu\alpha}$ [see Eq. (3)], we can express the operators $\hat{B}^\dagger_\sigma$ in terms of creation operators for particles in a certain hyperfine state located at point $\mathbf{r}$, $\hat{\psi}^{\dagger}_{\mu}(\mathbf{r})$, or their Fourier components corresponding to free moving particles, $\hat{c}^{\dagger}_{\mu\mathbf{k}}=\Sigma_{\mathbf{r}}e^{i\mathbf{kr}}\hat{\psi}^{\dagger}_{\mu}(\mathbf{r})$. The time evolution after the release can be easily described in terms of time evolution for the $\hat{c}^{\dagger}_{\mu\mathbf{k}}$ operators, $\hat{c}^{\dagger}_{\mu\mathbf{k}}(t)=\exp(-i\epsilon_{\mathbf{k}}t)\,\hat{c}^{\dagger}_{\mu\mathbf{k}}$, where $\epsilon_{\mathbf{k}}=k^2/(2m)$ is the free particle spectrum. Consequently, the many-body state $\|\tilde{\Psi}_N(t)\rangle$ can be obtained by making in Eq. (22) the substitution $\hat{B}^{\dagger}_{\sigma}\rightarrow\Sigma_{\mathbf{r},\mu}Q^{\sigma}_{\mu}(\mathbf{r},t)\hat{\psi}^{\dagger}_{\mu}(\mathbf{r})$ with

$$Q^{\sigma}_{\mu}(\mathbf{r},t)=\sum_{\alpha,\mathbf{k},\mathbf{r}'}[\vec{\phi}_{\sigma 0}]_{\alpha}(\mathbf{r}')R^{*}_{\mu\alpha}(\mathbf{r}')e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}e^{-i\epsilon_{\mathbf{k}}t}. \quad (36)$$

Finally, introducing this expression of $Q^{\sigma}_{\mu}$ in Eq. (34) we obtain for the measured density profile the expression

$$\rho(\mathbf{r},t)=N\frac{\Gamma^3}{[2\pi(1+\tau^2)]^{3/2}}e^{-\Gamma^2(y^2+z^2)/(1+\tau^2)}\times\Bigg[\sin^2\theta e^{-\Gamma^2/(1+\tau^2)(x-\lambda mvt/m)^2}+\frac{(1-\lambda\cos\theta)^2}{2}e^{-\Gamma^2/(1+\tau^2)[x-(\lambda mv+mv_a)t/m]^2}+\frac{(1+\lambda\cos\theta)^2}{2}e^{-\Gamma^2/(1+\tau^2)[x-(\lambda mv-mv_a)t/m]^2}\Bigg], \quad (37)$$

where $\Gamma=\sqrt{m\omega}$ is the inverse characteristic length of the trap potential, $\tau=\omega t$ is time in units of $\omega^{-1}$, $\theta\in[0\pi/2]$ and $v_a$ are tunable parameters characterizing the laser field, and $v=v_a\cos\theta$ [see the paragraph containing Eq. (3)]. In Eq. (37) the density was normalized so that $\int d^3r\rho(\mathbf{r},t)=N$. The density profile for a left-moving density distribution ($\sigma=-1$) is shown in Fig. 3 for three different times after the release of the boson cloud. The parameters of the calculation are $\theta=\pi/3$ and $v_a=6\sqrt{\omega/m}$. Notice the three-peak structure of the density, corresponding to the three exponential terms in Eq. (37). The relative weights of the peaks are $\cos^4(\theta/2)$ (large peak), $1/2\sin^2\theta=2\sin^2(\theta/2)\cos^2(\theta/2)$ (middle peak), and $\sin^4(\theta/2)$ (small peak) and their characteristic velocities are $-\sigma v_a(1-\cos\theta)$, $\sigma v_a\cos\theta$, and $\sigma v_a(1+\cos\theta)$, respectively. The left- and right-moving distributions are symmetric with respect to a $x\rightarrow -x$ reflection (see also Fig. 4). Notice that the total momentum corresponding to a distribution described by Eq. (37) vanishes. By analyzing the transformation (3) to the dark state basis, we observe that $\sin\theta$ is the coefficient of the hyperfine state $|3\rangle$. Consequently, the middle peak in the density distribution (37) consists of particles in this particular hyperfine state. The other two peaks contain mixtures of states $|1\rangle$ and $|2\rangle$. A state-selective measurement of particles in the hyperfine state $|3\rangle$ would reveal a single peak structure moving to the left or to the right with a velocity $v=v_a\cos\theta$. The dependence of the density profile $\rho(\mathbf{r},t)/N$ on $\theta$ and on the ratio $\gamma=v_a/\sqrt{\omega/m}$ for $\mathbf{r}=(x,0,0)$ and $t=\omega^{-1}$ is shown in Fig. 4.

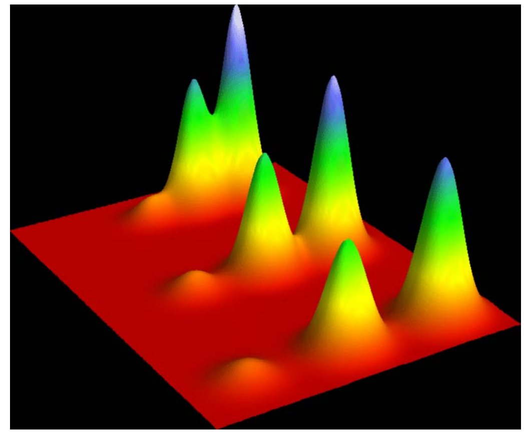

FIG. 3. (Color online) Density of particles at three different moments, $t_1=0.4\omega^{-1}$, $t_2=0.6\omega^{-1}$, and $t_3=0.8\omega^{-1}$, after both the trap and the laser fields are removed at $t=0$. For clarity, the density distributions are shifted along the $y$ axis. This time-of-flight expansion corresponds to a many-body ground state (22) and is obtained using the single-particle eigenfunctions for the exactly solvable model of trapped bosons with Ising-type spin-orbit coupling ($v\neq v'=0$) [10] with $v_a=6(\omega/m)^{1/2}$. This left-moving density distribution is measured with a probability $w_L$, while there is a $w_R$ probability to observe a right-moving distribution which corresponds to a $x\rightarrow -x$ reflection (see also Fig. 4). Notice the characteristic three-peak structure. To resolve the BEC peaks, the spin-orbit coupling energy scale should be larger than the trap level spacing, i.e., $mv^2\gg\omega$. In the opposite limit the phenomenon of real-space BEC separation is smeared out by finite-size effects (22).

## V. SUMMARY AND CONCLUSIONS

To summarize, in this paper we have introduced and discussed in detail a type of many-body system consisting of pseudo-spin-1/2 bosons with spin-orbit interactions. We have shown that at low temperatures the system condenses into a type of entagled BEC, the spin-orbit coupled Bose-

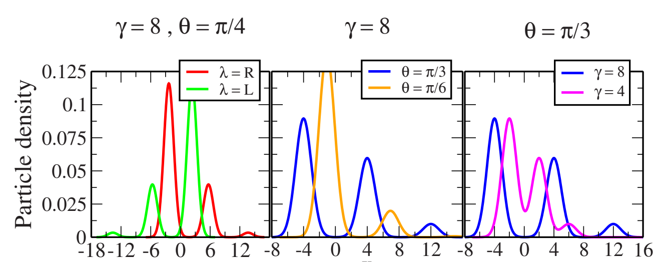


FIG. 4. (Color online) Density profiles $\rho(\mathbf{r},t)/N$ for $\mathbf{r}=(x,0,0)$ and $t=\omega^{-1}$. The position $x$ is measured in units of $\Gamma^{-1}$. Left panel: right-moving vs left-moving distributions. Notice that the "center of mass" of the distributions is always at $x=0$. Middle panel: Dependence on the angle $\theta$. At small angles all the weight concentrates in the large peak which is centered near $x=0$. In the limit $\theta\rightarrow\pi/2$ the strength of the SO interaction vanishes at $v\rightarrow 0$ and the present analysis is not valid. Left panel: Dependence on the relative strength of the spin-orbit coupling, $\gamma=v_a/\sqrt{\omega/m}$. To resolve the peak structure, the spin-orbit coupling energy scale should be larger than the trap level spacing. In the opposite limit interference effects become important (see main text).

Einstein condensate (SOBEC). The difference in this state stems from the coupling of an internal degree of freedom, the pseudo-spin created as a result of an atom interacting with a spatially modulated laser field, to the real space motion of the particles. As a result, the single-particle spectrum is characterized by degenerate minima at finite momenta and, consequently, the bosons condense at low temperatures into an entangled quantum state with nonzero momentum. For an arbitrary spin-orbit coupling, the single particle spectrum has a double-well structure in momentum space (see Fig. 1) with minima at nonzero momenta. In this case, a system of $N$ noninteracting bosons is characterized by a large $(N+1)$-fold degeneracy of the many-body ground state. Weak density-density interactions reduce this large degeneracy to a twofold degeneracy. The corresponding ground-state wave function describes a superposition of left-moving and right-moving condensates with weights $w_L$ and $w_R=1-w_L$, respectively. Performing a time-of-flight expansion of the condensed bosons results in a characteristic three-peak structure (see Fig. 4). The total momentum of the density profile is identically zero, but the peaks are moving along the $x$ direction with velocities proportional to the $k$ vector of the laser field modulation in that direction. The probability of measuring a left- (right-) moving condensate is $w_L$ ($w_R$) and the signature of a left- (right-) moving state consists of the middle and small peaks moving left (right), while the large peak moves in the opposite direction.

In conclusion, the spin-orbit coupled BEC can be viewed as a state occurring at the interface between spintronics and cold atom physics, with nontrivial properties that have a significant potential for applications. We note here that the ground state of the double-well SOBEC [see Eq. (22)] represents a $N00N$ state [23,31], similar to those recently proposed for the construction of gravimeter bases on atom interferometry [32]. Therefore the study of a SOBEC state in the context of quantum entanglement and quantum interference is highly relevant. In addition, the double degeneracy associated with the pseudo-spin degree of freedom makes this state a natural candidate for a qubit. A possible way to measure such a qubit was described in the last section. Time-dependent laser fields [similar to those which lead to the spin-orbit-coupled Hamiltonian (1)] could be used as "gates" to perform unitary rotations in the space of degenerate ground states. Note that the coupling of the spin to the orbital motion yields a protecting mechanism against decoherence, due to momentum conservation, and suggests that the spin-orbit coupled condensates are interesting candidates for fault tolerant quantum computation. Finally, we note that for a symmetric Rashba-type spin-orbit coupling the system is characterized by a single-particle spectrum that has a continuous set of minima along a circle in momentum space. This results in a huge degeneracy that may lead to possible phases with nontrivial topological properties, making the study of the symmetric SOBEC a potentially very interesting problem.

## ACKNOWLEDGMENT

V.G. is grateful to C. Clark, I. Satija, I. Spielman, and J. Vaishnav for useful discussions and JQI for financial support.


[1] A. J. Leggett, Rev. Mod. Phys. **73**, 307 (2001).
[2] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[3] W. Pauli, Phys. Rev. **58**, 716 (1940).
[4] G. Juzeliunas and P. Ohberg, Phys. Rev. Lett. **93**, 033602 (2004).
[5] D. Jaksch and P. Zoller, New J. Phys. **5**, 56 (2003).
[6] J. Ruseckas, G. Juzeliunas, P. Ohberg, and M. Fleischhauer, Phys. Rev. Lett. **95**, 010404 (2005).
[7] K. Osterloh, M. Baig, L. Santos, P. Zoller, and M. Lewenstein, Phys. Rev. Lett. **95**, 010403 (2005).
[8] S. L. Zhu, H. Fu, C. J. Wu, S. C. Zhang, and L. M. Duan, Phys. Rev. Lett. **97**, 240401 (2006).
[9] I. I. Satija, D. C. Dakin, and C. W. Clark, Phys. Rev. Lett. **97**, 216401 (2006).
[10] T. D. Stanescu, C. Zhang, and V. M. Galitski, Phys. Rev. Lett. **99**, 110403 (2007).
[11] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur, and W. Ketterle, Nature (London) **396**, 345 (1998); T.-L. Ho, Phys. Rev. Lett. **81**, 742 (1998); T. Ohmi and K. Machida, J. Phys. Soc. Jpn. **67**, 1822 (1998); C. K. Law, H. Pu, and N. P. Bigelow, Phys. Rev. Lett. **81**, 5257 (1998); T.-L. Ho and S.-K. Yip, *ibid.* **84**, 4031 (2000).
[12] S. Ashhab and A. J. Leggett, Phys. Rev. A **68**, 063612 (2003).
[13] E. D. Siggia and A. E. Ruckenstein, Phys. Rev. Lett. **44**, 1423 (1980).
[14] P. Fazekas and P. Entel, Z. Phys. B: Condens. Matter **50**, 231 (1983).
[15] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **81**, 1539 (1998); D. S. Hall, M. R. Matthews, C. E. Wieman, and E. A. Cornell, *ibid.* **81**, 1543 (1998).
[16] H. J. Lewandowski, J. M. McGuirk, D. M. Harber, and E. A. Cornell, Phys. Rev. Lett. **91**, 240404 (2003); D. M. Harber, H. J. Lewandowski, J. M. McGuirk, and E. A. Cornell, Phys. Rev. A **66**, 053616 (2002).
[17] M. G. Moore and P. Meystre, Phys. Rev. A **59**, R1754 (1999).
[18] A. Sorensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Nature (London) **409**, 63 (2001).
[19] H. Pu and P. Meystre, Phys. Rev. Lett. **85**, 3987 (2000).
[20] K. Helmerson and L. You, Phys. Rev. Lett. **87**, 170402 (2001).
[21] S. Raghavan, H. Pu, and N. P. Bigelow, Opt. Commun. **188**, 149 (2001).
[22] M. Erhard, H. Schmaljohann, J. Kronjager, K. Bongs, and K. Sengstock, Phys. Rev. A **69**, 032705 (2004); A. Widera, O. Mandel, M. Greiner, S. Kreim, T. W. Hansch, and I. Bloch, Phys. Rev. Lett. **92**, 160406 (2004).
[23] B. C. Sanders, Phys. Rev. A **40**, 2417 (1989).
[24] A. M. Polyakov, Nucl. Phys. B **120**, 429 (1977).
[25] A. A. Abrikosov, L. P. Gor'kov, and E. Dzyaloshinskii, *Meth-*

*ods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs, NJ, 1963).
[26] E. I. Rashba, Sov. Phys. Solid State **2**, 1224 (1960).
[27] S. A. Brazovskii, Sov. Phys. JETP **41**, 85 (1975).
[28] J. Schmalian and M. Turlakov, Phys. Rev. Lett. **93**, 036405 (2004); A. V. Chubukov, V. M. Galitski, and V. M. Yakovenko, *ibid.* **94**, 046404 (2005); B. Binz, A. Vishwanath, and V. Aji, *ibid.* **96**, 207202 (2006).
[29] A. Posazhennikova, Rev. Mod. Phys. **78**, 1111 (2006).
[30] A. B. Kuklov and B. V. Svistunov, Phys. Rev. Lett. **89**, 170403 (2002).
[31] P. Kok, H. Lee, and J. P. Dowling, Phys. Rev. A **65**, 052104 (2002).
[32] J. B. Fixler, G. T. Foster, J. M. McGuirk, and M. A. Kasevich, Science **315**, 74 (2007).