# Local noise can enhance two-qubit teleportation

Ye Yeo
*Department of Physics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Singapore*
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For single-qubit teleportation, it has been shown that there is a family of two-qubit mixed states whose teleportation fidelity can be enhanced by subjecting one of the qubits to an amplitude damping channel. This is an interesting result, since noise in general degrades quantum entanglement. It is believed that this enhancement is due to an improvement in the classical correlations of the two-qubit states. Here, we consider two-qubit teleportation using a family of four-qubit mixed states as a resource. In this context, we show that one can again achieve enhancement in teleportation fidelity via dissipative interactions with the local environment. For a rather general class of input states, we find that this improvement implies an enhancement in the quantum discord of some teleported states. We conjecture that an improvement in some quantum property of the four-qubit mixed states could have resulted from the local interactions. We expect that our analysis will make an important case study for future investigations on the different aspects of composite quantum systems.

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## I. INTRODUCTION

Teleportation [1] is a way to send quantum information about object(s) to other (distant) object(s) using entanglement, where the spatially separated sender (Alice $\mathcal{A}$) and receiver (Bob $\mathcal{B}$) are only allowed to perform local quantum operations and communicate among themselves via a classical channel. Two-qubit teleportation via two independent, equally entangled Werner states [2] was first studied by Lee and Kim [3]. They showed that entanglement of a two-qubit input state is lost during the teleportation even when the Werner states have nonzero entanglement, and in order to teleport any nonzero entanglement the channel states should possess a critical value of minimum entanglement. Two-qubit teleportation therefore not only demands more stringent conditions on the channel states, but could also reveal important aspects of the nature of two-qubit states. It definitely deserves more detailed studies.

Composite systems of two or more quantum objects $A,B,\ldots$ have interesting properties that are absent in quantum systems composed of a single object. Specifically, the principle of quantum superposition gives rise to the phenomenon of *entanglement*—a mysterious connection between separated quantum objects, which Einstein, Podolsky, and Rosen [4] pointed out was a feature of quantum mechanics. It is believed that many of the profound results in quantum information theory [5] are impossible without the resource of entanglement. Recent investigations, however, have indicated that there are other important properties associated with, say, two-qubit states, besides entanglement. For instance, although it is well known that entanglement must grow with the system size for pure-state quantum computation to have exponential speedup [6], it is less evident that entanglement is responsible for the better performance of mixed-state quantum computation. This has motivated Datta, Shaji, and Caves [7] to propose quantum discord [8] as the figure of merit for characterizing the resources present in such mixed-state quantum computational model as the DQC1, introduced by Knill and Laflamme [9]. Clearly, a complete understanding of these different aspects of composite quantum systems is both fundamentally and practically important.

In this paper, we consider the teleportation of two-qubit states via a class of four-qubit entangled (channel) states $\Xi(\alpha,\beta)$ [Eq. (17)]. Our analysis of the effects of local noise on the "usefulness" of $\Xi(\alpha,\beta)$ shows that the corresponding *generalized singlet fraction* [10] can be enhanced by subjecting Alice's qubits to dissipative interaction with the environment via a pair of *time-correlated amplitude damping channels* [11]. Interestingly, this enhancement implies an enhancement in the quantum discord of some teleported states, even though the negativities [12,13] of these states are actually decreased. Our result thus demonstrates the importance of studying "quantum correlations" other than entanglement in two-qubit teleportation. It will make an important case study for future investigations on the different aspects of composite quantum systems.

Our paper is organized as follows. In Sec. II, we provide a short introduction to the negativity (a measure of entanglement) and the quantum discord. To set the stage, we briefly review how local noise can enhance fidelity of quantum teleportation via a certain class of two-qubit entangled states in Sec. III. We present our results in Sec. V, after a summary of the two-qubit teleportation scheme $\mathcal{E}_0$ [14] in Sec. IV. Finally, in Sec. VI, we conclude with some remarks.

## II. QUANTUM CORRELATIONS

A density operator $\rho_{AB}$ is *separable* if it can be written as a convex sum of separable pure states [2],

$$\rho_{AB} = \sum_k p_k |\psi^k\rangle_A\langle\psi^k| \otimes |\phi^k\rangle_B\langle\phi^k|, \tag{1}$$

where $\{p_k\}$ is a probability distribution and $|\psi^k\rangle_A$ and $|\phi^k\rangle_B$ are vectors belonging to Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. For two-qubit systems, a necessary and sufficient condition for separability is that a matrix, obtained by partial transposition of $\rho_{AB}$, has non-negative eigenvalue(s) [12]. Here, as a measure of the amount of entanglement associated with a given two-qubit state $\rho_{AB}$, we consider the negativity [13]

$$\mathcal{N}[\rho_{AB}] \equiv \max\left\{0,-2\sum_m \lambda_m\right\}, \tag{2}$$

where $\lambda_m$ is a negative eigenvalue of $\rho_{AB}^{T_B}$, the partial transposition of $\rho_{AB}$. The locally unitarily equivalent Bell basis states

$$|\Psi_{\mathrm{Bell}}^{\mu}\rangle_{AB} = (u_A^{\mu} \otimes u_B^0)\frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}) \tag{3}$$

are a class of maximally entangled states, with $\mathcal{N}[|\Psi_{\mathrm{Bell}}^{\mu}\rangle_{AB}\langle\Psi_{\mathrm{Bell}}^{\mu}|]=1$. Here, $u^0$ is the $2\times 2$ identity matrix; $u^1=\sigma^1$, $u^2=i\sigma^2$, $u^3=\sigma^3$, and $\sigma^j$ ($j=1,2,3$) are the Pauli matrices.

Recently, Groisman *et al.* [15] argued that $|\Psi_{\mathrm{Bell}}^{0}\rangle_{AB}$ contains one bit of "quantum correlation" and one bit of "classical correlation." The total amount of correlation in a bipartite quantum state $\rho_{AB}$, $\mathcal{C}_{\mathrm{total}}[\rho_{AB}]$, is equal to the quantum mutual information $I(A:B)\equiv S[\rho_A]+S[\rho_B]-S[\rho_{AB}]$, where $\rho_A=\mathrm{tr}_B(\rho_{AB})$, $\rho_B=\mathrm{tr}_A(\rho_{AB})$, and von Neumann entropy $S[\rho]=-\mathrm{tr}[\rho \log_2 \rho]$. It follows that, for $|\Psi_{\mathrm{Bell}}^{0}\rangle_{AB}$, we have $\mathcal{C}_{\mathrm{total}}[|\Psi_{\mathrm{Bell}}^{0}\rangle_{AB}]=2$. To obtain the amount of classical correlation associated with $|\Psi_{\mathrm{Bell}}^{0}\rangle_{AB}$, they determined $\mathcal{C}_{\mathrm{total}}[\sigma_{AB}]$, where $\sigma_{AB}=(|00\rangle_{AB}\langle 00|+|11\rangle_{AB}\langle 11|)/2$ is the state resulting from the erasure of the entanglement between $A$ and $B$. That is, $\mathcal{C}_{\mathrm{classical}}[|\Psi_{\mathrm{Bell}}^{0}\rangle_{AB}]=\mathcal{C}_{\mathrm{total}}[\sigma_{AB}]=1$; or $\mathcal{C}_{\mathrm{quantum}}[|\Psi_{\mathrm{Bell}}^{0}\rangle_{AB}]=\mathcal{C}_{\mathrm{total}}[|\Psi_{\mathrm{Bell}}^{0}\rangle_{AB}]-\mathcal{C}_{\mathrm{classical}}[|\Psi_{\mathrm{Bell}}^{0}\rangle_{AB}]=1$. Clearly, this one bit of quantum correlation refers to the quantum entanglement associated with $|\Psi_{\mathrm{Bell}}^{0}\rangle_{AB}$.

Another information-theoretic measure of the quantum nature or "quantumness" of the correlations between $A$ and $B$ was introduced by Ollivier and Zurek [8]. It is the quantum discord

$$\mathcal{D}_A(A{:}B) \equiv \sum_{m=0}^{1} \pi_m S[\rho_{A|\Pi_B^m}] + S[\rho_B] - S[\rho_{AB}], \tag{4}$$

where the projectors $\Pi_B^m=|\pi^m\rangle_B\langle\pi^m|$ (with $|\pi^0\rangle\equiv\cos\theta|0\rangle+e^{i\phi}\sin\theta|1\rangle$, $|\pi^1\rangle=e^{-i\phi}\sin\theta|0\rangle-\cos\theta|1\rangle$, and $-\pi\leqslant\theta\leqslant\pi$, $0\leqslant\phi\leqslant 2\pi$) describe perfect measurements of $B$; $\rho_{A|\Pi_B^m}=\mathrm{tr}_B(\Pi_B^m\rho_{AB}\Pi_B^m)/\pi_m$ is the state of $A$ after the measurement outcome $m$ has been detected; $S[\rho_{A|\Pi_B^m}]$ is the missing information about $A$, and probability $\pi_m=\mathrm{tr}[\Pi_B^m\rho_{AB}]$. In general, this quantity depends both on $\rho_{AB}$ and $\{\Pi_B^m\}$, and is asymmetric under the change $A\leftrightarrow B$. And, it may not exactly quantify the quantum correlation in Ref. [15]. However, we note that for the density operator we will mostly be concerned with in this paper,

$$\tau_{AB} = t_{00}|00\rangle_{AB}\langle 00| + t_{01}|00\rangle_{AB}\langle 11| + t_{10}|11\rangle_{AB}\langle 00| + t_{11}|11\rangle_{AB}\langle 11|,$$

the minimum discord

$$\mathcal{D}_{\min}[\tau_{AB}] = \mathcal{C}_{\mathrm{quantum}}[\tau_{AB}] \tag{5}$$

when $|\pi^0\rangle=|0\rangle$ and $|\pi^1\rangle=|1\rangle$. $t_{00}$, $t_{11}$ are real coefficients that satisfy $t_{00}+t_{11}=1$, while $t_{01}$, $t_{10}$ may be complex with $t_{10}=t_{01}^*$. Here, we use the minimum discord to characterize the nonclassical correlations in a given quantum state. In addition to its possible role in mixed-state quantum computation (see Ref. [7]), Zurek [16] has shown that quantum Maxwell's demons can extract more work than classical ones from correlations between a pair of quantum systems and that the difference is given by the discord.

## III. SINGLE-QUBIT TELEPORTATION

### A. Singlet fraction

When Alice and Bob share an arbitrary two-qubit mixed state $\chi_{AB}$ as a resource, the standard teleportation protocol of Bennett *et al.*, $\mathcal{T}_0$, acts as a generalized depolarizing channel $\Lambda_B^{\chi,\mathcal{T}_0}$, with probabilities given by the maximally entangled components of the resource [17,18],

$$\rho_B^{\mathrm{out}} \equiv \Lambda_B^{\chi,\mathcal{T}_0}(|\psi\rangle_B\langle\psi|) = \sum_{\mu=0}^{3} \langle\Psi_{\mathrm{Bell}}^{\mu}|\chi|\Psi_{\mathrm{Bell}}^{\mu}\rangle u_B^{\mu\dagger}|\psi\rangle_B\langle\psi|u_B^{\mu}. \tag{6}$$

Here, $|\psi\rangle_B=a_0|0\rangle_B+a_1|1\rangle_B$, with $a_0,a_1\in\mathcal{C}^1$ and $|a_0|^2+|a_1|^2=1$, is an arbitrary "unknown" (input) state of a qubit. Consequently, at Bob's end, the teleported (output) state $\rho_B^{\mathrm{out}}$ can only be a distorted copy of the state $|\psi\rangle_A$ initially held by Alice. The reliability for teleportation of a given channel state $\chi_{AB}$ is quantitatively measured by the teleportation fidelity,

$$\Phi[\Lambda_B^{\chi,\mathcal{T}_0}] \equiv \int d\psi_B\langle\psi|\rho_B^{\mathrm{out}}|\psi\rangle_B = \frac{1}{3} + \frac{2}{3}\mathcal{F}[\chi], \tag{7}$$

where the singlet fraction

$$\mathcal{F}[\chi] \equiv \langle\Psi_{\mathrm{Bell}}^{0}|\chi|\Psi_{\mathrm{Bell}}^{0}\rangle. \tag{8}$$

The maximum teleportation fidelity depends on the maximal singlet fraction [18,19]: $\Phi[\Lambda_B^{\chi,\mathcal{T}_{\mathrm{opt}}}]=1/3+2\mathcal{F}_{\max}[\chi]/3$, where $\mathcal{F}_{\max}[\chi]\equiv\max_u\langle\Psi_{\mathrm{Bell}}^{0}|(u^0\otimes u)\chi(u^0\otimes u^\dagger)|\Psi_{\mathrm{Bell}}^{0}\rangle$. The maximization is over the set of all unitary operations $u$ on $\mathcal{C}^2$. Clearly, a necessary condition for faithful teleportation is that Alice and Bob share *a priori* a maximally entangled channel state. In order to be useful for $\mathcal{T}_0$, $\chi_{AB}$ must have $\mathcal{F}_{\max}[\chi]>1/2$ [19,20]. If the channel state $\chi_{AB}$ is mixed too much ($\mathcal{F}_{\max}[\chi]\leqslant 1/2$), it will not provide for any better teleportation fidelity than that of an ordinary classical communication protocol. Teleportation can indeed serve as a fundamentally important operational test of not only the presence but also the "quality of entanglement."

### B. Local environment can enhance fidelity of single-qubit teleportation

Interactions with the environment and imperfections of preparation result in noisy or mixed states described by density operators. For instance, $|\Psi_{\mathrm{Bell}}^{\mu}\rangle_{AB}\langle\Psi_{\mathrm{Bell}}^{\mu}|\rightarrow\chi_{AB}=\mathcal{E}(|\Psi_{\mathrm{Bell}}^{\mu}\rangle_{AB}\langle\Psi_{\mathrm{Bell}}^{\mu}|)=\sum_\nu E_{AB}^{\nu}|\Psi_{\mathrm{Bell}}^{\mu}\rangle_{AB}\langle\Psi_{\mathrm{Bell}}^{\mu}|E_{AB}^{\nu\dagger}$, where $\mathcal{E}$ is a quantum operation or channel ($E_{AB}^{\nu}$'s are the corresponding

Kraus operators) [21], which mathematically describes the noise and the resulting decoherence. In general, the dissipative effects of noise degrade quantum entanglement and $\chi_{AB}$ may become separable. It is thus rather surprising when Badziag *et al.* [22] presented an interesting class of two-qubit entangled states, which may be made useful or "more useful" for single-qubit teleportation by subjecting one of the qubits to dissipative interaction with the environment via an amplitude damping channel. They considered the following one-parameter family of two-qubit mixed states:

$$\xi_{AB} \equiv \sum_{\nu=0}^{1} (u_A^0 \otimes K_B^{\nu})|\Psi_{\rm Bell}^0\rangle_{AB}\langle\Psi_{\rm Bell}^0|(u_A^0 \otimes K_B^{\nu\dagger}), \tag{9}$$

where

$$K^0 = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & 1 \end{pmatrix}, \quad K^1 = \begin{pmatrix} 0 & 0 \\ \sqrt{1-q} & 0 \end{pmatrix} \tag{10}$$

with $0 \leqslant q \leqslant 1$, are Kraus operators that define an amplitude damping channel. Hereafter, $|0\rangle$ and $|1\rangle$ denote the excited and ground states, respectively. The amplitude damping channel is characterized by the parameter $q$, with $1-q$ denoting the dissipation strength when a qubit interacts with the environment via this channel. Badziag *et al.* [22] (see also Bandyopadhyay [23]) showed that subjecting $\xi_{AB}$ to local noise at Alice's site

$$\xi_{AB} \to \xi'_{AB} = \sum_{\mu=0}^{1} (K_A^{\mu} \otimes u_B^0)\xi_{AB}(K_A^{\mu\dagger} \otimes u_B^0) \tag{11}$$

may improve the maximal singlet fraction. That is, there exist values of $q$ such that though $\mathcal{F}_{\max}[\xi] < 1/2$ we can have $\mathcal{F}_{\max}[\xi'] > 1/2$, and also $1/2 < \mathcal{F}_{\max}[\xi] \leqslant \mathcal{F}_{\max}[\xi']$. This is intriguing because the dissipative interaction with qubit $B$, which degrades entanglement in the first place is utilized to improve the quality of $\xi_{AB}$ by applying it to qubit $A$. Bandyopadhyay reasoned qualitatively that given any mixed channel state $\chi_{AB}$, the corresponding maximal teleportation fidelity is determined by both the amount of entanglement $\mathcal{N}[\chi]$, and the "classical correlations" between Alice's qubit $A$ and Bob's qubit $B$; and since $\mathcal{N}[\chi]$ cannot be increased by Alice's local operations (in fact, $\mathcal{N}[\xi'] < \mathcal{N}[\xi]$), her action, Eq. (11), would only have enhanced the "classical correlations." According to Bandyopadhyay, the enhancement in the maximal singlet fraction is thus due to improved classical correlations. In the context of single-qubit teleportation, it seems that this issue cannot be pursued further.

## IV. TWO-QUBIT TELEPORTATION AND THE GENERALIZED SINGLET FRACTION

In Ref. [14], we gave an explicit protocol $\mathcal{E}_0$ for faithfully teleporting arbitrary two-qubit states, $|\Psi\rangle_{A_1A_2} = \Sigma_{i,j=0}^{1} a_{ij}|ij\rangle_{A_1A_2}$ with $a_{ij} \in \mathcal{C}^1$ and $\Sigma_{i,j=0}^{1}|a_{ij}|^2 = 1$, employing genuine four-qubit entangled states

$$|\Upsilon^{00}(\theta_{12},\phi_{12})\rangle_{A_3A_4B_1B_2} \equiv \frac{1}{2}\sum_{J=0}^{3} |J\rangle_{A_3A_4} \otimes |J'\rangle_{B_1B_2}. \tag{12}$$

$\{|J\rangle = S|ij\rangle\}$ and $\{|J'\rangle = T|ij\rangle\}$ are orthonormal bases, with

$$S(\theta_1,\phi_1) \equiv \begin{pmatrix} \cos\theta_1 & 0 & 0 & -\sin\theta_1 \\ 0 & \cos\phi_1 & -\sin\phi_1 & 0 \\ 0 & \sin\phi_1 & \cos\phi_1 & 0 \\ \sin\theta_1 & 0 & 0 & \cos\theta_1 \end{pmatrix},$$

$$T(\theta_2,\phi_2) \equiv \begin{pmatrix} \cos\theta_2 & 0 & 0 & -\sin\theta_2 \\ 0 & \sin\phi_2 & \cos\phi_2 & 0 \\ 0 & \cos\phi_2 & -\sin\phi_2 & 0 \\ \sin\theta_2 & 0 & 0 & \cos\theta_2 \end{pmatrix}. \tag{13}$$

Here, $-\pi/2 < \theta_{12} \equiv \theta_1 - \theta_2 < \pi/2$ and $-\pi/2 < \phi_{12} \equiv \phi_1 - \phi_2 < \pi/2$, since $0 < \theta_1, \theta_2, \phi_1, \phi_2 < \pi/2$. Whenever $\theta_{12} = \phi_{12} = 0$, $|\Upsilon^{00}\rangle$ is reducible to a tensor product of two Bell states: $|\Upsilon^{00}\rangle_{A_3A_4B_1B_2} = |\Psi_{\rm Bell}^0\rangle_{A_3B_2} \otimes |\Psi_{\rm Bell}^0\rangle_{A_4B_1}$. Alice performs a complete projective measurement jointly on $A_1A_2A_3A_4$ in the following basis of 16 orthonormal states:

$$|\Pi^{\mu\nu}(\theta_{12},\phi_{12})\rangle_{A_1A_2A_3A_4} \equiv (U_{A_1A_2}^{\mu\nu} \otimes U_{A_3A_4}^{00}) \times |\Pi^{00}(\theta_{12},\phi_{12})\rangle_{A_1A_2A_3A_4}, \tag{14}$$

with $|\Pi^{00}(\theta_{12},\phi_{12})\rangle_{A_1A_2A_3A_4} \equiv \frac{1}{2}\Sigma_{K=0}^{3}|K'\rangle_{A_1A_2} \otimes |K\rangle_{A_3A_4}$ and $U^{\mu\nu} \equiv u^{\mu} \otimes u^{\nu}$. Upon receiving classical information of her measurement result, Bob can always succeed in recovering an exact replica of the original state of Alice's particles $A_1A_2$, by applying the appropriate recovery unitary operations to his particles $B_1B_2$. If Alice and Bob share *a priori* two pairs of particles, $A_3A_4$ and $B_1B_2$, in an arbitrary four-qubit mixed state $\Xi_{A_3A_4B_1B_2}$ as a resource, $\mathcal{E}_0$ acts as a generalized depolarizing bichannel [10]: $\Lambda_{B_1B_2}^{\Xi,\mathcal{E}_0}(|\Psi\rangle_{B_1B_2}\langle\Psi|) = \Sigma_{\mu,\nu=0}^{3}\langle\Upsilon^{\mu\nu}|\Xi|\Upsilon^{\mu\nu}\rangle U_{B_1B_2}^{\mu\nu\dagger}|\Psi\rangle_{B_1B_2}\langle\Psi|U_{B_1B_2}^{\mu\nu}$, where we define $|\Upsilon^{\mu\nu}\rangle \equiv (U^{00} \otimes U^{\mu\nu\dagger})|\Upsilon^{00}\rangle$. The fidelity of teleportation

$$\Phi[\Lambda_{B_1B_2}^{\Xi,\mathcal{E}_0}] \equiv \int d\Psi_{B_1B_2}\langle\Psi|\Lambda_{B_1B_2}^{\Xi,\mathcal{E}_0}(|\Psi\rangle_{B_1B_2}\langle\Psi|)|\Psi\rangle_{B_1B_2} = \frac{1}{5} + \frac{4}{5}\mathcal{G}[\Xi], \tag{15}$$

where the generalized singlet fraction

$$\mathcal{G}[\Xi] \equiv \max_{\theta_{12},\phi_{12}} \{\langle\Upsilon^{00}(\theta_{12},\phi_{12})|\Xi|\Upsilon^{00}(\theta_{12},\phi_{12})\rangle\}, \tag{16}$$

in contrast to Eqs. (7) and (8). We note that the $\theta_{12}$ and $\phi_{12}$, which give $\mathcal{G}[\Xi]$, determine Alice's measurement, Eq. (14). This is in contrast to the teleportation scheme of Lee and Kim, where Alice's joint measurement is decomposable into two independent Bell measurements and which is really a straightforward generalization of $\mathcal{T}_0$. $\Xi$ is useful for $\mathcal{E}_0$ if $\mathcal{G}[\Xi] > 1/2$ and $\Phi[\Lambda_{B_1B_2}^{\Xi,\mathcal{E}_0}] > 3/5$.

TABLE I. $q_{\rm crit}$ decreases with increasing $\alpha$.

| $\alpha$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 |
|---|---|---|---|---|---|---|---|---|
| $q_{\rm crit}$ | 0.034 | 0.032 | 0.028 | 0.022 | 0.015 | 0.0079 | 0.0027 | 0.000099 |

## V. RESULTS

### A. Local environment can enhance the fidelity of two-qubit teleportation

Consider the four-qubit state

$$\Xi_{A_1A_2B_1B_2}(\alpha,\beta)=\sum_{\nu=0}^{1}(U^{00}_{A_1A_2}\otimes K^{\nu\nu}_{B_1B_2})\times|\Upsilon^{00}(\alpha,\beta)\rangle_{A_1A_2B_1B_2}\langle\Upsilon^{00}(\alpha,\beta)|(U^{00}_{A_1A_2}\otimes K^{\nu\nu\dagger}_{B_1B_2}), \quad (17)$$

which can be obtained in the following way: Alice prepares the four-qubit state $|\Upsilon^{00}(\alpha,\beta)\rangle$ [Eq. (12)] locally in her laboratory and sends qubits $B_1$ and $B_2$ to Bob simultaneously across a pair of time-correlated amplitude damping channels described by the Kraus operators [11]

$$K^{00}=\begin{pmatrix}\sqrt{q}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad K^{11}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\\sqrt{1-q}&0&0&0\end{pmatrix}. \quad (18)$$

Quantum channels with correlated noise was defined in Ref. [24], where the problem of quantum channels with memory was first introduced. In particular, they considered the correlated depolarizing channels described by the Kraus operators: $\sqrt{1-p}(u^0\otimes u^0)$ and $\sqrt{p/3}(u^j\otimes u^j)$ $(j=1,2,3)$, with $0\leqslant p\leqslant 1$. Using the method described in Ref. [25], these operators can be derived by solving the quantum master equation with the Lindblad operators describing correlated Pauli rotations. The Kraus operators for the time-correlated damping channels, Eq. (18), are similarly derived by solving the quantum master equation with the Lindblad operators describing correlated decays [11].

The generalized singlet fraction of $\Xi_{A_1A_2B_1B_2}(\alpha,\beta)$,

$$\mathcal{G}[\Xi(\alpha,\beta)]=\max_{\theta_{12},\phi_{12}}\Big\{\tfrac{1}{16}[3+\sqrt{q}+(\sqrt{q}+q)\cos 2(\theta_{12}-\alpha)+4(1+\sqrt{q})\cos(\theta_{12}-\alpha)\cos(\phi_{12}-\beta)+2\cos 2(\phi_{12}-\beta)]\Big\}=\tfrac{1}{16}(3+\sqrt{q})^2, \quad (19)$$

is independent of both $\alpha$ and $\beta$, and is a simple function of $q$. Clearly, the noisy channels have a detrimental effect even though when $q=0$, $\Xi(\alpha,\beta)$ is still useful for $\mathcal{E}_0$ [since $\mathcal{G}[\Xi(\alpha,\beta)]=9/16>1/2$]. In addition, we note that as far as the generalized singlet fraction is concerned, this effect is the same regardless of if $|\Upsilon^{00}(\alpha,\beta)\rangle$ is a tensor product of two Bell states. Lastly, we emphasize that the time-correlated amplitude damping channels are not decomposable into a tensor product of amplitude damping channels.

Now, applying the prescription, similar to that in Refs. [22,23]: Alice allows her pair of qubits $A_1$ and $A_2$ to interact with the local environment via a pair of time-correlated amplitude damping channels of the same strength as above; we obtain

$$\Xi'_{A_1A_2B_1B_2}(\alpha,\beta)=\sum_{\mu=0}^{1}(K^{\mu\mu}_{A_1A_2}\otimes U^{00}_{B_1B_2})\Xi_{A_1A_2B_1B_2}(\alpha,\beta)\times(K^{\mu\mu\dagger}_{A_1A_2}\otimes U^{00}_{B_1B_2}). \quad (20)$$

To determine the corresponding generalized singlet fraction, we demand that $\phi_{12}=\beta$ and $\theta_{12}$ satisfies

$$0=2(1-q)^2\sin 2\theta_{12}+q(1+\sqrt{q})^2\sin 2(\theta_{12}-\alpha)+2(1+\sqrt{q})^2\sin(\theta_{12}-\alpha)+2(1-\sqrt{q})^2\sin(\theta_{12}+\alpha)+q(1-\sqrt{q})^2\sin 2(\theta_{12}+\alpha). \quad (21)$$

$\theta_{12}$ is therefore, in general, a very complicated function of both $\alpha$ and $q$. Note that Alice's joint measurement is now not only dictated by $\alpha$ and $\beta$, but also by $q$. However, for $\alpha=\beta=0$, we have

$$\mathcal{G}[\Xi'(0,0)]=\max_{\theta_{12},\phi_{12}}\Big\{\tfrac{1}{16}[3+q+(1-q+2q^2)\cos 2\theta_{12}+4(1+q)\cos\theta_{12}\cos\phi_{12}+2\cos 2\phi_{12}]\Big\}=\tfrac{1}{8}(5+2q+q^2). \quad (22)$$

Both $\mathcal{G}[\Xi(0,0)]$ and $\mathcal{G}[\Xi'(0,0)]$ are always strictly greater than $1/2$, and we have $\mathcal{G}[\Xi'(0,0)]\geqslant\mathcal{G}[\Xi(0,0)]$ if $0\leqslant q\leqslant q_{\rm crit}\approx 0.033\,845\,4$. $q_{\rm crit}$ is the critical value of $q$ beyond which $\mathcal{G}[\Xi'(0,0)]<\mathcal{G}[\Xi(0,0)]$.

Suppose $\theta_{12}=\theta$ is the solution to Eq. (21), then

$$\mathcal{G}[\Xi'(\alpha,\beta)]=\frac{5+q}{16}+\frac{(1-q)^2}{16}\cos 2\theta+\frac{q(1+\sqrt{q})^2}{32}\times\cos 2(\theta-\alpha)+\frac{(1+\sqrt{q})^2}{8}\times\cos(\theta-\alpha)+\frac{(1-\sqrt{q})^2}{8}\cos(\theta+\alpha)+\frac{q(1-\sqrt{q})^2}{32}\cos 2(\theta+\alpha). \quad (23)$$

Equation (23) reduces to Eq. (22) if $\theta=\alpha=0$. In contrast to Eq. (19), $\mathcal{G}[\Xi'(\alpha,\beta)]$ depends on $\alpha$, but not on $\beta$. Hence, Eq. (22) also holds for arbitrary $\beta$ when $\theta=\alpha=0$ and $\phi_{12}=\beta$. Furthermore, from Eqs. (21) and (23), we deduce that the range of values of $q$ for which $\mathcal{G}[\Xi'(\alpha,\beta)]\geqslant\mathcal{G}[\Xi(\alpha,\beta)]$ shrinks as $\alpha$ differs more and more from zero (see Table I). In fact, when $\alpha=\alpha_{\max}=\cos^{-1}3/4$, $q_{\rm crit}=0$. That is, beyond $\alpha_{\max}$, $\mathcal{G}[\Xi'(\alpha,\beta)]<\mathcal{G}[\Xi(\alpha,\beta)]$. So, as

TABLE II. $q=0.01$ and $\mathcal{N}[\Lambda^{\Xi,\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]=0.55\sin 2\epsilon$. For each $\alpha$, we obtain the corresponding $\theta$ from Eq. (21).

| $\alpha$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 |
|---|---|---|---|---|---|
| $\theta$ | 0 | 0.010 108 1 | 0.020 270 9 | 0.030 544 5 | 0.040 987 2 |
| $\mathcal{N}[\Lambda^{\Xi',\mathcal{E}_0}_{B_1B_2}(\|\Psi\rangle_{B_1B_2}\langle\Psi\|)]$ | $0.5050\sin 2\epsilon$ | $0.5049\sin 2\epsilon$ | $0.5046\sin 2\epsilon$ | $0.5041\sin 2\epsilon$ | $0.5035\sin 2\epsilon$ |

$|\Upsilon^{00}(\alpha,\beta)\rangle$ differs more and more from a tensor product of two Bell states, enhancement in teleportation fidelity is possible only with noisier channels. An interesting question is what exactly does this improvement in generalized singlet fraction mean physically?

### B. Enhancement does not imply better teleportation of entanglement

In order to answer the above question, we consider input states

$$|\Psi\rangle_{B_1B_2}=\cos\epsilon|00\rangle+\sin\epsilon|11\rangle \tag{24}$$

with $0\leqslant\epsilon\leqslant\pi/4$. For $\Xi(\alpha,\beta)$, the output states are $\Lambda^{\Xi,\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)=\tau_{B_1B_2}$, with $t_{00}=\gamma_+$, $t_{01}=t_{10}=1/4(1+\sqrt{q})\sin 2\epsilon$, $t_{11}=\gamma_-$, and $\gamma_\pm=1/2\pm 1/4[(1+q)\cos 2\epsilon]$. Straightforward calculations yield

$$\mathcal{N}[\Lambda^{\Xi,\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]=\frac{1}{2}(1+\sqrt{q})\sin 2\epsilon. \tag{25}$$

We note that the negativity of the output state decreases as the channels become more noisy, i.e., $q\to 0$, and when $q=0$, we have the smallest $\mathcal{N}[\Lambda^{\Xi,\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]=0.5\sin 2\epsilon$.

For $\alpha=\beta=0$, we have

$$\mathcal{N}[\Lambda^{\Xi',\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]=\frac{1}{2}(1+q)\sin 2\epsilon\leqslant\frac{1}{2}(1+\sqrt{q})\sin 2\epsilon \tag{26}$$

for $0\leqslant q\leqslant 1$. Note that we have equality when $q=0$. That is, at maximum dissipation strength, addition of noise does not further decrease the amount of teleported entanglement. Otherwise, the addition of noise decreases the amount of teleported entanglement further. For a given $0<\epsilon\leqslant\pi/4$, this decrease grows with $q$ increasing from zero and reaching a maximum at $q=1/4$. And, for a fixed $0<q<1$, input states with less entanglement is affected less by the addition of noise.

Now, suppose $\alpha=0.1\pi$ and $q=0.02<q_{\mathrm{crit}}$. We derive, from $\Xi'(0.1\pi,\beta)$, $\Lambda^{\Xi',\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)=\tau_{B_1B_2}$, but with

$$t_{00}\approx 0.989215\cos^2\epsilon+0.0107852\sin^2\epsilon,$$

$$t_{01}=t_{10}\approx 0.508132\cos\epsilon\sin\epsilon,$$

$$t_{11}\approx 0.0107852\sin^2\epsilon+0.989215\cos^2\epsilon.$$

This gives

$$\mathcal{N}[\Lambda^{\Xi',\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]\approx 0.508132\sin 2\epsilon<0.570711\sin 2\epsilon \approx\mathcal{N}[\Lambda^{\Xi,\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)].$$

In general, we can numerically verify that $\mathcal{N}[\Lambda^{\Xi',\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]\leqslant\mathcal{N}[\Lambda^{\Xi,\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]$ (see, for instance, Table II). This is not unexpected since with the addition of further noise to the channel state $\Xi$, the resulting generalized depolarizing bichannel $\Lambda^{\Xi',\mathcal{E}_0}_{B_1B_2}$ becomes noisier, which degrades the teleported entanglement more. It is consistent with the results of Ref. [3].

### C. Enhancement in generalized singlet fraction implies improved quantum discord

One may conclude that, as in the case of single-qubit teleportation, the enhancement in the generalized singlet fraction is solely due to an improvement in the classical correlations and hence would not bring about an enhancement in any quantum property, such as entanglement, of the output states. Surprisingly, we can show that there is enhancement in the quantum discord [Eq. (5)] of some output states whenever there is an enhancement in the generalized singlet fraction. To this end, we calculate

$$\mathcal{D}_{\min}[\Lambda^{\Xi,\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]=-\gamma_+\log_2\gamma_+-\gamma_-\log_2\gamma_- +\Gamma_-\log_2\Gamma_-+\Gamma_+\log_2\Gamma_+, \tag{27}$$

where $\gamma_\pm$ is as given above and $\Gamma_\pm=1/2\pm\sqrt{2}/8[2+2\sqrt{q}+3q+q^2-(2\sqrt{q}-q-q^2)\cos 4\epsilon]^{1/2}$. Like negativity, $\mathcal{N}[\Lambda^{\Xi,\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]$ [Eq. (25)], the quantum discord of the output state for an input state with nonzero $\epsilon$ decreases as $q\to 0$, and remains nonzero even at $q=0$. $\mathcal{D}_{\min}[\Lambda^{\Xi,\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]$ is zero only if $\epsilon=0$, when the input state has zero quantum correlations and zero entanglement. These are consistent with the understanding that quantum discord describes quantum correlations including but not limited to entanglement.

For $\alpha=\beta=0$, we have

$$\mathcal{D}_{\min}[\Lambda^{\Xi',\mathcal{E}_0}_{B_1B_2}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]=-\sigma_+\log_2\sigma_+-\sigma_-\log_2\sigma_- +\Sigma_-\log_2\Sigma_-+\Sigma_+\log_2\Sigma_+, \tag{28}$$

where

$$\sigma_\pm=\frac{1}{2}[1\pm(1-q+q^2)\cos 2\epsilon],$$

$$\Sigma_{\pm} = \frac{1}{8}[4 \pm \sqrt{2}\sqrt{5 - 6q + 13q^2 - 8q^3 + 4q^4 + (3 - 10q + 11q^2 - 8q^3 + 4q^4)\cos 4\epsilon}].$$

We note that when $q=0$, we have

$$\mathcal{D}_{\min}[\Lambda_{B_1B_2}^{\Xi',\mathcal{E}_0}(|\Psi\rangle_{B_1B_2}\langle\Psi|)] > \mathcal{D}_{\min}[\Lambda_{B_1B_2}^{\Xi,\mathcal{E}_0}(|\Psi\rangle_{B_1B_2}\langle\Psi|)] \quad (29)$$

for $0<\epsilon<\pi/4$. This, together with Eqs. (19), (22), and (26), shows that enhancement in generalized singlet fraction implies better teleportation of quantum discord. If $0<q\leq q_{\text{crit}} \approx 0.033\,845\,4$, then the inequality in Eq. (29) holds for $0<\epsilon<\epsilon_{\text{thres}}$, where $\epsilon_{\text{thres}}$ is some threshold value of $\epsilon$ beyond which the inequality is reversed. $\epsilon_{\text{thres}}$ decreases with increasing $q$ (see Table III). Lastly, we note that this is a necessary but not sufficient condition, since when $q_{\text{crit}}<q<0.1153$ there are values of $\epsilon$ such that the above inequality still holds.

The above phenomenon is not specific to the case when $\alpha=\beta=0$. For definiteness, we consider $\alpha=0.1\pi$ and $q=0.02<q_{\text{crit}}$, which from Eq. (21) we derive $\theta \approx 0.045\,735\,0$. Straightforward calculations then yield $\mathcal{D}_{\min}[\Lambda_{B_1B_2}^{\Xi',\mathcal{E}_0}(|\Psi\rangle_{B_1B_2}\langle\Psi|)]$ as a function of $\epsilon$,

$$\mathcal{D}_{\min}[\Lambda_{B_1B_2}^{\Xi',\mathcal{E}_0}(|\Psi\rangle_{B_1B_2}\langle\Psi|)] = -\lambda_+\log_2\lambda_+ - \lambda_-\log_2\lambda_- + \Lambda_-\log_2\Lambda_- + \Lambda_+\log_2\Lambda_+, \quad (30)$$

where

$$\lambda_+ \approx 0.989215\cos^2\epsilon + 0.0107852\sin^2\epsilon,$$

$$\lambda_- \approx 0.0107852\sin^2\epsilon + 0.989215\cos^2\epsilon,$$

$$\Lambda_{\pm} \approx 0.5 \pm 0.489215\sqrt{0.634854 + 0.365146\cos 4\epsilon}.$$

Obviously, for $0<\epsilon<0.336\,138$, we have Eq. (29).

### D. Uncorrelated amplitude damping channels

We end with showing that by subjecting her qubits $A_1$ and $A_2$ to a pair of uncorrelated amplitude damping channels, Alice does not improve the fidelity of two-qubit teleportation. The resulting state is

$$\Xi''_{A_1A_2B_1B_2}(\alpha,\beta) = \sum_{\lambda,\mu=0}^{1}(K_{A_1}^{\lambda}\otimes K_{A_2}^{\mu}\otimes U_{B_1B_2}^{00})\Xi_{A_1A_2B_1B_2}(\alpha,\beta) \times(K_{A_1}^{\lambda\dagger}\otimes K_{A_2}^{\mu\dagger}\otimes U_{B_1B_2}^{00}). \quad (31)$$

For simplicity, we assume the channels [Eq. (10)] to have the same $q$ as $\Xi_{A_1A_2B_1B_2}(\alpha,\beta)$ in Eq. (17). From here on, we focus on the $\alpha=\beta=0$ case. The corresponding generalized singlet fraction is then

$$\mathcal{G}[\Xi''(0,0)] = \frac{1}{16}(2 + 4\sqrt{q} + q + 2\sqrt{q^3} + 7q^2). \quad (32)$$

Obviously, it is smaller than $\mathcal{G}[\Xi(\alpha,\beta)]$ in Eq. (19). Considering again input states given in Eq. (24), we obtain

$$\Lambda_{B_1B_2}^{\Xi'',\mathcal{E}_0}(|\Psi\rangle_{B_1B_2}\langle\Psi|) = 1/4[1 + q^2 + (1 - q + 2q^2)\cos 2\epsilon] \times|00\rangle_{B_1B_2}\langle 00| + q/4(1+\sqrt{q}) \times\sin 2\epsilon(|00\rangle_{B_1B_2}\langle 11| + |11\rangle_{B_1B_2}\langle 00|) + 1/4(1-q^2)(|01\rangle_{B_1B_2}\langle 01| + |10\rangle_{B_1B_2}\langle 10|) + 1/4[1 + q^2 - (1 - q + 2q^2)\cos 2\epsilon]|11\rangle_{B_1B_2}\langle 11|$$

. As a result of the additional diagonal terms, the minimum discord is no longer given by Eq. (5) and its computation is more involved. However, our numerical results clearly indicate that the amount of teleported quantum discord always decreases if Alice subjects her qubits to a pair of uncorrelated amplitude damping channels. This is not surprising since the quantum channels are not correlated. More importantly, they are a tensor product of two channels in contrast to Eq. (18). In fact, it can also be shown that the generalized singlet fraction of the pair of channel states $\xi_{A_1B_1}\otimes\xi_{A_2B_2}$ [Eq. (9)] is greater than that of $\xi'_{A_1B_1}\otimes\xi'_{A_2B_2}$ [Eq. (11)] for $0.194146<q<1$ [26]. That is, enhancement in the maximal singlet fraction of individual channel state $\xi$ is not sufficient to result in an improvement of the generalized singlet fraction. It is not difficult to see that, in this case, we do not have better teleportation of quantum discord too. In other words, enhanced classical correlations in the individual channel state $\xi$ do not yield output states with improved quantum correlations. We, therefore, conjecture that an improvement in some quantum property of the four-qubit mixed states $\Xi_{A_1A_2B_1B_2}(\alpha,\beta)$ [Eq. (17)] could have resulted from the local interactions via a pair of time-correlated amplitude damping channels, Eq. (18).

TABLE III. $\epsilon_{\text{thres}}$ decreases with increasing $q$.

| $q$ | 0 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 |
|---|---|---|---|---|---|---|---|
| $\epsilon_{\text{thres}}$ | $\pi/4$ | 0.402162 | 0.344089 | 0.305542 | 0.274645 | 0.247371 | 0.221728 |

## VI. REMARKS AND CONCLUSION

In conclusion, we have shown that a dissipative interaction with the local environment via a pair of time-correlated amplitude damping channels can enhance the generalized singlet fraction of a class of entangled four-qubit mixed states [Eq. (17)]. Predictably, the introduction of noise does not increase the entanglement of output states. However, surprisingly, we demonstrate that this enhancement implies an improvement in the quantum discord of some output states. Quantum discord describes quantum correlations. Hence, the enhancement in generalized singlet fraction could correspond to an improvement in quantum correlations of the four-qubit channel state. This is in contrast to the case of single-qubit teleportation, where the enhancement in the maximal singlet fraction is thought to be due to improved classical correlations in the two-qubit channel state. While there is no doubt that further work needs to be done to verify the above conjecture, our results certainly reveal interesting aspects of bipartite as well as multipartite entanglement. It is hoped that they will lead to a better understanding of multipartite entanglement in the future.

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