Quantum tunneling and the resonant states

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Using model systems we construct exact solutions of the Schrödinger equation for the tunneling effect in terms of the so-called resonant states. Two models of atomic ionization by an electrostatic field E are studied in detail with a special emphasis on strong fields where the decay exponents become proportional to $E^{2/3}$. The tunneling process, initiated by the field, is presented by an infinite sum of exponentially decaying terms without a usual slow decaying component at longer times. The sum of this series in some cases, including our models, lead to a nonexponential decay. The normalization techniques of the spatially divergent resonant wave functions via regularization of divergent integrals are considered. In particular, we apply an approximate semiclassical cutoff procedure which allows us to place the resonant states on a similar footing as the usual bound states and even use the standard normalization of the probability density. An application of such approach can illuminate and simplify calculation schemes for the study not only of ionization in electrostatic fields but also the multiphoton ionization by low frequency laser beams.

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I. INTRODUCTION

The quantum tunneling through a classically forbidden region is, on the one hand, one of the most spectacular phenomena in quantum mechanics and, on the other hand, it is one of the most universal and important processes in nature and in devices created by modern technology. The radioactive decay, semiconductor devices, Josephson junctions in superconductors are only a few such examples. The field emission caused by very strong external electric fields as a controllable source of electrons presents a new challenge for the theory which traditionally has been developed in a spirit of perturbation technique. Note that some forms of the perturbation theory can be effective in strong fields.

One of such treatments of tunneling is the quasiclassical approximation, which yields useful results [1,2] in cases when the potential barrier is "strong," i.e., wide and/or high enough. Otherwise, especially in strong external fields created by pulsed sources and lasers, numerical methods are used and more careful theoretical analyses are required. For simple binding potentials nonperturbative methods were developed in [3,4] and a general approach is worked out in a set of papers [5] by Moshinsky, Garcia-Calderon, and coworkers, where, based on Gamow's [6] notion of resonant states and following exploration of their properties [7,2], the authors developed a scheme for computing both short- and long-term decay probabilities when the potential function of escaping particles is of finite support. Note that such techniques are developed mostly for nuclear decay [8] and scattering via compound states when the potential field is of compact support or rapidly decreasing to zero. The resonant state method became even more attractive when an effective technique for its application was found. This was named by different authors "complex scaling," "complex coordinate rotation," "dilation analytic formalism" etc. This method goes further than Gamow's original work in abandoning Hermitian Hamiltonians by replacing one of the spacial coordinates, say x, with a new complex one $xe^{i\theta}, 0<\theta<2\pi/3$ [9]. By adjusting the angle θ the resonant wave functions $u(x)$ can be made square integrable, i.e., normalizable, in the new complex frame. This removes the obstacle of using a divergent u. In addition this method provides a more general approach for computing resonances without matching boundary conditions of u (this becomes difficult when the shape of a potential function is not simple) and also turns out to be quite suitable for numerical computations [10].

Our goal here is to simplify the computations of tunneling and make them more universal and transparent physically by generalizing the wave-function expansion in terms of the resonant states in the spirit of [6,7,5]. For clarity we study tunneling in the models of (i) the δ-function repulsive potential barrier as in [5], (ii) the δ-function binding potential modified by a uniform electric field as in [3], and (iii) an atomic model in the form of a rectangular well in an electric field. By matching the boundary conditions in a physical frame we find the exact solutions of the Schrödinger equation, i.e., the resonant states of the corresponding Hamiltonians, and use them to construct the leading terms which determine the process of dissociation. Our focus is on the strong field effects and the methods of normalization of the resonant states, which are shown in particular to be approximately treatable in terms of probability density and normalizable even in the traditional form. Our approach provides a physically transparent and relatively simple for calculations way to study the tunneling probability for such important processes as the electron emission by solid surfaces in strong fields and can be used for the multiphoton ionization of deep atomic levels by a strong laser radiation when the usual Floquet states method requires quite sophisticated calculations.

In Sec. II we study the tunneling of a particle bound initially between a wall and a δ-function barrier especially for the case when the binding is weak. Section III is devoted to generalizing the resonant function expansion to more general potentials than ones of compact support, in particular when the external force is created by a uniform electric field. The application of this technique is used in Sec. IV for studying the ionization of model systems by electrostatic fields where the binding potentials are an attractive δ function and a rectangular well. The normalization methods are presented in

Sec. V which is followed by a short discussion of results in Sec. VI.

II. TUNNELING THROUGH A δ-FUNCTION BARRIER

This problem is comprehensively explored in [5]. We need to study this tunneling here again for paying more attention to a weak binding, which makes the decay rapid, and to see how and where the theory should be modified for the case when the potential function does not decay at infinity and the normalization of the resonant states is unknown.

We consider a particle in one dimension between a wall with the infinite potential for $x<0$ and repulsive δ-function potential at $x=1$. The evolution process is governed by the dimensionless Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + b\delta(x-1)\psi, \quad x>0, \tag{1}$$

where the initial state $\psi(x,0)$ is assumed to live in the interval $[0,1]$, $b>0$, and the particle mass is $1/2$ while $\hbar$ and charge e are taken equal to 1. Qualitatively when b is small we call the barrier weak, a strong barrier is created by large b. When $b=\infty$ Eq. (1) yields the normalized stationary states

$$\psi_m^b(x,t) = \sqrt{2}e^{-i(\pi m)^2 t}\sin(\pi m x), \quad x\in[0,1] \quad (m=1,2,\ldots,\infty). \tag{2}$$

The solution of (1) can be expressed [5] in terms of the resonant states $u_n(x)$ in the following form:

$$\psi(x,t) = \sum_{n=-\infty}^{\infty} C_n u_n(x) M(k_n,t), \tag{3}$$

where

$$C_n = \int_0^1 \psi(x,0)u_n(x)dx, \quad M(k_n,t) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ik^2t}}{k-k_n}dk. \tag{4}$$

The functions u_n introduced in [5] are the eigenstates of the stationary equation (with b finite) associated with (1),

$$-\frac{d^2u_n(x)}{dx^2} + [b\delta(x-1) - k_n^2]u_n(x) = 0, \tag{5}$$

which describes only the outgoing waves when $x\to\infty$, i.e., in our case u_n are subject to the boundary conditions

$$u_n(0)=0, \quad u_n(x) = \text{const}\exp(ik_nx) \quad \text{for } x>1. \tag{6}$$

As it was shown by Gamow [6] all of the eigenvalues k_n in such a setting are complex numbers. It is easy to see in particular that for (5) and (6) they are solutions of the equation

$$1 - e^{2ik_n} = \frac{2ik_n}{b} \tag{7}$$

and the resonant functions have the form

$$u_n(x) = N_n\begin{cases}\sin k_n x, & 0<x\leq 1,\\ \sin k_n e^{ik_n(x-1)}, & 1<x<\infty,\end{cases} \tag{8}$$

where N_n is the normalization coefficient. Our way of solving Eq. (7) (the authors of [5] who solved it earlier did not give their routine) is straightforward: By dropping temporarily the subscript and denoting $b-2ik=\rho\exp(i\phi)$ we obtain from (7),

$$f(\phi) \equiv \ln(-\phi\csc\phi) - \phi\cot\phi = b + \ln b, \quad \rho = -\phi\csc\phi. \tag{9}$$

The form of Eq. (9) places its positive solutions for ϕ_j in the intervals $[(2j-1)\pi, 2j\pi]$, $j=1,2,\ldots$, where $f(\phi)$ grows monotonically from $-\infty$ to ∞ and thus guarantees exactly one solution inside each interval for any positive b. Clearly $-\phi_j$ is a solution too. All $\text{Im}(k_n)<0$ and the k_n asymptotics is given by the following equations:

$$\text{Re}(k_n)\to\pi n, \quad \text{Im}(k_n)\to -\ln|n|, \quad n\in Z. \tag{10}$$

When the potential barrier is strong, $b/|n|\gg 1$, we have approximately

$$\text{Re}(k_n)\approx \pi n\left(1-\frac{1}{b+1+4\pi^2n^2/3b}\right), \quad \text{Im}(k_n)\approx -\frac{\pi^2n^2}{b^2}. \tag{11}$$

The resonant wave functions u_n represent the solution of (1) in the forms (3) and (4) because they are orthogonal in the following sense:

$$\int_0^\infty u_n(x)u_m(x)dx = 0, \quad n\neq m, \tag{12}$$

and normalized. Their normalization is covered in the literature, see, for example, [2,5,11] and references therein, and in the case of compact support of $V(x)$, $0\leq x\leq R$ the normalization equation is

$$\int_0^R u_n^2(x)dx + \frac{i}{2k_n}u_n^2(R) = 1. \tag{13}$$

We will discuss (12) and (13) below, but here we note that for the δ-function binding by using (7) and (8) Eq. (13) can be reduced to

$$N_n^2\left(1+\frac{1}{b-2ik_n}\right) = 2.$$

This means that when b is large $N\approx\sqrt{2}$ for all n, i.e., almost the same normalization as for the stationary states (2). The same is true for big n and therefore $|k_n|\gg 1$ due to (10). Thus, for $b\gg 1$ in virtue of (2), (3), and (8) the time evolution of the initial state, which corresponds to the lowest bound state $n=0$, is described approximately by the wave function

$$\psi(x,t) = \sqrt{2}e^{-ik_0^2t}\begin{cases}\sin k_0 x, & 0<x\leq 1,\\ \sin k_0 e^{ik_0(x-1)}, & 1<x<\infty,\end{cases} \tag{14}$$

because $C_0(\pi)\approx 1$ in (3) while $C_n(\pi)\approx -2/b(n-n^{-1})$, see [5].

TABLE I. Data for imaginary parts of k_n and for decay exponents Γ_n.

n $b\rightarrow$	0.1	0.5	2.0	5.0	10	20
0	−2.01 15.8	−1.12 9.67	−0.462 4.56	−0.178 1.93	−0.0665 0.766	−0.0205 0.246
1	−2.38 50.3	−1.55 33.3	−0.854 18.8	−0.441 10.0	−0.2065 4.825	−0.0744 1.783
2	−2.59 88.0	−1.77 60.7	−1.078 37.2	−0.637 22.3	−0.3478 12.36	−0.1457 5.274
3	−2.74 128	−1.93 90.3	−1.233 58.1	−0.783 37.1	−0.4697 22.47	−0.2215 10.74
4		−2.05 121	−1.351 80.6	−0.898 53.8	−0.5722 34.48	−0.2950 17.94

In the case of large b the tunneling from the ground level $\psi^b(x,0)=\psi_0^b(x,0)$ is determined therefore by a single term and the survival probability decays exponentially

$$\left| \int_0^1 \psi_0^b(x,0)\psi(x,t)dx \right|^2 \approx \exp(-\Gamma_0 t),$$

where by Eqs. (11) $\Gamma_0\approx-2\,\mathrm{Im}(k_0^2)=4\pi^3/b^2$. By substituting (11) in (14) the asymptotics of the probability density of the resonant state can be written as

$$|\psi(x,t)|^2 \rightarrow \frac{\Gamma_0}{2\pi}e^{-\Gamma_0(t-x/v)}, \quad x\rightarrow\infty. \qquad (15)$$

In the exponent of (15) $x/2\pi$ was replaced by x/v where v may be treated as velocity: $v=\hbar k_0/m$ in our units $\hbar=2m=1$ and $\mathrm{Re}\,k_0=\pi$. Equations (14) and (15) are not valid for (i) very long times when the slow asymptotics of $M(k,t)$ takes over in (3) and (ii) for short time also in the case of weaker barrier (smaller b), when C_0 is not dominant and thus some time interval is needed until the faster decaying resonances die out. Note that the parameters $|C_n|^2$ are not exactly probabilities of corresponding resonant states (which are not exactly "states" too), but they describe $\psi(x,t)$ in an obvious way (3) and are close to the probabilities especially when there is a significant disparity in the decay exponents.

Up to now our analysis is based on the results of [5], though we are always tracking dominant terms of the process from the rest when this is possible. Let us consider the cases of smaller b.

The right-hand terms in each cell of Table I give the corresponding parameters Γ_n. Though the absolute values of $\mathrm{Im}\,k_n$ increase slowly with n for small b, but the decay exponents Γ grow fast enough to neglect, after an initial time interval, all the terms with $n\neq 0$ in (3) even when $b=0.1$, i.e., use Eq. (14) for the dominant term of the wave function and Eq. (15) for $|\psi|^2$ with a proper correction of the overlap integral C_0. While the authors of [5] studied the case $b=100$ in their numerical example and included contributions of 500 terms in (3) it is clear that the first 3–4 terms would give a very good precision after a short initial time for all $b>20$. Note that the normalization parameter in $\psi(x,t)$ will stay almost equal to $\sqrt{2}$ according to (13) and Table I. For example, when $b=0.1$ the absolute values of N_n differ from $\sqrt{2}$ by factors 1.065, 1.016, 1.007, 1.004 for the resonant states with $n=0,1,2,3$, respectively, and the imaginary parts of these factors are less than 0.08.

For the long-time behavior of the wave function one may keep only the first term in Eq. (3) and use the relation [5] $M(k_n,t)=e^{-ik_n^2t}-M(-k_n,t)$, where M has the power-law asymptotics

$$M(-k_n,t) \approx \frac{1}{4\sqrt{\pi i}}\left[2i\left(\frac{1}{k_n t^{1/2}}\right)-\left(\frac{1}{k_n t^{1/2}}\right)^3\right] \quad \text{for } t\rightarrow\infty,$$

which represents a t^{-3} decay of the survival probability for very large t. The theory, developed in [5], implies another form of (3) which underlines the wave-function structure

$$\psi(x,t)=\sum_{n=0}^{\infty} C_n u_n(x)e^{-ik_n^2 t} - \sum_{n=0}^{\infty}[C_n u_n(x)M(-k_n,t)-\bar{C}_n^* u_n^*(x)M(-k_n^*,t)], \qquad (16)$$

where for calculation of $\bar{C}\,\psi$ is replaced by its complex conjugate ψ^* in (4). The complex eigenvalues k_n in (16) are all in the fourth quadrant and the bound states are absent. The second series in (16) is clearly responsible for the very-long-time tail of $\psi(x,t)$ (terms with $t^{-1/2}$ disappear, see [5]).

III. GENERALIZATION OF THE RESONANCE STATE APPLICATIONS

The wave-function expansion in terms of the resonant states (3), (4), and (16) for potentials of the compact support [3,5] can be used also in the cases of rapidly decaying potentials [2,7]. In fact this approach is valid (i.e., has the same level of validation) for potentials which even grow when $|x|\rightarrow\infty$, as, for example, in the homogeneous electric field. We consider here the tunneling from a potential well created by an attractive potential $V(x)$ which is modified by an external electric field $-E$ ($E>0$). The evolution of an initial state $\psi(x,0)$ is governed by the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}(x,t)=H\psi(x,t), \quad H=H_0-Ex, \quad H_0=-\frac{d^2}{dx^2}+V(x), \qquad (17)$$

where H_0 has bound states (this is not necessary) and the system can be studied on the whole axis $(-\infty,\infty)$ or for $x\geqslant 0$ like in [4]. The requirement to have only the outgoing waves provides the boundary conditions for the stationary problem $Hu(x)=k^2u(x)$ whose solutions are the resonant states $u_n(x)$ and the corresponding discrete eigenvalues k_n^2 are always complex [5] and presumably different. The functions

$u_n(x)$ are oscillating and divergent when $x \to \infty$. The same approach in [7,11] needed potentials vanishing at ∞ because the authors avoided the infinite limits for integrations involving $u_n(x)$. We believe that some form of regularizing divergent integrals with the resonant wave functions is always present anyway, see [2]. Keeping this in mind, i.e., implicitly assuming the presence of decaying factors, which disappear after taking limits, or by an analytic continuation of integrals, we use the infinite limits below. This method does not allow to integrate absolute values of u_n on the whole line, but with a special cutoff this is done in Sec. V.

The functions $u_n(x)$, $n \in Z$ compose a complete set, i.e.,

$$\sum_{n=-\infty}^{\infty} u_n(x)u_n(x') = \delta(x-x'), \tag{18}$$

and a continuous function $\phi(x)$, which satisfies the same boundary conditions as u_n, can be expanded as

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n u_n(x).$$

The mutual orthogonality of u_n can be shown by taking two equations

$$Hu_n(x) = k_n^2 u_n(x), \quad Hu_m(x) = k_m^2 u_m(x),$$

multiplying them by u_m and u_n, respectively, and subtracting the results. Integrating the difference yields

$$(k_n^2 - k_m^2)\int_{-\infty}^{\infty} u_n(x)u_m(x)dx = 0, \tag{19}$$

where the regularization of integrals turns all functions into zeros at ∞. Choosing the normalization

$$\int_{-\infty}^{\infty} u_n^2(x)dx = 1 \tag{20}$$

allows us to obtain the coefficients for the $\phi(x)$ expansion

$$c_n = \int_{-\infty}^{\infty} \phi(x)u_n(x)dx.$$

The outgoing Green function for the operator $H-k^2$ is a solution of the equation

$$(H-k^2)G^{+}(x,x';k) = \delta(x-x'), \tag{21}$$

with the same boundary conditions as u_n. It can be written in the form

$$G^{+}(x,x';k) = \sum_{n=-\infty}^{\infty} \frac{u_n(x)u_n(x')}{k_n^2 - k^2}. \tag{22}$$

This is obvious if one substitutes (22) and (21), uses (18) and symmetry. The next step is to solve the Schrödinger equation (17) assuming that the boundary conditions hold for $\psi(x,0)$ too and using the temporal Green function as follows:

$$\psi(x,t) = \int_{-\infty}^{\infty} g(x,x';t)\psi(x',0)dx'. \tag{23}$$

The function $g(x,x';t)$ is a solution of the same equation (17) as for ψ,

$$i\frac{\partial g}{\partial t}(x,x';t) = Hg(x,x';t), \tag{24}$$

it obeys the same boundary conditions in x,x' as $u_n(x)$ and also the initial condition

$$g(x,x';0) = \delta(x-x'). \tag{25}$$

By applying the Laplace transform to (24) one comes to the equation

$$(H-ip)\bar{g}(x,x';p) = -i\delta(x-x'), \tag{26}$$

where

$$\bar{g}(x,x';p) = \int_0^{\infty} e^{-pt}g(x,x';t)dt, \quad \mathrm{Re}\, p > 0. \tag{27}$$

Comparing Eqs. (26) and (21) we conclude that the functions G^{+} and $i\bar{g}$, which have the same boundary conditions, are identical if $ip=k^2$. Therefore, the inverse Laplace transform for (27) and use of (22) yields

$$g(x,x';t) = \frac{i}{2\pi}\sum_{n=-\infty}^{\infty} u_n(x)u_n(x')\int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{e^{-ik^2t}}{k^2-k_n^2}dk^2, \quad \epsilon > 0, \tag{28}$$

where the path of integration is determined by rotation $k^2 = e^{i\pi/2}p$. Finally using (23) and (28), and the notation (4) for the overlap integrals C_n we obtain the solution of (17) in the form

$$\psi(x,t) = \sum_{n=-\infty}^{\infty} C_n u_n(x) e^{-ik_n^2 t}. \tag{29}$$

Note that in an electric field, which occupies the whole space, the form of running waves is determined by k^2 (see below) unlike in the usual situation with a short-range potential when the tunneling particle becomes free at large distances and the wave form is $e^{\pm ikx}$. Hence the wave number k is irrelevant here and the whole treatment is somewhat simpler, in particular the integration over k^2 in (28) does not need a special shape [5] of the contour, but can be done by closing the path in (28) around the whole lower half-plane of k^2, where e^{-ik^2t} rapidly decays, and collecting the residues at all the points $k^2=k_n^2$, $n \in \mathbb{Z}$. Contrary to (16), Eq. (29) does not have the usual addition which describes a power-law decay at very long times. If the imaginary parts of the resonance energies are strictly negative [12] there is only a set of exponentially decaying terms in the series (29). The electric field after releasing a charged particle from the potential well then will accelerate and push it far away not allowing the particle to be trapped again (namely this is the physical nature of the slow decaying component, see [13]). In an alternating external field the situation is different and a long surviving term does exist. In the case when the resonance

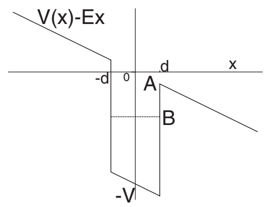

FIG. 1. Attractive rectangular well in electric field. The dashed line shows the Stark shifted real part of the resonance energy.

energies asymptotically approach to the real axis a nonexponential term of a different nature can be formed by (29). This happens in our models, see below.

IV. IONIZATION OF A MODEL ATOM IN ELECTRIC FIELD

This problem with different binding potentials $V(x)$ has been solved by many authors [1–4], we consider a simple special case in one dimension $-\infty < x < \infty$ when the binding potential is a rectangular well

$$V(x) = \begin{cases} -V & \text{if } |x| \leqslant d, \\ 0 & \text{if } d < |x| < \infty. \end{cases}$$

The external electric field $-E$, $(E>0)$ drastically modifies the potential function, see Fig. 1. When $E=0$ the time-independent Schrödinger equation takes the form

$$-\frac{d^2\phi(x)}{dx^2} + V(x)\phi(x) = \varepsilon\phi(x). \qquad (30)$$

This equation produces, along with the states in continuum, a set of discrete eigenstates with negative energies $\varepsilon_n = -q_n^2$, which are solutions [14] of one of the following equations:

$$\sqrt{V-q_n^2}\tan(d\sqrt{V-q_n^2}) = q_n, \quad q_n \tan(d\sqrt{V-q_n^2}) = -\sqrt{V-q_n^2}.$$

There is always at least one even bound state and their total number is $1+\|d\pi^{-1}\sqrt{V}\|$. The number of odd bound states $\|d\pi^{-1}\sqrt{V}+1/2\|$, can be zero when the parameter $G=d^2V$, which determines the spectrum, is not large enough.

If at time $t=0$ the field is turned on, $E\neq 0$, all the bound states disappear and the system evolution is governed by the equation

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + [V(x) - Ex]\psi(x,t),$$
$$E>0 \quad (t>0, -\infty < x < \infty), \qquad (31)$$

subjected to some initial condition, which in a natural way can be one of the bound states $\psi(x,0)=\phi_n(x)$. The whole x axis is open here for the particle whose tunneling clearly goes only to the right, i.e., toward $x\to+\infty$.

Using the techniques of resonant functions we notice that $u_n(x)$ must be exponentially decaying on the left, $x\to-\infty$, but give only outgoing waves on the right. They can be written up to the normalization parameter N in the form

$$u_n(x) = N\begin{cases} \mathrm{Ai}(y), & -\infty < x \leqslant -d, \\ a\,\mathrm{Ai}(z) + b\,\mathrm{Bi}(z), & -d < x \leqslant d, \\ c[\mathrm{Ai}(y) - i\,\mathrm{Bi}(y)], & d < x < \infty, \end{cases} \qquad (32)$$

where Ai(*), Bi(*) are the Airy functions, $y=-E^{-2/3}(Ex+k_n^2)$, $z=-E^{-2/3}(Ex+V+k_n^2)$, k_n^2 is the complex energy of the resonant state. The parameters a, b, c, k_n are to be determined from the requirement of continuity of u_n and its derivative at $x=\pm d$. We used in (32) the condition of keeping only the outgoing waves for $x>d$ and asymptotics of the Airy functions [15] for Re $y\to\infty$ and $|\arg y|<\pi$,

$$\mathrm{Ai}(y) \to \frac{\exp(-\zeta)}{2\sqrt{\pi}y^{1/4}}F(-\zeta), \quad \mathrm{Bi}(y) \to \frac{\exp(\zeta)}{\sqrt{\pi}y^{1/4}}F(\zeta),$$

$$\mathrm{Ai}(-y) - i\mathrm{Bi}(-y) \to \frac{\exp[i(\zeta-\pi/4)]}{\sqrt{\pi}y^{1/4}}, \qquad (33)$$

where $\zeta=2y^{3/2}/3$ and $F(\zeta)$ is an asymptotic series in ζ^{-1},

$$F(\zeta) = 1 + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \cdots, \quad \text{with } c_j = \frac{(2j+1)(2j+3)\cdots(6j-1)}{216^j j!}.$$

Using straightforward manipulations we obtain an equation for the resonance energies k_n^2 in terms of the Airy functions and their derivatives

$$\frac{\mathrm{Ai}'(y_+) - i\,\mathrm{Bi}'(y_+)}{\mathrm{Ai}(y_+) - i\,\mathrm{Bi}(y_+)} = \frac{\mathrm{Ai}(y_-)[\mathrm{Ai}'(z_+)\mathrm{Bi}'(z_-) - \mathrm{Ai}'(z_-)\mathrm{Bi}'(z_+)] - \mathrm{Ai}'(y_-)[\mathrm{Ai}'(z_+)\mathrm{Bi}(z_-) - \mathrm{Ai}(z_-)\mathrm{Bi}'(z_+)]}{\mathrm{Ai}(y_-)[\mathrm{Ai}(z_+)\mathrm{Bi}'(z_-) - \mathrm{Ai}'(z_-)\mathrm{Bi}(z_+)] - \mathrm{Ai}'(y_-)[\mathrm{Ai}(z_+)\mathrm{Bi}(z_-) - \mathrm{Ai}(z_-)\mathrm{Bi}(z_+)]}, \qquad (34)$$

where $y_\pm=-E^{-2/3}(\pm Ed+k_n^2)$, $z_\pm=-E^{-2/3}(\pm Ed+V+k_n^2)$ correspond to the right-hand (+) and left-hand (−) boundaries of the potential well. When the field is weak, $E\ll 1$, there is a direct correspondence $k_n^2\approx\varepsilon_n$ between the stationary states—solutions ϕ_n of Eq. (30)—and resonances. As a result, the usual approach to study the atomic ionization, say in [1,16], is often reduced to finding just a single resonant state, i.e., one term in (29), which is originated by the bound state of the initial condition. This cannot be adequate when the external field is strong.

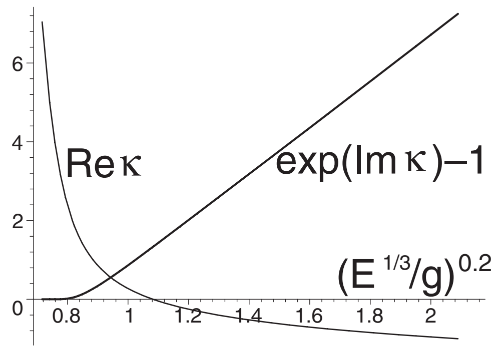

FIG. 2. Solutions of Eq. (35) for the first resonant state $\kappa_0 = -E^{2/3}k_0^2$.

A. Case of δ-function binding

We begin by considering a simpler form of the binding potential $V(x) = -g\delta(x)$ which comes as a limit when $d \to 0$, $2Vd \to g$. This problem was considered earlier by several authors in [17], where they present an expansion in terms of binding potential for treating strong field effects, in [18] for studying resonances in the double δ quantum well, in Sec. III A of [3] by using the Green function technique for identifying the principal resonance, and other works like [19] devoted to the Stark shift and tunneling. We want to pay more attention to the effects of electric fields in a very wide range, including very strong fields, on the whole set of resonance energies and identify their distribution in the complex energy plane. The stationary Eq. (30) yields a single normalized bound state with the energy $-g^2/4$ while the parameters of all resonances can be found from a reduced version of Eq. (34) as its limit

$$\mathrm{Ai}(\kappa)[\mathrm{Ai}(\kappa) - i\,\mathrm{Bi}(\kappa)] = -i\frac{E^{1/3}}{\pi g}, \tag{35}$$

where $\kappa = -E^{-2/3}k_n^2$. The decaying in time character of tunneling in (29) requires $\mathrm{Im}\,\kappa > 0$ in (35).

In the case of a weak external field $E \ll 1$, when $|\kappa|$ may be expected to be large, we substitute in (35) the asymptotics of the Airy functions, whose leading terms are given by Eqs. (33). The result comes to an equation

$$1 + \frac{5}{32\kappa^3} + \frac{i}{2}\exp(-4\kappa^{3/2}/3) = 2\sqrt{\kappa}\frac{E^{1/3}}{g},$$

whose solution yields the complex energy k_0^2 of the principal resonant state

$$k_0^2 \approx -\frac{g^2}{4} - 5\frac{E^2}{g^4} - i\frac{\Gamma_0}{2}, \quad \Gamma_0 = \frac{g^2}{2}\exp\left(\frac{-g^3}{6E}\right). \tag{36}$$

It is shifted down (compared with the bound state energy) by the Stark effect proportionally to E^2/g^4 and has a negative imaginary part as it should.

In Fig. 2 we plot trajectories of the real and imaginary parts of the numerical solutions of Eq. (36) as functions of the parameter $Q = E^{1/3}/g$ which describes the relative strength of the external field in a very wide range. The limiting points of these trajectories when $E \to 0$ correspond to the single unperturbed state, $\mathrm{Re}\,k_0^2 \to -g^2/4$, $\mathrm{Im}\,k_0^2 \to 0$. Using κ_n instead of k_n^2 ($n=0$ here) allows the curves in Fig. 2 to be applicable for different values of g, i.e., though the problem involves two parameters g and E the main part of computing k_n^2 needs in fact only their combination Q.

One can see in Fig. 2 that when $E^{1/3}/\pi g$ on the right-hand side of (35) exceeds 0.2 the graph of $\exp(\mathrm{Im}\,\kappa)$ is very close to a straight line $-4 + 5.9Q^{1/5}$. Thus, we come to an approximate empiric formula for the decay rate of the lowest resonant state in the case of a strong electric field

$$\Gamma_0 = 2g^2Q^2 \ln(5.9Q^{1/5} - 4) \quad \text{for } Q > 0.6, \tag{37}$$

which shows that the decay rate is roughly proportional to $E^{2/3}$. For weaker electric fields acceptable results for Γ_0 are given by (36). Equations (36) and (37) produce the same Γ when $Q=0.63$ and their results differ in $\pm 30\%$ for $0.57 < Q < 0.7$. On the other hand, the numerical solutions of Eq. (35) are close to the asymptotics (36) if $Q < 0.25$. Though Γ_0 from Eq. (36) differs less than in 50% from the exact one in the interval $0.25 < Q < 0.75$, it becomes irrelevant for larger Q.

Table II gives, along with k_0^2, locations of higher resonances in the complex plane of energy function k^2. When the field is not strong (say $Q < 0.5$) it is easy to locate the principal resonance which is originated by the ground level, its energy is denoted k_0^2. In stronger fields one can follow this resonance by continuity, as shown in Table II, where there are six more resonances. Three of them have positive ($n > 0$) and another three have negative ($n < 0$) real part of energy for $E \to 0$. If a trajectory of some of them moves from one half-plane to another the "initial" location with $E \to 0$ is chosen to identify n. The indexes n of the resonant states are conditional in the table except $n=0$. The principal resonance shows significantly slower decay, $\Gamma_0 < \Gamma_1/3$, ($\Gamma \sim E^{2/3}\,\mathrm{Im}\,\kappa$) for all $Q < 0.5$. If the field is stronger one should include more resonances, but in such cases a comparison of C_n is needed, though when $Q > 0.2\pi$ higher resonances with $\mathrm{Re}\,k_n^2 > 0$ (and maybe much smaller C_n) decay even slower and determine eventually the tunneling character. According to Table II in this case the decay might be almost exponential because Γ_n for different $n > 0$ are close to each other.

A very interesting picture presents the behavior of $\mathrm{Re}\,\kappa_0$ responsible for the Stark effect in Fig. 2 and especially the plot of $\mathrm{Re}\,k_0^2$ in Fig. 3.

When $Q \to 0$ the energy level approaches to $-g^2/4$ and for $Q < 0.8$ it follows the pattern of weak fields, see (36), i.e., down-shift increasing with E. The broken line in Fig. 3 shows the real part of the weak field approximation (36). When E is small the Stark shift $-5E^2/g^2$ given by Eq. (36) is very close to the exact results from (35): For $Q < 0.25$, the difference is within $\pm 16\%$ up to $Q=0.4$, but for larger Q the curves in Fig. 3 rapidly diverge. In stronger fields the down shift decreases and disappears when $Q=1.5$. Then the field growth makes the Stark shift positive and monotonically increasing with E. Thus, the principal resonance moves to the right half-plane of complex energies.

TABLE II. Parameters of higher resonances for δ-function binding.

$E^{1/3}/g$	0.06π	0.08π	0.1π	0.2π	0.3π	0.6π	1.0π	5.0π	10.0π
$\mathrm{Im}(k_0/g)^2$	-3.72×10^{12}	-6.10×10^{-6}	-8.69×10^{-4}	−0.1309	−0.5216	−3.446	−11.93	−448.1	−2011.0
$g^{-2}\,\mathrm{Im}\,k_{-1}^2$	−0.0740	−0.1373	−0.2281	−1.052	−2.470	−10.42	−29.89	−816.7	−3386.0
$g^{-2}\,\mathrm{Im}\,k_{-2}^2$	−0.1293	−0.2377	−0.3823	−1.618	−3.715	−15.26	−43.12	−1331.0	−4616.0
$g^{-2}\,\mathrm{Im}\,k_{-3}^2$	−0.1746	−0.3179	−0.5053	−2.090	−4.763	−19.39	−54.48	−1408.0	−5708.0
$g^{-2}\,\mathrm{Im}\,k_{1}^2$	−0.001767	−0.005087	−0.01115	−0.1020	−0.3237	−1.988	−6.980	−289.0	−1349.0
$g^{-2}\,\mathrm{Im}\,k_{2}^2$	−0.002066	−0.005680	−0.01198	−0.09857	−0.2995	−1.760	−6.062	−244.4	−1137.0
$g^{-2}\,\mathrm{Im}\,k_{3}^2$	−0.002227	−0.005956	−0.01229	−0.09556	−0.2844	−1.633	−5.565	−220.7	−1023.0
$\mathrm{Re}(k_0/g)^2$	−0.2502	−0.2513	−0.2557	−0.3035	−0.2864	0.4503	3.403	198.2	940.1
$g^{-2}\,\mathrm{Re}\,k_{-1}^2$	−0.0505	−0.0962	−0.1552	−0.5604	−1.155	−3.956	−9.722	−151.0	−445.7
$g^{-2}\,\mathrm{Re}\,k_{-2}^2$	−0.0817	−0.1486	−0.2311	−0.8652	−1.864	−6.923	−18.19	−377.4	−1381.0
$g^{-2}\,\mathrm{Re}\,k_{-3}^2$	−0.1069	−0.1911	−0.2961	−1.130	−2.471	−9.417	−25.25	−562.1	−2133.0
$g^{-2}\,\mathrm{Re}\,k_{1}^2$	0.08918	0.1612	0.2553	1.059	2.421	9.865	27.67	710.0	2875.0
$g^{-2}\,\mathrm{Re}\,k_{2}^2$	0.1510	0.2706	0.4254	1.729	3.914	15.77	43.96	1108.0	4450.0
$g^{-2}\,\mathrm{Re}\,k_{3}^2$	0.2016	0.3604	0.5652	2.283	5.804	20.71	57.64	1448.0	5799.0

The normalization coefficients N_n of $u_n(x)$ can be written in a closed form by using (20) and (35) and the integration properties of the Airy functions [20]

$$N_n^{-2} = -\frac{g}{E}\mathrm{Ai}(\kappa_n)[\mathrm{Ai}(\kappa_n) + E^{1/3}\,\mathrm{Ai}'(\kappa_n)]. \tag{38}$$

In weak fields the normalization parameter N_0 is very close to the normalization of the unperturbed by the field E eigenfunction. Using (32) we obtain $N_0 \approx \sqrt{g/2}/\mathrm{Ai}(\kappa)$, where $\mathrm{Ai}(\kappa)$ can be taken for $E \ll 1$ in its asymptotic form (22) with $\kappa \approx g^2E^{-2/3}/4$. With the help of N_0 and Eqs. (32) and (35) the principal resonant function for $x>0$ can be presented in the form

$$u_0(x) = i\pi g^{3/2}E^{-1/3}\,\mathrm{Ai}(\kappa)[\mathrm{Ai}(y) - i\,\mathrm{Bi}(y)]/\sqrt{2},$$

which should be relevant not only for very small E. Far away from the well we have

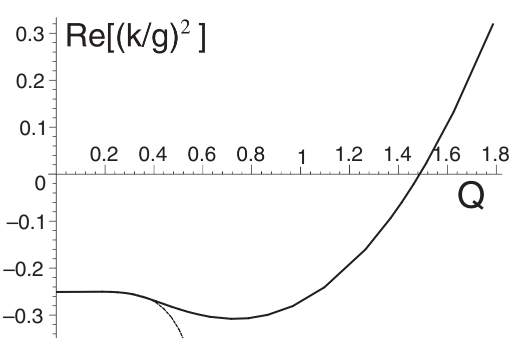

FIG. 3. The real part of the first resonance energy vs the electric field strength

$$u_0(x) \approx \frac{\sqrt{\Gamma_0}}{(4Ex)^{1/4}}\exp\left[i\left(\frac{\pi}{4} + \frac{2}{3}\sqrt{E}x^{3/2}\right) + \frac{\Gamma_0}{2}\sqrt{\frac{x}{E}}\right],$$

$$x \gg \frac{g^2}{4E}. \tag{39}$$

At the point $x_0 = g^2/4E$, where the tunneling particle enters the classically accessible region, the resonant function for $E \ll 1$ is already very small,

$$u_0(x_0) \approx 0.355e^{i\pi/6}gE^{-1/6}\sqrt{2\pi\Gamma_0}.$$

Note that asymptotic (39) is based on inequality $Q^{-2} = g^2E^{-2/3} \gg 1$ which leads also to (36) and produces a small prefactor $\sqrt{\Gamma_0}$ in (39), i.e., a low permeability of the barrier created by a weak electric field. The last term in the exponent of (39) is responsible for the spacial divergence of the resonant state $u_0(x)$, discussed in Sec. II.

We emphasize that all the spatially divergent outgoing resonant waves in a static electric field are not of type e^{ikx}, but in virtue of (33) they have the asymptotic form

$$u_n(x) \to \exp\left[i\sqrt{x}E^{1/6}\left(\frac{2}{3}xE^{1/3} - \kappa_n\right)\right], \tag{40}$$

where x is large enough to dominate the second term in the exponent and thus the direction of propagation is determined not by the sign of k but by the field E. Meanwhile the parameter Γ_n, which defines both the time decay and spatial divergence of the wave, is a part of κ_n and comes in (40) as a factor for $\sqrt{x}$ (but not for x as in the usual theories where the emitted particle is asymptotically free).

Let us outline the distribution of resonant energies, i.e., the roots $\kappa_n \propto k_n^2$ of Eq. (35) in the complex plane k^2. The assignment of indexes n is described above. When $E^{-2/3}\,\mathrm{Re}\,k^2 > 1$ and increasing we use the asymptotics (33) and rewrite (35) approximately as

$$i + 2Q\xi + \exp(4i\xi^3/3) = 0,$$

where temporarily $\xi = k_n/g$. Clearly the imaginary part of ξ cannot grow fast with n, therefore assuming $\xi = A - ia, A \gg a > 0$, both A and a real, we obtain $e^{4A^2 a} \approx \sqrt{4A^2Q^2 + (1-2aQ)^2}$. Therefore, $a \sim A^{-2}\ln(2AQ)/4$ and $4A^3 \sim 3\pi(n+1/2)$ for large positive n.

If Re κ is positive (Re $k^2 < 0$) and not small while arg $\kappa < \pi/3$, the left-hand side of(35) behaves as const $\kappa^{-1/2}$, but it grows rapidly with $|\kappa|$ when $\pi/3 < \arg\kappa < \pi/2$. This means that in the left half-plane, Eq. (35) is satisfied by k_n^2 which is close to the ray $\arg(k^2) = -2\pi/3$. Finally, this asymptotic analysis yields a symmetric form

$$k_n^2 \to (3\pi|n|E/2)^{2/3}\begin{cases}(1 - i\ln|n|/9\pi n), & n > 0,\\ e^{-2\pi i/3}(1 + i\ln|n|/9\pi|n|), & n < 0.\end{cases} \quad (41)$$

The resonances with small $|n|$ are not placed obviously by this simple pattern, in particular in weak fields, where the bound states produce the resonant energies which are only slightly shifted from their stationary locations and thus are close to the negative real axis. Nevertheless, surprisingly, the distribution of resonances in Table II has a clear signature of the asymptotics (41). In strong fields the decay exponent grows as $E^{2/3}$ not only for the higher resonances (41) but for the principal one as well (37).

We illustrate the properties of the wave function, expanded in terms of the resonant states (29), by considering the survival of our single bound state in the form of the projection

$$P(t) = \int_{-\infty}^{\infty} \psi^*(x,0)\psi(x,t)dx = \sum_{n=-\infty}^{\infty} \bar{C}_n C_n e^{-ik_n^2 t}, \quad (42)$$

with the same notations as in (16) and with $\psi(x,0) = \sqrt{g/2}\exp(-g|x|/2)$. The computation of the parameters C_n using (4) (on the whole axis) and (38) is straightforward, but for a general statement about the evolution of $P(t)$ one needs to have their explicit asymptotic form for studying the convergence of (42). The result reads as

$$C_n = \bar{C}_n \approx \begin{cases}\frac{1}{4}gE^{1/3}e^{-i\pi/4}(2/3\pi n)^{5/6}, & n \to +\infty,\\ -(3\pi|n|)^{-1/2}, & n \to -\infty,\end{cases} \quad (43)$$

where we used (41) and the asymptotics of Airy functions [15]. Substituting C_n in (42) allows to conclude that $P(t) = P_1(t) + P_2(t) + P_3(t)$, where P_1 consists of a finite number of exponentially decaying terms, P_2 is given by a series whose sum decays faster than any component of $P_1(t)$, and

$$P_3(t) \approx \sum_{n=M}^{\infty} \frac{e^{it(3\pi nE/2)^{2/3}}}{n^{(5/3)+tE^{2/3}(3\pi n/2)^{-1/3}}}. \quad (44)$$

Here $M \gg 1$ and clearly the series representing $P_3(t)$ is absolutely convergent. A crude estimate of (44) (neglecting oscillations in the numerator) shows that $P_3(t)$ decays at least as t^{-2},

$$|P_3(t)| < \sum_{n=M}^{\infty} n^{-5/3}e^{-tE^{2/3}(3\pi n/2)^{-1/3}\ln n} < \int_{M-1}^{\infty} x^{-5/3}e^{-tE^{2/3}(3\pi x/2)^{-1/3}\ln x}dx. \quad (45)$$

Denoting $q = M-1$ and $s = E^{2/3}(3\pi/2)^{-1/3}$ the positive integral in (45) can be bound in the following way:

$$\int_q^{\infty} x^{-5/3}(\ln x/3 - 1)e^{-tsx^{-1/3}\ln x}dx = -\frac{e^{-tsq^{-1/3}\ln q}}{stq^{1/3}} + \frac{1}{3st}\int_q^{\infty} x^{-4/3}e^{-tsx^{-1/3}\ln x}dx < \frac{1}{3st}\int_q^{\infty} x^{-4/3}(\ln x/3 - 1)e^{-tsx^{-1/3}\ln x}dx < \frac{1}{3(st)^2}.$$

$P_3(t)$ comes from the resonances with positive real parts of energy. The physical nature and a more precise mathematical form of its time behavior require additional work.

B. Rectangular well

When $E=0$, Eq. (34) can be reduced to the form

$$\left(\frac{\sqrt{V-q_n^2}}{q_n} - \frac{q_n}{\sqrt{V-q_n^2}}\right)\tan(2d\sqrt{V-q_n^2}) = 2. \quad (46)$$

Though both Eqs. (46) and (34) (even in the case of small E) cannot be solved analytically and thus the Stark shift is difficult to find in a closed form, the decay exponent for the resonant state with the energy $-k_n^2 \approx \varepsilon_n$ can be evaluated as

$$\Gamma_n^W = \frac{2q_n^2(V-q_n^2)}{V(1+q_n d)}\exp\left(-\frac{4(q_n^2 - Ed)^{3/2}}{3E}\right). \quad (47)$$

Equation (47) when $d \to 0$ is consistent with (36) for the δ-function binding and is a good approximation in the case of a rectangular well when $E \to 0$. The term Ed in the exponent of (47) clearly cannot be dropped. Therefore, the usual form $\exp(-2|\varepsilon|^{3/3}/3E)$ for the decay parameter, which often can be seen in the literature, say in [1], sometimes needs corrections.

We choose a rectangular well with $G = Vd^2 < \pi^2$ and therefore with a single stationary symmetric bound state to compare behavior of the resonances here with the case of the δ-function well which always has only one bound state. If $G=2$ and $E=0$ Eq. (46) yields $(q_0 d)^2 = (k_0 d)^2 \approx 1.2078$ for this unperturbed by E bound state. Like it was done in Sec. IV A the actual number of parameters for solving Eq. (34) can be reduced by combining them: Let $\kappa = -E^{-2/3}k_n^2$ be the quantity to be evaluated, but instead of Q the independent variable representing the field strength is parametrized as $S = E^{1/3}d$ and thus in (34) $y_\pm = \kappa \mp S, z_\pm = y_\pm - G/S^2$. For example, when $G=2$ and E is small Eq. (47) gives $\Gamma_0^W \approx 0.456d^{-2}\exp(-1.77/d^3E)$. The trajectory of Im κ_0 vs $dE^{1/3}$ in Fig. 4 is similar to the δ-function case and for $S > 1$ empirically $\exp(\mathrm{Im}\,\kappa) \approx 4.2\sqrt{S} - 3.3$. Therefore, for $Vd^2 = 2$ we have again "the $E^{2/3}$ law"

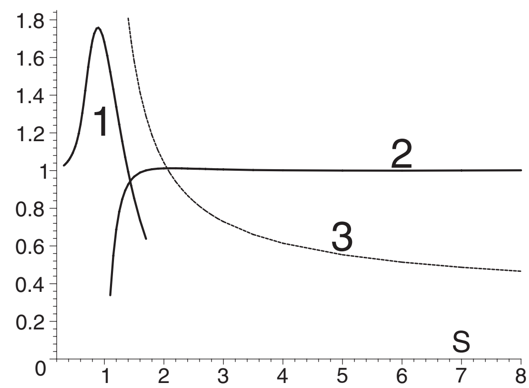

FIG. 4. The decay exponent approximation (47) and (48): Curves 1 and 2, respectively. Broken line 3 is the plot of $d^2E^{2/3}$. All curves are normalized by the exact Γ.

$$\Gamma_0^S(E) \approx 2E^{2/3}\ln(4.2\sqrt{E^{1/3}}d - 3.3), \tag{48}$$

with a very good precision (better than 5%) for $S>1.5$.

Equation (48) holds in our study up to $S=8$, i.e., when the field strength E varies more than by the factor 200 and probably more, but the exponential decay of a single state becomes irrelevant in stronger fields. The validity of these approximations is illustrated by Fig. 4.

One can see in Fig. 4 that Eqs. (47) and (48) describe the decay; each in its interval of S. On the interval $1.4<S<1.6$ they agree with each other and with the exact result within 20%. A relation similar to (48) seems possible for any G. More importantly the general approximation in strong fields $\Gamma \sim \text{const}\, E^{2/3}$, which is represented by curve 3 in Fig. 4 (without additional parameters, simply as $d^2E^{2/3}$), gives a satisfactory description of Γ for $S>1.5$. For weak fields it is important to note the term Ed in the exponent of (47) which comes from the field variation inside the potential well. A form of Γ^W similar to (36) and (47) is widely used and one should be careful especially when the potential well is not very narrow, say in the Coulomb field: In our computations Eq. (47) without this term would underestimate Γ by a factor ~ 10.

The trajectories of the real and imaginary parts of κ for $G=2$ are shown in Figs. 5 and 6.

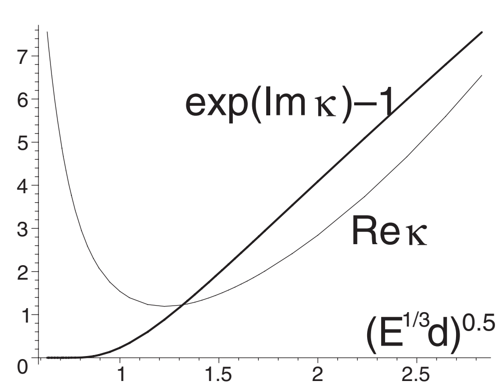

FIG. 5. Solutions of Eq. (34) for the main resonant state $\kappa_0 = -E^{2/3}k_0^2$ as functions of $S=E^{1/3}d$.

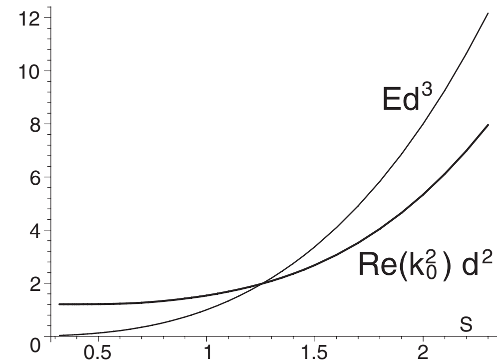

FIG. 6. The Stark effect for the principal resonant state and down shift Ed^3 of the right-hand edge of the rectangular well as functions of $S=E^{1/3}d$.

The real part of κ_0 in Fig. 5 behaves differently compared with the δ-function case in Fig. 2. The Stark effect is represented more directly by the dependence of $S^2\,\text{Re}\,\kappa_0 = -d^2\,\text{Re}\,(k_0^2)$ on S. This curve in Fig. 6 is monotonically increasing which means that unlikely the δ-function binding the "energy level" is always shifted down.

Each point of this curve corresponds to the distance from B to the x axis multiplied by d^2 in Fig. 1. The curve for Ed^3 in Fig. 6 represents the location of point A in Fig. 1, i.e., the edge of the potential well, scaled by the same factor. One can see that for $S>1.25$ the real part of the "energy level," point B, is above point A in Fig. 1. For this case with only one bound state of the unperturbed Hamiltonian the locations of several resonances (which are close to the original level) in increasing field are given in a Table III.

When field is not strong (say $S<1$) it is easy to locate the principal resonance k_0^2 which is originated by the ground state as in Table I. Six more resonances in Table II are identified in such a way as to make them closer to the principal one. The decay of resonant states, which is determined by the corresponding product $d^2\,\text{Im}\,k_n^2$, is much slower when $S\leqslant 1$ for the principal resonance than for higher ones, but in stronger fields the states with $n>0$, whose real part of energy is above the potential barrier, surprisingly decay slower than ones with $k_n^2<0$. The dependence $\Gamma \propto E^{2/3}$ found for $n=0$ analytically is more distinct for higher resonances and clearly seen in Tables II and III when $Q>0.5$, $S>1.5$ respectively.

The Stark effect $\text{Re}\,k_0^2 - 1.2078$ being small for $S<0.5$ becomes huge (more than 400) for $S=8$, although the bottom of the potential well goes down even lower, as in Fig. 6 where the field is not very strong yet. Note that the electric field in Table III varies more than by the factor 3×10^4.

In conclusion we may say that Eq. (29) together with the results of this section gives the exact theory of the wave-function evolution for all times in the field of an arbitrary strength via decay of multiple resonances. The resonant functions u_n for the rectangular well case are defined analytically (32) and (34) as well as parameters C_n (4) as soon as the initial state is chosen. When the fields are not strong one can have a good precision keeping only a few of them in (29) and the evaluation of the main parameters is simple. The distribution of higher resonances is found to be similar to the

TABLE III. Parameters of resonances for a rectangular well with $G=2$.

$E^{1/3}d$	0.35	0.5	0.6	0.8	1.0	1.5	3.0	5.0	8.0
d^2 Im k_0^2	-1.46×10^{-18}	-1.30×10^{-6}	−0.0005	−0.0383	−0.2079	−1.4227	−12.345	−45.175	−137.34
d^2 Im k_{-1}^2	−0.2666	−0.5690	−0.8556	−1.6579	−2.7275	−6.5518	−28.505	−83.710	−225.14
d^2 Im k_{-2}^2	−0.4557	−0.9634	−1.4227	−2.6196	−4.1795	−9.6930	−40.490	−115.89	−304.85
d^2 Im k_{-3}^2	−0.6097	−1.2796	−1.8723	−3.3979	−5.3773	−12.337	−50.804	−144.03	−440.31
d^2 Im k_1^2	−0.0776	−0.2760	−0.4071	−0.6749	−0.9650	−1.8902	−10.040	−28.776	−76.678
d^2 Im k_2^2	−0.0947	−0.1420	−0.1622	−0.2449	−0.4141	−2.0050	−9.0954	−27.154	−73.568
d^2 Im k_3^2	−0.0778	−0.0899	−0.1145	−0.2110	−0.4430	−1.6755	−8.7975	−25.922	−68.838
d^2 Re k_0^2	−1.2086	−1.2149	−1.2300	−1.3313	−1.5385	−2.6816	−18.073	−93.423	−419.13
d^2 Re k_{-1}^2	−0.1603	−0.3870	−0.6133	−1.2129	−2.0155	−5.2223	−31.515	−131.75	−516.24
d^2 Re k_{-2}^2	−0.2777	−0.6201	−0.9289	−1.7373	−2.8370	−7.1575	−39.701	−155.40	−578.54
d^2 Re k_{-3}^2	−0.3698	−0.8008	−1.1812	−2.1800	−3.5339	−8.7635	−46.400	−174.53	−672.26
d^2 Re k_1^2	0.2729	0.4600	0.5635	0.8068	1.106	2.1100	0.6451	27.809	30.858
d^2 Re k_2^2	0.4562	0.8123	1.1754	2.1258	3.363	7.1156	14.599	57.853	97.586
d^2 Re k_3^2	0.5927	1.2106	1.7551	3.1537	4.952	9.5930	26.624	85.646	221.37

δ-function case: being located in the lower half of the energy plane they approach asymptotically to the rays arg $(k^2)=0$ and arg $(k^2)=-2\pi/3$, Eq. (41). Table III shows that states with Re $k_n^2<0$, which may be expected to have large representation [larger C_n because for them $u_n(x)$ are concentrated near $x=0$], have larger Im k_n^2 and thus they die out much faster than the resonances with Re $k_n^2>0$. These states ($n>0$) behave like the running waves, their $u_n(x)$ are spread out and thus the overlap integrals with the initial state, i.e., prefactors C_n in (29), are small. As the result the principal resonance can keep its dominance in the decay behavior for quite a long time. In realistic situations the functions u_n are not given analytically but numerically and it is difficult to have enough information of their phases for integration in (20) which is needed for normalization. The regularization in (20) is a problem too. In such situations the normalization parameters can be approximately evaluated by using a quasiclassical approach of the next section.

V. ON NORMALIZATION OF THE RESONANT STATES

The normalization of the resonant wave functions has been always an important topic in the literature [1,2,5,7,11]. We believe that expansions (16) and (29) are adequate representations of the wave functions in terms of resonant states in physical space in spite of "nonphysical" spacial structure of $u_n(x)$. Their normalization (13) is implied [2,5] by the Green's function method for the time-independent Schrödinger equation with a potential function of a finite support on an interval $|x|\leqslant R$. It strikingly differs from the usual normalization involving the probability density $|\psi(x)|^2$ which is not integrable for the resonant functions $u_n(x)$, but such a normalization as well as the orthogonality (19) are necessary [7] because the problem's operator is not Hermitian. As it was pointed out by Zel'dovich [2] some form of regularization of divergent integrals is always present in these calculations, maybe in an implicit form. For example, the normalization (13) can be obtained by the following steps:

$$\int_0^\infty u_n^2(x)dx=\int_0^R u_n^2(x)dx+u_n^2(R)\int_R^\infty e^{2i(x-R)k_n}dx, \quad (49)$$

where k_n is the wave number of the state $u_n(x)$ and the outgoing wave is const e^{ik_nx} in free space. The second integral with a divergent but oscillating integrand, can be regularized if one multiplies the integrand by $e^{-\alpha x}$, such that $\alpha-2\,\mathrm{Im}\,k_n>0$, and finds the limit $\alpha\to0$ after integration which yields (13). Another way to get the same result is the direct integration assuming Im $k_n>0$ and the analytic continuation of the result $-1/2ik_n$ on the whole right one-half of the k plane, because Re $k_n>0$ for the outgoing waves. Note that introduced in [5] Eq. (16), which uses (4), holds in fact by an implicit assumption of (20).

Similar techniques can be used for neutralizing divergence in problems with an electric field on the line $-\infty<x<\infty$. The resonant functions decay as $\exp[-E^{1/2}(-x)^{3/2}]$ when $x\to-\infty$, see (33). The divergent part of the normalization integral for the function (32) is proportional to

$$I=\int_{x_0}^\infty \exp\left(\frac{4}{3}i\sqrt{E}(x+k_n^2/E)^{3/2}\right)\frac{dx}{\sqrt{x+k_n^2/E}}$$
$$=E^{-1/2}\int_{x_0E+k_n^2}^{\infty+k_n^2}\exp\left(\frac{4}{3E}iy^{3/2}\right)\frac{dy}{\sqrt{y}},$$

where Im $k_n^2<0$ and $x_0>0$ is finite. Assuming temporarily Im $k_n^2>0$ makes the integral over y convergent at its upper limit and in the spirit of regularization k_n^2 can be dropped there. The resulting integral can be differentiated in k_n^2 as many times as we wish and thus it is an analytic function in k_n^2, which has a perfect sense for Im $k_n^2<0$ too and can be treated as the analytic continuation of I. This integral can be also expressed explicitly in terms of gamma functions. Such a reasoning can be applied for binding potentials with a rapid decay at large x, but for the Coulomb well it needs a modification.

Let us try to treat the resonant states exactly as the "normal" wave functions. The decay of a bound state should describe a single particle tunneling from a previously stationary state. In the case of a potential with a compact support, the particle leaving the range of this potential becomes free and its wave function $u(x)\to\text{const}\, e^{ikx}$ is divergent when $x\to+\infty$ because $\text{Im}\, k<0$. Note that the components $e^{-ik_n^2 t}u_n(x)$ of full wave functions in (14), (16), and (29) are decaying in time. Somewhat naively one can say that for any finite t the particle cannot go infinitely far from the origin, i.e., though the function $u(x)$, defined for all $0\leqslant x<\infty$, is a useful tool, the "physical" wave function $\psi(x,t)=u(x)e^{-ik^2 t}$ must have a spacial cutoff of the form $x<x_{\max}(t)$. The coordinate $x_{\max}(t)$ cannot be introduced unambiguously because of the uncertainty principle, but in a semiquantitative way this is possible. In addition such an approach clarifies the physical nature of the resonant states. In two of the models considered above, the tunneling for both Eqs. (16) and (38) or (40) suggest the same: $x_{\max}(t)=\int_0^t v(t')dt'$ is the terminal point of the classical trajectory for the tunneling particle. In Sec. II $v=\hbar k/m\approx 2\pi$, see Eq. (15), while in the electric field $v(t)=eEt/m$, therefore due to $\text{Re}\, k\approx k$ in weak fields and using our units $\hbar=1$, $m=1/2$ for $x_{\max}(t)$ we have vt and $eEt^2/2m$, respectively.

The time evolution of a resonant state with a complex eigenenergy cannot be unitary and this is manifested by its spacial divergence. By using a cutoff we can treat the resonant wave function conditionally as a "normal" normalizable eigenfunction. Under such condition the unitary evolution would require

$$\int_0^{x_{\max}} |\psi(x,t)|^2 dx = 1. \tag{50}$$

In the case of the δ barrier, Eq. (50) yields a normalization different from (13),

$$\left(\int_0^R |u(x)|^2 dx + i|u(R)|^2 \frac{1-e^{(k-\bar{k})x_{\max}}}{k-\bar{k}}\right) e^{-i(k-\bar{k})(k+\bar{k})t} = 1. \tag{51}$$

Clearly Eq. (50) cannot hold exactly with any $x_{\max}(t)\neq 0$, but to see the picture it is worth studying the tunneling from the ground state in the problems (5)–(11) with large b. Dropping the terms of order of b^{-2} we reduce (51) to the form

$$\frac{N^2}{2}\left(\frac{3}{b}e^{-4\pi^3 t/b^2} + e^{2\pi^2(x_{\max}-2t\,\text{Re}\,k)/b^2}\right) = 1, \tag{52}$$

where $\text{Re}\, k=\pi+O(1/b)$ The first term in (52) is always much smaller than the second one and it decays in time. If we choose $x_{\max}(t)=2t\,\text{Re}\, k$ the second term becomes time independent and $N\approx\sqrt{2}$ will be the normalization constant. Thus, we have a single particle on the interval $0\leqslant x<x_{\max}$ with the moving boundary $x_{\max}(t)$. The increase of $|u(x)|^2$ with x has a clear physical sense: A particle with larger x was emitted at an earlier time when the flow from the well was denser because the probability of the particle staying trapped was larger.

Equation (52) suggests that when the barrier is not strong, i.e., b is not large for the δ barrier or small E in Sec. IV, this normalization and the whole picture with a sharp cutoff becomes relevant only after the trapping well is sufficiently depopulated [say, when the first term in (52) is much smaller than 1, i.e., when t satisfies the inequality $t \gg b^2\ln(3/b)/4\pi^3$, which is not very limiting for any value of b].

We should note that this treatment of resonant states has its limitations as the individual terms in the wave-function expansions (16) and (29) cannot be considered as "states" of any quantum system unlikely to a similar expansion for a ψ function in the standard scattering theory. In the latter case each term of the expansion has a definite momentum and a perfect sense while only the whole series (16) or (29) describes a quantum system while a term of the series is not an eigenfunction of a Hermitian operator. Obviously for any finite time there is no flow on $x=\infty$ though the separate terms of (16) and (29) do not decay. Using the truncated $u_n(x)$ in the expansions (16) and (29) gives a physical sense to their individual terms and clearly distorts the corresponding $\psi(x,t)$, but not too much under some conditions.

The cancellation of contributions from different $u_n(x)$ when $x\to\infty$, can be seen as an intuitive justification of using regularization, i.e., "taming" divergent terms in (19), (20), and (49): These disappear anyway with the help of timing factors and rapidly decreasing coefficients C_n to produce a physical wave function. Our technique of using the density $|u_n|^2$ with the cutoff to obtain (51) and (52) is a somewhat crude realization of this idea.

VI. CONCLUSION

(1) We constructed an exact theory in one dimension of the decay via tunneling in a uniform stationary electric field E in terms of the resonant states. The solution is analytical in the case of a rectangular well as the binding potential.

(2) In the case of tunneling initiated by an electrostatic field, the wave function of an initially bound particle is presented as a series composed from exponentially decaying in time terms of the resonant waves without a separate power-law asymptotic tail. In the complex energy plane the resonances are asymptotically approaching the positive real axis, i.e., their decay rates go to zero.

(3) For a model of a particle trapped by the δ-function barrier we showed that its tunneling can be effectively treated by a small number of the resonant states even when the barrier is weak.

(4) The distribution of the resonance energies, the overall decay character, and the strong field effects, such as the decay exponent and the Stark shift proportional to $E^{2/3}$, found in our models, should be relevant also for a much wider set of binding potentials. In particular $\Gamma\propto E^{2/3}$ comes from the properties of the Airy functions which always determine the wave-function asymptotic in the electric field.

(5) Two methods of normalization in the physical frame for the divergent resonant states are explored: (i) An exact

one by using a regularization procedure, which is similar to the usual technique for short-range potentials, and (ii) an approximate "traditional" normalization of the probability density. The second method puts the resonant states on the same footing as the "normal" wave functions. Both normalizations agree with each other in our models after some initial time.

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