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Scalable method to estimate experimentally the entanglement of multipartite systems

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We present an experimentally efficient and scalable method for the estimation of entanglement of mixed quantum states in terms of simple parity measurements.

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I. INTRODUCTION

Entanglement has been identified as a key resource for quantum-information processing tasks. Furthermore, it is clear that the speed-up gained in computation times using quantum mechanical systems instead of classical ones to process information emerges only in the limit of a large, or very large, number of system components. For example, quantum simulators are expected to be composed of a few tens of qubits; and a quantum computer must run on quantum registers of at least several hundreds (or several thousands, if error correction is to be used) of qubits to outperform its present-day classical counterpart. This explains the tremendous effort dedicated during the last few years to the experimental production and coherent manipulation of multiparticle entangled states of photons [1–6], ions [8–11], in cavity QED devices [7], and in coupled solid state quantum devices [12–15].

Also the experimental quantification of multipartite entanglement has thus become a major issue of interest. In principle, such quantification can be carried out through quantum state tomography [9,16,17], i.e., the complete reconstruction of the state's density matrix via the measurement of a complete set of observables, followed by the subsequent evaluation of a valid entanglement measure. In practice, however, tomography rapidly saturates the available resources and is thus no viable strategy under the perspective of scalability. Clear evidence of this is given by the experimental characterization of genuine multiparticle entangled states of up to eight ions [11]: Ten hours of data aquisition—implementing measurements in $3^8=6561$ detection bases, each corresponding to a different experimental setting—were followed by computationally expensive data processing, to reconstruct the eight-ion density matrix of the experimentally prepared state. Therefore, full tomography of entangled ion chains composed of more than only eight ions appears largely impracticable.

Quantum nonlocality tests [1,2,4–7] and entanglement witnesses [3,8–11,18–21] provide alternative means to assess the degree of entanglement of a quantum state, and were used in several experiments. Both these techniques require the measurement of only a few observables, but allow the detection of entanglement of only a small class of states. This implies that some *a priori* knowledge on the state to be analyzed is necessary. A simple entanglement measurement scheme for arbitrary mixed states is therefore highly desirable.

First steps in this direction were taken in Refs. [22,23] for the two qubit case and for the multipartite case in Refs. [24,25], where multipartite concurrence [26] was shown to be directly accessible through projective measurements on two identically prepared quantum states [27,28]. The original approach [24,25,31]—experimentally demonstrated for twin photon entanglement [29,30]—was restricted to the ideal case of pure states, and a first generalization for mixed states was given in Ref. [32], yet applicable only for bipartite systems. Here we give the extension of this approach to direct experimental entanglement estimation for mixed states of quantum systems with an arbitrary number of constituents. Our procedure, based on local parity measurements, features excellent scaling properties: The number of required observables—which can all be probed in one single experimental setting—is equal to the number of subsystems.

II. OBSERVABLE LOWER BOUND

Consider two copies of an arbitrary mixed state ϱ of an N-partite quantum system with Hilbert space $\mathcal{H}$. We introduce an observable V such that

$$C^2(\varrho) \geqslant \mathrm{Tr}(\varrho \otimes \varrho V), \tag{1}$$

i.e., that allows us to experimentally bound the concurrence of the state from below. The Hermitian operator V acts on the composite Hilbert space associated with the twofold copy of the system $\mathcal{H}\otimes\mathcal{H}$, and has the two following remarkable properties: (i) It can be detected through projective measurements of only N two-particle observables; and (ii) a single experimental setting is required throughout the detection process.

The required two-particle measurements are simultaneous parity measurements on each particle and its copy. In each run of the experiment, measuring the local parity state of all N pairs defines an event in which each pair is projected onto either a symmetric or an antisymmetric state. From all possible events we distinguish three types:

(i) The entire system together with its copy is projected onto a globally symmetric state—symmetric with respect only to the exchange of both copies of the entire system.

(ii) The entire system and copy are projected onto a globally antisymmetric state—antisymmetric with respect only

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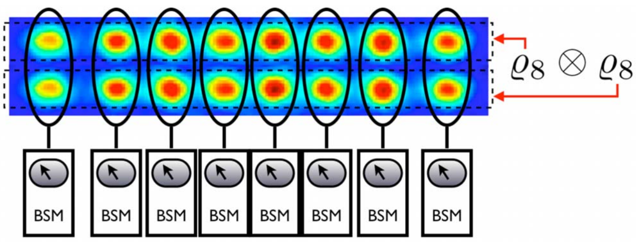

FIG. 1. (Color online) An eight-qubit state ϱ, together with its copy, is encoded, for example, in strings of two-level ions and is subject to Bell-state measurements (BSMs) on each pair. The entanglement of ϱ is obtained from the joint probabilities of appearance of singlets and triplets.

to the exchange of both copies of the entire system.

(iii) System and copy are projected onto a full locally symmetric state in which all N-particle-copy pairs are simultaneously found to be in a symmetric state [which is a particular case of (i)].

The probabilities of these three events suffice to obtain the expectation value of V, as described below. The measurement protocol is sketched in Fig. 1, for two strings of eight ions, reminiscent of the experimental situation in Ref. [11]. Each particle, together with its counterpart in the copy, is subject to a local parity measurement, which reduces to a Bell-state measurement, since the particles are qubits in this example. Recording the abundance of singlets in the string of eight ion pairs allows to infer the probabilities of the three above events immediately.

From a more technical point of view, the system's Hilbert space $\mathcal{H}$ is a tensor product $\mathcal{H}\equiv\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$ of the single particle Hilbert spaces $\mathcal{H}_i$, $1\leqslant i\leqslant N$. The symmetric and antisymmetric subspaces $\mathcal{H}_i\odot\mathcal{H}_i$ and $\mathcal{H}_i\wedge\mathcal{H}_i$ of the Hilbert space $\mathcal{H}_i\otimes\mathcal{H}_i$ of two copies of the ith single-particle subsystem are defined as the subspaces spanned by all states that acquire a phase shift of 0 or π, respectively, upon exchange of the single-particle copies. These two subspaces are associated with the local two-particle projectors P_+^i and $P_-^i=1-P_+^i$. The globally symmetric and antisymmetric subspaces $\mathcal{H}\odot\mathcal{H}$ and $\mathcal{H}\wedge\mathcal{H}$ are the subspaces of all states that are symmetric and antisymmetric with respect to the exchange of two copies of the entire system, and not only of some subsystems, and are in turn associated with the global projectors $\mathbf{P}_+$ and $\mathbf{P}_-=1-\mathbf{P}_+$. In terms of these, our observable can be explicitly expressed as

$$V=4[\mathbf{P}_+-P_+^1\otimes\cdots\otimes P_+^N-(1-2^{1-N})\mathbf{P}_-]. \tag{2}$$

Since the symmetric (antisymmetric) global projector $\mathbf{P}_+$ ($\mathbf{P}_-$) can be decomposed into a sum of all products of N local projectors with an even (odd) number of antisymmetric local projectors, it suffices to measure the parity of the N pairs of copies to reconstruct $\mathrm{Tr}(\varrho\otimes\varrho V)$.

Finally, it is important to note that V, as defined in (2), has an equivalent interpretation to that of its bipartite analogue [32], just with a much more intricate combinatorial structure: The expectation value of $A=4(\mathbf{P}_+-P_+^1\otimes\cdots\otimes P_+^N)$ yields the concurrence of pure states [25]. For a general state ϱ, however, a positive expectation value of A can have two causes: Entanglement or mixedness of ϱ. In turn, the operator $\mathbf{P}_-$ quantifies the degree of mixing of ϱ, $1-\mathrm{Tr}(\varrho^2)=2\,\mathrm{Tr}(\varrho\otimes\varrho\mathbf{P}_-)$. The linear combination of A and $\mathbf{P}_-$ in Eq. (2) therefore rescales the expectation value of A with respect to the state's intrinsic impurity, and thus provides an estimate of the inscribed entanglement through a lower bound of multipartite concurrence, as elaborated in the Appendix.

III. TIGHTNESS OF THE BOUND

Let us finally test the tightness of the observable bound on mixed random states. In Figs. 2 and 3 we plot the expectation value of the operator (2), versus concurrence in quasipure approximation [33] (which is known to yield very good approximations for weakly mixed states), for 10^5 random states of 4-qutrit and 5-qubit systems, respectively, and for different degrees of mixing. Mixed states of different purity were obtained by acting with the generalized depolarizing channel (which essentially mixes a pure state with the identity) [34] onto 10^5 random pure states, for three different coupling

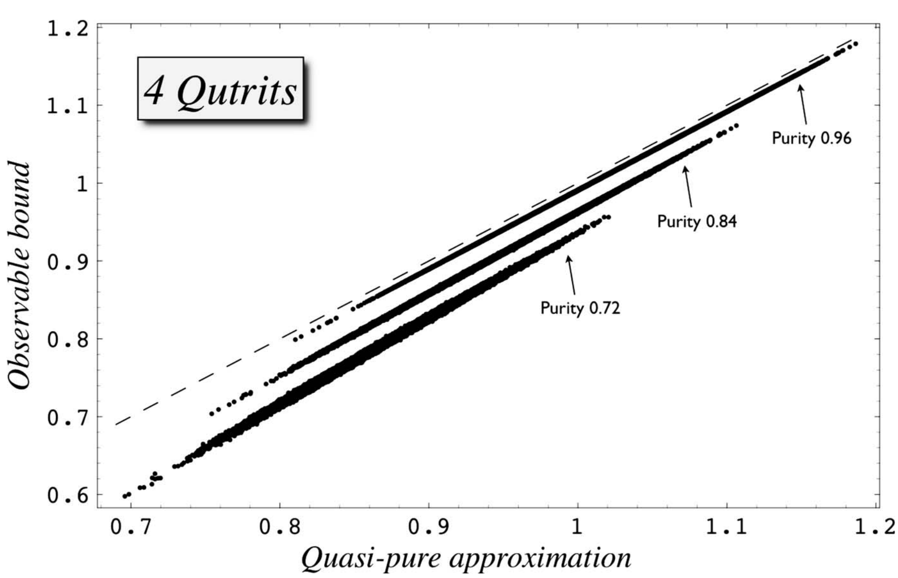

FIG. 2. Observable lower bound versus concurrence, in quasipure approximation, for 10^5 four-qutrit density matrices with strong, intermediate, and weak mixing. The dashed line indicates equality of our present measurable bound and of entanglement in quasipure approximation. The tightness of the observable bound is excellent for strongly entangled or weakly mixed states, but remains surprisingly good even for strong mixing.

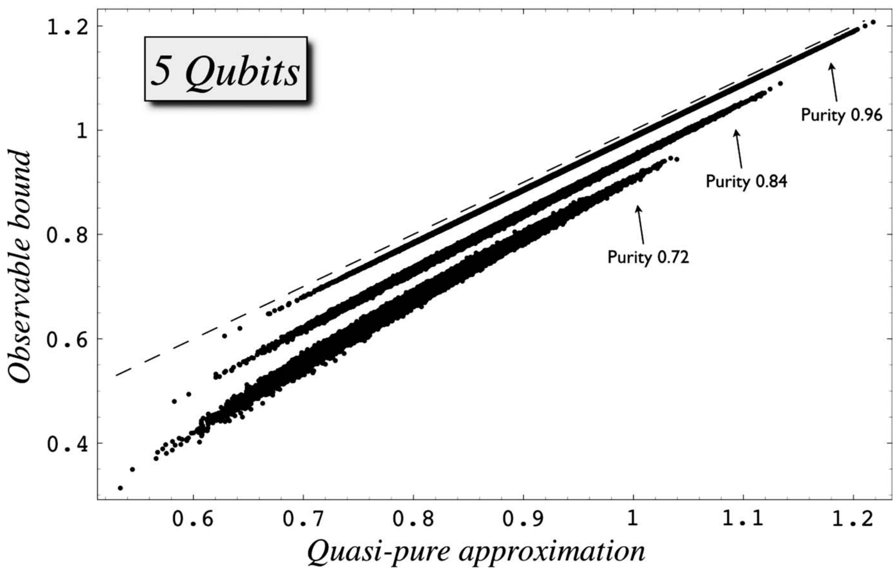

FIG. 3. Same as Fig. 2, now for 10^5 five-qubit density matrices at various levels of purity. Once more, the tightness of the observable bound is excellent for strongly entangled or weakly mixed states, and still surprisingly good for strongly mixed states.

strengths. As spelled out by the comparison in Figs. 2 and 3, the observable bound is hardly weaker than the quasipure approximation. In fact, the comparison is excellent for weakly mixed or highly entangled states. On the other hand, for some very strongly mixed or very weakly entangled states other techniques involving few measurements, such as tailored witnesses [35], may be used to improve the tightness of the entanglement estimation if some *a priori* knowledge of the state is available. The expectation value of (2), however, provides a directly observable nontrivial bound for any unknown multipartite state's concurrence.

IV. CONCLUSIONS

We have derived a general lower bound for the entanglement of mixed quantum states, which provides a hierarchy of observable entanglement measures. As such, our result has the essential virtue of scalability for unknown, multipartite mixed quantum states in arbitrary finite dimensions. Given a twofold copy of the state to be analyzed, our bounds are experimentally accessible, with linear scaling of the experimental overhead with the number of system constituents. While derived for a specific type of multipartite concurrence [26,36], equivalent expressions can be found for other observable multipartite concurrences with the same algebraic structure [24,36]. This defines a versatile toolbox for the experimental probing of quantum correlations inscribed into ever larger multicomponent quantum systems, an essential prerequisite for scaling up quantum-information technology.

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APPENDIX

Here we prove that the observable defined in (2) satisfies Eq. (1), for any state ϱ: The concurrence of ϱ is given by the convex roof [37] $C(\varrho)=\inf \sum_j C(\psi_j)$, i.e., the minimal average concurrence over all (subnormalized) pure-state decompositions $\varrho=\sum_j |\psi_j\rangle\langle\psi_j|$. If $\sum_{jk} C(\psi_j)C(\psi_k) \geq \mathrm{Tr}(\varrho\otimes\varrho V) = \sum_{jk}\langle\psi_j|\otimes\langle\psi_k|V|\psi_j\rangle\otimes|\psi_k\rangle$ holds for all decompositions $\{|\psi_j\rangle\}$, then it also holds for the optimal convex-roof decomposition, and inequality (1) is automatically satisfied. Therefore, we seek V such that

$$C(\psi)C(\phi) \geq \langle\psi|\otimes\langle\phi|V|\psi\rangle\otimes|\phi\rangle \tag{A1}$$

holds for any two arbitrary pure states $|\psi\rangle,|\phi\rangle \in \mathcal{H}$. Such an observable is known for the bipartite concurrence c: $v = 4[\mathbf{P}_+ - P_+^1\otimes P_+^2 - \frac{1}{2}(P_-^1\otimes P_+^2 + P_+^1\otimes P_-^2)]$ [32]. Now, we can make use of the fact that the N-partite concurrence can be decomposed into bipartite terms as

$$C(\Psi) = 2^{1-N/2}\sqrt{\sum_i c_i^2(\Psi)}, \tag{A2}$$

where the sum is taken over the bipartite concurrencies c_i corresponding to each subdivision of the entire system into two subsystems. This allows us to bound our quantity of interest from below as

$$C(\Psi)C(\Phi) = 2^{2-N}\sqrt{\sum_i c_i^2(\Psi)}\sqrt{\sum_i c_i^2(\Phi)} \tag{A3}$$

$$\geq 2^{2-N}\sum_i c_i(\Psi)c_i(\Phi) \tag{A4}$$

$$\geq 2^{2-N}\sum_i \langle\psi|\otimes\langle\phi|v_i|\psi\rangle\otimes|\phi\rangle, \tag{A5}$$

where we made use of the Cauchy-Schwarz inequality $\sqrt{\sum_i x_i^2}\sqrt{\sum_i y_i^2} \geq \sum_i x_i y_i$ and the above knowledge on bipartite systems. It is now a matter of straightforward algebra to show that $V=\sum_i v_i$, which finishes the proof of Eq. (1).

[1] D. Bouwmeester, J. W. Pan, M. Daniell, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **82**, 1345 (1999).
[2] J.-W. Pan *et al.*, Nature (London) **403**, 515 (2000).
[3] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Guhne, P. Hyllus, D. Bruss, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **92**, 087902 (2004).
[4] Z. Zhao *et al.*, Nature (London) **430**, 54 (2004).
[5] J.-W. Pan, M. Daniell, S. Gasparoni, G. Weihs, and A. Zeilinger, Phys. Rev. Lett. **86**, 4435 (2001).
[6] M. Eibl, N. Kiesel, M. Bourennane, C. Kurtsiefer, and H. Weinfurter, Phys. Rev. Lett. **92**, 077901 (2004).
[7] A. Raschenbeutel *et al.*, Science **288**, 2024 (2000).
[8] C. A. Sackett *et al.*, Nature (London) **404**, 256 (2000).
[9] C. F. Roos *et al.*, Science **304**, 1478 (2004).
[10] D. Leibfried *et al.*, Nature (London) **438**, 639 (2005).
[11] H. Häffner *et al.*, Nature (London) **438**, 643 (2005).
[12] M. Bayer *et al.*, Science **291**, 45 (2001).
[13] X. Wang, M. Feng, and B. C. Sanders, Phys. Rev. A **67**, 022302 (2003); M. Abdel-Aty, M. R. B. Wahiddin, and A.-S. F. Obada, e-print arXiv:quant-ph/0608208; T. P Spiller, I. D'Amico, and B. W Lovett, New J. Phys. **9**, 20 (2007); C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, e-print arXiv:quant-ph/0707.1889.
[14] J. Majer *et al.*, Nature (London) **449**, 443 (2007).
[15] I. P. Radu *et al.*, Science **320**, 899 (2008).
[16] A. G. White, D. F. V. James, P. H. Eberhard, and P. G. Kwiat, Phys. Rev. Lett. **83**, 3103 (1999).
[17] C. F. Roos, G. P. T. Lancaster, M. Riebe, H. Haffner, W. Hansel, S. Gulde, C. Becher, J. Eschner, F. Schmidt-Kaler, and R. Blatt, Phys. Rev. Lett. **92**, 220402 (2004).
[18] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[19] B. Terhal, Phys. Lett. A **271**, 319 (2000).
[20] N. Kiesel, C. Schmid, G. Toth, E. Solano, and H. Weinfurter, Phys. Rev. Lett. **98**, 063604 (2007).
[21] F. A. Bovino, G. Castagnoli, A. Ekert, P. Horodecki, C. M. Alves, and A. V. Sergienko, Phys. Rev. Lett. **95**, 240407 (2005).
[22] P. Horodecki and A. Ekert, Phys. Rev. Lett. **89**, 127902 (2002).
[23] P. Horodecki, Phys. Rev. Lett. **90**, 167901 (2003).
[24] F. Mintert, M. Kuś, and A. Buchleitner, Phys. Rev. Lett. **95**, 260502 (2005); F. Mintert, A. R. R. Carvalho, M. Kuś, and A. Buchleitner, Phys. Rep. **415**, 207 (2005).
[25] L. Aolita and F. Mintert, Phys. Rev. Lett. **97**, 050501 (2006).
[26] A. R. R. Carvalho, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. **93**, 230501 (2004).
[27] A. Peres and W. K. Wootters, Phys. Rev. Lett. **66**, 1119 (1991).
[28] T. A. Brun, Quantum Inf. Comput. **4**, 401 (2004).
[29] S. P. Walborn *et al.*, Nature (London) **440**, 1022 (2006).
[30] S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, F. Mintert, and A. Buchleitner, Phys. Rev. A **75**, 032338 (2007).
[31] G. Romero, C. E. Lopez, F. Lastra, E. Solano, and J. C. Retamal, Phys. Rev. A **75**, 032303 (2007).
[32] F. Mintert and A. Buchleitner, Phys. Rev. Lett. **98**, 140505 (2007).
[33] F. Mintert and A. Buchleitner, Phys. Rev. A **72**, 012336 (2005).
[34] M. L. Aolita, I. García-Mata, and M. Saraceno, Phys. Rev. A **70**, 062301 (2004).
[35] O. Gühne, M. Reimpell, and R. F. Werner, Phys. Rev. Lett. **98**, 110502 (2007).
[36] R. Demkowicz-Dobrzański, A. Buchleitner, M. Kuś, and F. Mintert, Phys. Rev. A **74**, 052303 (2006).
[37] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).