Number squeezing, quantum fluctuations, and oscillations in mesoscopic Bose Josephson junctions

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We use a two-mode Bose-Hubbard Hamiltonian to determine the ground state and dynamical evolution for a Bose Josephson junction realized by an ultracold Bose gas in a double-well trap. We identify Mott-like lobes where number fluctuations are suppressed and the interference fringes in the momentum distribution are strongly reduced. Different from superconducting Josephson junctions, the lobes size increases at increasing wells imbalance. Upon a sudden rise of the barrier between the two wells, an initially phase-coherent state evolves into a coherent superposition of phase states, leading to destructive interference in the time-dependent momentum distribution.

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I. INTRODUCTION

Superconductor Josephson junctions are a paradigmatic example of macroscopic quantum coherence. The underlying physical mechanism is the Josephson effect [1]: two superconductors connected by a weak link have coherent dynamical behavior determined by the relative macroscopic phase of the superconducting condensates. Josephson junctions have been used to discuss fundamental concepts in quantum mechanics [2] and perform precision measurements [3], and are now promising candidates to implement quantum-information devices [4]. Importantly, superconducting Josephson junctions allow one to precisely control the state of the system by varying external parameters—e.g., a gate voltage or magnetic flux.

Bose Josephson junctions have been only recently proposed [5] and realized [6], and many issues remain open. In the simplest configuration a Bose Josephson junction is realized by confining an ultracold Bose gas in a double-well potential. This can be described using a two-mode model in which the bosons occupy the lowest level in each well. In the classical regime of large particle numbers and weak repulsive interactions the gas is well described by the Gross-Pitaevskii equation. Within the two-mode model, it can be recast in the form of generalized Josephson equations for the dynamics of the relative phase and population imbalance between the wells [5]. These equations differ from the ones used for superconducting junctions [3] by the presence of a nonlinear coupling among the phase and population-imbalance variables, obtained from boson-boson interactions in the mean-field approximation, which gives rise to a rich dynamical behavior [5].

In this paper we focus on the mesoscopic quantum regime beyond the Gross-Pitaevskii equation in the limit of strong interactions and/or smaller values of N. This gets within reach of current experiments [7]. As interactions are increased phase fluctuations become more important while number fluctuations are suppressed; the ground state of the system approaches a regime similar to a mesoscopic Mott insulator. During the time evolution the phase coherence first degrades (phase diffusion [8]), but in a closed quantum system, periodically revives, as demonstrated experimentally [9]. At intermediate times between phase collapse and revival, superpositions of phase states are predicted to form [10], but are not easily observable in superconducting Josephson junctions [11].

The quantum behavior of superconducting Josephson junctions is usually accounted for by the standard phase model [3,12]. This model has been proposed to study the quantum fluctuations in a Bose Josephson junction [15] and has been extended for large particle numbers N with subleading $1/N$ corrections [16]; however, it does not account for large population imbalance among the two sides of the junction. In this work we overcome this limitation. Using the quantum two-mode Bose-Hubbard Hamiltonian we investigate the ground-state properties of the junction which we summarize in a "phase diagram" obtained by studying number fluctuations at varying well asymmetry and interaction strength. Quite remarkably, the phase diagram differs from the one known for superconducting Josephson junctions. We furthermore study the equilibrium and time-dependent momentum distribution which allow us to determine experimentally the phase coherence of the system.

II. MODEL

We use the two-mode Bose-Hubbard Hamiltonian

$$H = E_1^0 \hat{a}_1^\dagger \hat{a}_1 + E_2^0 \hat{a}_2^\dagger \hat{a}_2 + \frac{U_1}{2}\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \frac{U_2}{2}\hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 - K(\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2), \quad (1)$$

where $\hat{a}_i$ and $\hat{a}_i^\dagger$ with $i=1,2$ are bosonic field operators satisfying $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$, E_i^0 are the energies of the two wells, $U_i > 0$ are the boson-boson repulsive interactions, and K is the tunnel matrix element—i.e., the Rabi oscillation energy in the case of a noninteracting model. The Heisenberg equations of motion for this model yield

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$$i\hbar\partial_t\hat{a}_1 = E_1^0\hat{a}_1 + U_1\hat{n}_1\hat{a}_1 - K\hat{a}_2,$$

$$i\hbar\partial_t\hat{a}_2 = E_2^0\hat{a}_2 + U_2\hat{n}_2\hat{a}_2 - K\hat{a}_1, \quad (2)$$

where $\hat{n}_i=\hat{a}_i^\dagger\hat{a}_i$. This is the quantum equivalent of the two-mode model used in the mean-field approximation $\langle\hat{a}_i\rangle=\sqrt{N_i}\exp(i\theta_i)$ and $\langle\hat{n}_i\rangle=N_i$ to describe Bose-Josephson junctions [5]. Indeed, by defining $n=(N_1-N_2)/2$ and $\phi=\theta_2-\theta_1$ the above equations are readily transformed into the Josephson-like equations

$$\hbar\partial_t n = -2K\sqrt{(N/2)^2-n^2}\sin\phi,$$

$$\hbar\partial_t\phi = \Delta\tilde{E} + nU_s + Kn\cos\phi/\sqrt{(N/2)^2-n^2}, \quad (3)$$

where $\Delta\tilde{E}=[E_1^0+U_1(N-1)/2-E_2^0-U_2(N-1)/2]$ is related to the well asymmetry and $U_s=(U_1+U_2)$. To go beyond the mean-field model, we transform the Hamiltonian (1) into the exact quantum phase one by defining first the operators [17] $\hat{a}_i=\sqrt{\hat{n}_i+1}\hat{e}^{i\theta_i}$ and $\hat{a}_i^\dagger=\hat{e}^{-i\theta_i}\sqrt{\hat{n}_i+1}$ and then the relative-phase and relative-number operators $\hat{e}^{i\phi}=\hat{e}^{i\theta_2}\hat{e}^{-i\theta_1}$ and $\hat{n}=(\hat{n}_1-\hat{n}_2)/2$. Throughout the paper we work at fixed total particle number $\hat{N}=\hat{n}_1+\hat{n}_2=N$, which renders the operator $e^{i\phi}$ nonunitary [17]. In the new variables the Hamiltonian reads (up to a constant term)

$$H = U_s(\hat{n}-n_0)^2/2 - K\sqrt{N/2-\hat{n}+1}\hat{e}^{i\phi}\sqrt{N/2+\hat{n}+1} - K\sqrt{N/2+\hat{n}+1}\hat{e}^{-i\phi}\sqrt{N/2-\hat{n}+1}, \quad (4)$$

with $n_0=-\Delta\tilde{E}/U_s$. Equation (4) implies a nonlinear, $\hat{n}$-dependent Josephson coupling.

III. QUASICLASSICAL LIMIT

By using the commutation relations $[\sqrt{N/2+\hat{n}+1},\hat{e}^{i\phi}]=(\sqrt{N/2+\hat{n}+1}-\sqrt{N/2+\hat{n}})\hat{e}^{i\phi}$ and $[\sqrt{N/2-\hat{n}+1},\hat{e}^{-i\phi}]=(\sqrt{N/2-\hat{n}+1}-\sqrt{N/2-\hat{n}})\hat{e}^{-i\phi}$, and by expanding the Hamiltonian (4) for large N, we obtain

$$H = U_s(\hat{n}-n_0)^2/2 - KN\hat{\cos}\,\phi - K[1-(2\hat{n}^2+1/2)/N]\hat{\cos}\,\phi - 2iK(\hat{n}/N)\hat{\sin}\,\phi, \quad (5)$$

where $\hat{\cos}\phi=(\hat{e}^{-i\phi}+\hat{e}^{-i\phi})/2$ and $\hat{\sin}\phi=-i(\hat{e}^{i\phi}-\hat{e}^{-i\phi})/2$. The leading term in this expansion corresponds to the standard phase model used for superconductors, which reads $H_{SJJ}=E_C(\hat{n}-n_g)^2/2-E_J\cos\phi$, with E_C the charging energy, n_g the dimensionless gate charge, and E_J the Josephson energy. Hence we have $E_C\to U_s$, $E_J\to KN$ and $n_0\to n_g$. In analogy to the superconducting case, a fine-tuning of n_0 could be used to control the quantum state of the junction. The subleading terms in Eq. (5) renormalize the plasma-Josepshon frequency $\sqrt{U_sKN}$, so that in the limit $U_s\to 0$ it tends to the Rabi frequency $\omega_R=K$ as in the mean-field solution.

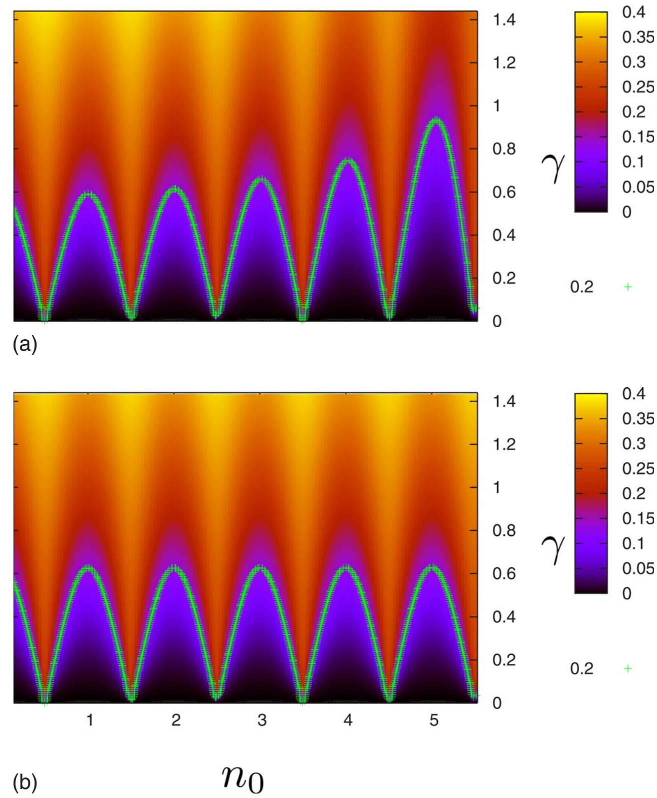

FIG. 1. (Color online) Relative-number fluctuations $\langle\Delta n^2\rangle$ for a Bose Josephson junction with $N=12$ (upper panel) and with $N=100$ (lower panel) in the plane (n_0,γ). The green line is a contour plot for $\langle\Delta n^2\rangle=0.2$ using a perturbative calculation.

IV. GROUND STATE OF THE QUANTUM HAMILTONIAN

The quantum Hamiltonian (4) contains two very different regimes depending on the ratio $\gamma=2KN/U_s$: for $\gamma\gg 1$ it yields a quasiclassical "superfluid" regime, where phase fluctuations are suppressed and the mean-field approximation applies, while for $\gamma\ll 1$ it yields the fully quantum "Mott-insulator-like" regime where number fluctuations are suppressed (number-squeezed states).

In view of the nonunitarity of the operator $e^{i\phi}$, it is more useful to represent the Hamiltonian by mapping it onto angular momentum variables in the subspace at fixed $J^2=N/2(N/2+1)$ [18] (let us choose for simplicity N even). By setting $\hat{J}_x=(\hat{a}_1^\dagger\hat{a}_2+\hat{a}_2^\dagger\hat{a}_1)/2$, $\hat{J}_y=-i(\hat{a}_1^\dagger\hat{a}_2-\hat{a}_2^\dagger\hat{a}_1)/2$, and $\hat{J}_z=(\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2)/2=\hat{n}$ we rewrite the Hamiltonian as

$$\hat{H} = U_s(\hat{J}_z-n_0)^2/2 - 2K\hat{J}_x. \quad (6)$$

For $n_0=0$ this Hamiltonian belongs to a class of models introduced by Lipkin, Meshkov, and Glick [19]. Figure 1 (upper panel) shows the numerical calculation for the number fluctuations on the ground state of the system in the plane (n_0,γ). For half-integer values of n_0 the number-squeezed regions are strongly suppressed even in the regime $\gamma\ll 1$ [14], because at these points the interaction energies of states with $\langle n\rangle=j$ and $j+1$ coincide favoring particle number fluctuations even if K is small. Note that the regions where number squeezing occurs increase with increasing imbalance n_0. This mesoscopic effect is a direct consequence of the effective nonlinear Josephson coupling, which decreases as n_0 approaches the boundary $\pm N/2$. This effect is absent when the particle number is large such that $N \gg n_0$, as shown in the lower panel of Fig. 1. This corresponds to the situation in superconducting junctions, which can be described by the standard phase model. The number-squeezed regions are reminiscent of, but distinct from, the Mott-insulator lobes of the phase diagram of the Bose-Hubbard model [13] (in the latter there is no imbalance and the size of the lobes decreases with increasing global chemical potential). In the two-mode case considered here, the lobes are cross over regions without a well-defined boundary; their shape can be captured qualitatively calculating $\langle \Delta n^2 \rangle$ perturbatively for small K/U_s [20], as shown in Fig. 1.

While in the limit $\gamma \ll 1$ the eigenvectors of Hamiltonian (6) are very close to the Fock states $|j\rangle$; in the opposite limit $\gamma \gg 1$, where large number fluctuations occur, its ground-state eigenvector is close to the angular momentum coherent state [18]

$$|\alpha\rangle = \sum_{m=-N/2}^{N/2} \binom{N}{m+N/2}^{1/2} \frac{\alpha^{m+N/2}}{(1+|\alpha|^2)^{N/2}} |m\rangle, \tag{7}$$

with $\alpha = \tan(\theta/2)\exp(-i\phi)$ and $\hat{J}_z|m\rangle = m|m\rangle$. Interestingly, the average energy on the state $|\alpha\rangle$ is given by $\langle\alpha|\hat{H}|\alpha\rangle = U_s(1-1/N)n^2/2 - 2K\sqrt{(N/2)^2 - n^2}\cos\phi$ with $n = -(N/2)\cos\theta$ and for $n_0=0$, which corresponds, up to $O(1/N)$, to the mean-field result.

V. MOMENTUM DISTRIBUTION OF THE BOSE JOSEPHSON JUNCTION

We use the exact quantum phase model to obtain the momentum distribution, which is one of the most accessible experimental observables. The field operator in the two-mode approximation reads $\hat{\Psi}(x) = \Sigma_{i=1}^{2} \Phi_i(x)\hat{a}_i$, where $\Phi_i(x)$ denotes the ground-state wave function of the well i. The one-body density matrix, defined as $\rho_1(x,x') = \langle \Psi^\dagger(x)\Psi(x')\rangle$, where the average is intended over the quantum state of the system, for the two-mode model reads $\rho_1(x,x') = \Sigma_{i,j=1}^{2} \Phi_j^*(x)\Phi_i(x')\langle a_j^\dagger a_i\rangle$. The momentum distribution is obtained from the one-body density matrix as $n(p) = \int dx \int dx' \exp[-ip(x-x')]\rho_1(x,x')$ and becomes then $n(p) = \Sigma_{i,j=1}^{2} \tilde{\Phi}_j^*(p)\tilde{\Phi}_i(p)\langle a_j^\dagger a_i\rangle$, with $\tilde{\Phi}_i(p)$ being the Fourier transform of $\Phi_i(x)$. For a symmetric well or a weakly asymmetric situation we choose $\Phi_1(x) = \Phi_0(x-d/2)$ and $\Phi_2(x) = \Phi_0(x+d/2)$, with d being the interwell distance, and hence we obtain

$$n(p) = |\tilde{\Phi}_0(p)|^2 (N + e^{-ipd}\langle \hat{J}_+\rangle + e^{ipd}\langle \hat{J}_-\rangle). \tag{8}$$

This is the generalization of the result derived by Pitaevskii

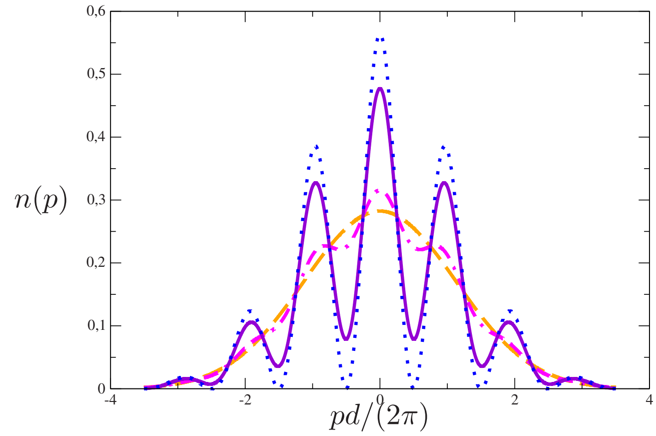

FIG. 2. (Color online) Momentum distribution for a Bose Josephson junction for various values of the interaction strength ($\gamma=100$, 1, 0.1, and 0.001 from top to bottom), for $N=10$, $n_0=0$, and $\Phi_0(x) \propto \exp(-x^2/\sigma^2)$ with $\sigma = 0.1d$.

and Stringari [15] for a Bose Josephson junction using the standard phase model and has also been used by Gati *et al.* [7] to quantify thermal decoherence in the experiment. In the quasiclassical regime $\gamma \gg 1$ we can evaluate the average in Eq. (8) using the coherent state (7); this yields $n(p) = |\tilde{\Phi}_0(p)|^2[N + 2\sqrt{(N/2)^2 - n^2}\cos(pd+\phi)]$; here, $n_0=0$ so that $n=0$ for the ground state. Hence for $\gamma \gg 1$ we expect interference fringes in momentum space [15], while in the fully quantum regime $\gamma \to 0$ the matrix elements $\langle \hat{J}_{+,-}\rangle$ are vanishingly small and the interferences are washed out. Notice that this implies averaging over repeated measurements, as a single measurement would still yield interference fringes [21]. Figure 2 shows the result of the full calculation from the exact quantum phase model.

VI. QUANTUM SUPERPOSITIONS OF PHASE STATES

Phase states can occur as a result of the time evolution following a sudden rise of the barrier between the two wells, starting from an initially coherent state (i.e., in the regime $\gamma \gg 1$ which is currently realized in experiments). For simplicity, we consider the symmetric case $n_0=0$. If at time $t=0_+$ we set the interwell coupling K in the Hamiltonian to zero, then the time evolution is governed by the term $U_s\hat{J}_z^2/2$ in the Hamiltonian. For each basis vector $|m\rangle$ of the coherent state, the time evolution is given by $|m(t)\rangle = \exp(-i2\pi m^2 t/T)|m\rangle$, where $T = 4\pi\hbar/U_s$ is the revival period [22] such that $|\alpha(T)\rangle = |\alpha\rangle$. Consider now the special times $T/2q$, q integer. In this case the phase factor governing the time evolution of the state $|m\rangle$ becomes $\exp(-i\pi m^2/q)$, which has the (anti)periodicity property $\exp[-i\pi(m+q)^2/q] = (-1)^q \exp(-i\pi m^2/q)$ depending on the parity of q. Hence for even q we can perform a discrete Fourier transform to obtain a state given by a superposition of q coherent states,

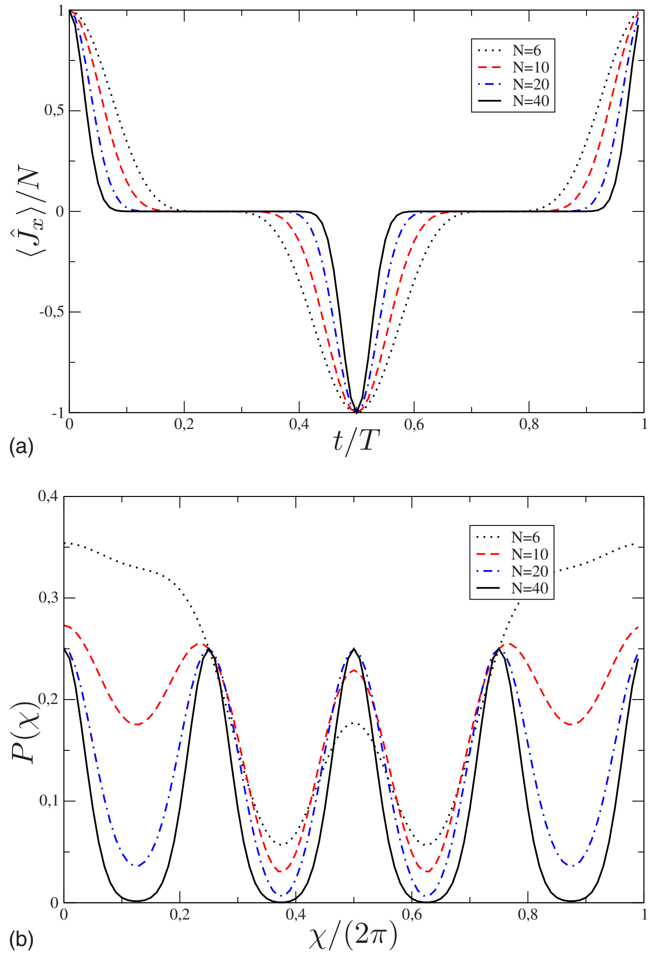

FIG. 3. (Color online) (a) Time evolution of $\langle J_x \rangle$ and (b) phase distribution at time $t=T/8$ after a sudden quench of the coupling constants starting from an initially coherent state with $\alpha=1$ for various values of N.

$$|\alpha(T/2q)\rangle = \sum_{k=0}^{q-1} u_k e^{i\pi kN/q} |e^{-i2\pi k/q}\alpha\rangle, \tag{9}$$

where $u_k=(1/q)\sum_{m=0}^{q-1} e^{-i\pi m^2/q} e^{i2\pi km/q}$, and similar states exist for odd values of q. In particular, for $q=2$ we have a superposition of two coherent states, $|\alpha(T/4)\rangle = [\exp(-i\pi/4)|\alpha\rangle + \exp(i\pi/4)(-1)^{N/2}|-\alpha\rangle]/\sqrt{2}$. This state was proposed both for light coherent states [10] and for Josephson junctions [11]. However, in the latter case it was not easy to probe it. Here, we show how such states affect the time-dependent momentum distribution. In particular, direct evaluation yields that the contrast in the momentum distribution vanishes exactly for the two-component state. Furthermore, with increasing N, the time intervals on which the contrast is reduced grow, as higher-order superposition states develop around $t=T/4$ [Fig. 3(a)]. For such states the phase distribution $P(\chi) \equiv \langle \alpha(0) \exp(i\chi) | \alpha(t) \rangle$ has equidistant equal peaks in the interval $[0, 2\pi]$ [Fig. 3(b)], which upon averaging strongly reduce the momentum-distribution contrast. Analogous phase contrast has been studied recently experimentally [7]. We have also verified that these features are robust with respect to small tunneling among the two wells.

VII. CONCLUDING REMARKS

Number-squeezed states are particularly important for atom-optics applications, as their phase-diffusion time is longer than for usual Bose condensates [7]. The measurement of the momentum distribution allows for the observation of quantum fluctuations [7] and superpositions of phase states on Bose Josephson junctions. One difficulty is to keep the number of atoms in the junction constant during the experiment. Atom losses may induce dephasing as discussed in [23]. The time required for the unitary time evolution into a superposition of phase states is experimentally feasible as demonstrated with the experiments in optical lattices [7,9].

Note added. Recently, we became aware of similar work by Averin *et al.* [24].

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$$\langle \Delta n^2 \rangle = (2K/U_s)^2[(N/2-\bar{n}_0)(N/2+\bar{n}_0+1)(1-\delta_{\bar{n}_0,N/2})/E_+^2 + (N/2+\bar{n}_0)(N/2-\bar{n}_0+1)(1-\delta_{\bar{n}_0,-N/2})/E_-^2],$$

with $E_\pm = \pm 2(n_0 - \bar{n}_0) - 1$ and $\bar{n}_0 = \text{int } n_0$. A similar result holds for the superconducting case.

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