Non-Markovian dynamics for bipartite systems

Bassano Vacchini*
Dipartimento di Fisica dell'Università di Milano and INFN, Sezione di Milano, Via Celoria 16, 20133 Milano, Italy
(Received 30 April 2008; published 25 August 2008)

We analyze the appearance of non-Markovian effects in the dynamics of a bipartite system coupled to a reservoir, which can be described within a class of non-Markovian equations given by a generalized Lindblad structure. A master equation is derived, which we term the quantum Bloch-Boltzmann equation, describing both motional and internal states of a test particle in a quantum framework. When due to the preparation of the system or to decoherence effects one of the two degrees of freedom is amenable to a classical treatment and not resolved in the final measurement, though relevant for the interaction with the reservoir, non-Markovian behaviors such as stretched exponential or power law decay of coherences can occur.

DOI: 10.1103/PhysRevA.78.022112 PACS number(s): 03.65.Yz, 05.20.Dd, 03.75.−b, 42.50.Lc

I. INTRODUCTION

The complete isolation of quantum mechanical systems, which should arise because of perfect shielding from the environment, can of course in general only be an idealization. The study of the dynamics of open quantum systems then naturally becomes of great interest [1], especially when it comes to a realistic description of experimental situations. While for the case of a closed quantum system the time evolution is given by a one-parameter unitary group characterized by a self-adjoint Hamiltonian, the situation is more involved for an open quantum system, where dynamical evolutions including irreversible effects like dissipation and decoherence must also be considered. The possible structures of dynamical equations for an open quantum system are not known in full generality, despite the huge efforts devoted to the problem. A well-known result of paramount importance has been obtained for Markovian dynamics, requiring the mapping giving the dynamics to be a completely positive quantum dynamical semigroup. The expression for the generators of such semigroups, which gives the master equation for the statistical operator of the system, has been fully characterized [2,3], providing a reference structure, often referred to as the Lindblad equation. Such Lindblad-type master equations ensure a well-defined time evolution, preserving in particular the positivity of the statistical operator. The different terms and operators appearing in the equations are often naturally amenable to a direct physical interpretation. Moreover, analytical approaches are often feasible, and when this is not the case numerical studies can always be performed, by considering Monte Carlo simulations of suitable stochastic differential equations associated with the master equation via a particular unraveling.

Such a general and physically transparent characterization is not available for master equations describing a non-Markovian dynamics. However, systems exhibiting non-Markovian dynamics, such as memory effects and decay behaviors other than exponential, are also of great interest both for practical applications and from a conceptual standpoint. In this spirit major efforts have been devoted to deriving possibly general classes of master equations, which, while providing well-defined time evolutions, also describe non-Markovian effects. Various difficulties appear in this connection. In particular, it is important to provide a link between the operators entering the structure of generalized master equations and quantities of physical relevance characterizing the environment and its coupling to the system. General classes of non-Markovian master equations have been obtained in the literature [4–6], also pointing to possible physical applications. In particular the analysis of the interaction of a quantum system with a structured reservoir, performed via a time-convolutionless projection operator technique relying on the use of correlated projection operators adapted to the structured reservoir [7], has led to point out a generalized Lindblad structure [8]. This generalized Lindblad structure describes a non-Markovian dynamics on states that are given by classical convex mixtures of subcollections, that is, positive trace class operators with trace equal to or less than 1, naturally appearing in the description of quantum experiments [9]. Master equations of this form have already been proposed in an utterly different context in order to introduce the notion of an event in the description of quantum mechanical systems [10], for the purpose of better understanding the interplay between classical and quantum descriptions of physical reality. More recently and to the point, similar equations have been considered for the statistical operator of an active atom interacting through collisions with a gas, when describing in a classical way the center of mass degrees of freedom [11].

General physical mechanisms leading to the appearance of such generalized Lindblad structures which can account for non-Markovian effects have already been conceived. This is the case if one studies the dynamics of an open system coupled to a structured reservoir using the above-mentioned time-convolutionless projection operator technique, provided the projectors used in obtaining the reduced equations of motion do project on classically correlated states between system and environment, rather than simply on a factorized state, as in the common wisdom [7,12,13]. Another natural situation leading to this generalized Lindblad structure appears in what has been called the generalized Born-Markov approximation [14]. Here one considers the usual second-order perturbation scheme, but once again the state of system and bath is supposed not to be factorized, but rather given by

*bassano.vacchini@mi.infn.it

a convex mixture of factorized states. The indices of the mixture are related to the structure of the bath and of the interaction Hamiltonian. Further work has traced back the derivation of non-Markovian equations of this form to the existence of extra unobserved degrees of freedom mediating the entanglement between the considered system and a Markovian reservoir [15,16]. Earlier work [17] also led to this kind of non-Markovian master equations for a system interacting with an environment with a finite heat capacity, so that energy exchanges between system and reservoir also affect the energy distribution of the reservoir. It has been recently shown that the same quantum master equation can also be derived in a physically more transparent manner by means of the projection superoperator technique [18].

In the present paper we show how such generalized master equations naturally arise by considering a bipartite quantum system interacting with a Markovian reservoir, whenever decoherence effects or superselection rules affect only one kind of degree of freedom of the bipartite system. A non-Markovian behavior then appears when the thus emerged classical label is not resolved in the final measurement. The analysis is done by means of a concrete and relevant physical example. We consider the dynamics of a quantum test particle, whose internal and center of mass degrees of freedom are both described quantum mechanically, interacting, e.g., with a gaseous background. The appearance of non-Markovian features is related to the involvement of both internal and center of mass degrees of freedom in the scattering amplitude which describes the coupling between bipartite system and environment. As a first step a quantum master equation is heuristically derived, which extends previous work on the quantum linear Boltzmann equation [19–22], focusing on a quantum description of the center of mass degrees of freedom, and on the Bloch-Boltzmann equation [11,23–26], which describes in a classical way the motion of the test particle, but retains a quantum expression for the dynamics of its internal degrees of freedom. According to this terminology the equation obtained is termed the quantum Bloch-Boltzmann equation. Two limiting situations then naturally appear. When decoherence affects more strongly or equivalently on a shorter time scale the center of mass degrees of freedom, a generalized Lindblad structure corresponding to the Bloch-Boltzmann equation appears, in which the momentum of the test particle is treated classically. This equation describes non-Markovian effects when the final measurement affects only the internal degrees of freedom. In a similar way, when the experimental effort is devoted to studying quantum superpositions of motional degrees of freedom, e.g., in interferometers for massive particles studying robustness of their quantum behavior, non-Markovian features can appear if the internal degrees of freedom influence the collisional scattering cross section but are later not observed in the assessment of the interference pattern.

The paper is organized as follows. In Sec. II we describe the mathematical framework and expression of the generalized Lindblad structure; in Sec. III we outline the derivation of the quantum Bloch-Boltzmann equation, involving both center of mass and internal degrees of freedom, starting from the classical linear Boltzmann equation and the known expression of the quantum linear Boltzmann equation involving only the motional degrees of freedom. Section IV is then devoted to considering the reduced dynamics of either internal or center of mass degrees of freedom, for which a generalized Lindblad structure follows, showing by means of example the appearance of non-Markovian behaviors. Finally, in Sec. V we briefly comment on our results.

II. GENERALIZED LINDBLAD STRUCTURE

We now want to introduce the above-mentioned non-Markovian generalization of the Lindblad structure, which is easily obtained considering the standard theorem of Gorini, Kossakowski, Sudarshan, and Lindblad for a special choice of Hilbert space for the open system and of the expression of its statistical operator. Let us consider a bipartite quantum system described on a Hilbert space $\mathcal{H}\otimes\mathcal{H}_B$, with $\mathcal{H}$ and $\mathcal{H}_B$ separable. Exploiting the isomorphism of $\mathcal{H}_B$ with either of the possible physically relevant choices of Hilbert space, such as $\mathbb{C}^n$, $l^2(\mathbb{C})$, or $L^2(\mathbb{R})$, the tensor product can be expressed as a direct sum or direct integral, thus naturally introducing a label α. For the case in which $\mathcal{H}_B$ describes a system with n degrees of freedom, one can consider the two equivalent constructions of the same bipartite Hilbert space

$$\mathcal{H}\otimes\mathbb{C}^n=\bigoplus_{\alpha=1}^{n}\mathcal{H}, \tag{1}$$

and similarly for $l^2(\mathbb{C})$, replacing the finite sum with a series. On similar grounds a continuous index appears for the case

$$\mathcal{H}\otimes L^2(\mathbb{R})=\int_{\mathbb{R}}^{\oplus} d\alpha\,\mathcal{H}, \tag{2}$$

exploiting the notion of a direct integral of Hilbert spaces (see, e.g., [27]). It is then of course possible to consider a statistical operator for the bipartite system, that is to say, a positive trace class operator on $\mathcal{H}\otimes\mathcal{H}_B$, normalized to 1. Considering for the sake of example the case of $\mathcal{H}\otimes\mathbb{C}^n=\oplus_{\alpha=1}^{n}\mathcal{H}$, and denoting by ϱ the statistical operator of the system, one can consider the general expression for the Lindblad equation for ϱ. Let us now restrict attention, however, to statistical operators whose matrix representation is block diagonal, so that they can equivalently be written as $\varrho=\sum_{\alpha=1}^{n}\rho_\alpha\otimes|\alpha\rangle\langle\alpha|$ or as $\varrho=(\rho_1,\ldots,\rho_\alpha,\ldots,\rho_n)$. The index α can now really be interpreted as a classical label indexing the various subcollections $\rho_\alpha\in\mathcal{TC}(\mathcal{H})$, which are given by positive trace class operators on $\mathcal{H}$ with trace less than or equal to 1. Such a block diagonal statistical operator

$$\varrho=(\rho_1,\ldots,\rho_\alpha,\ldots,\rho_n)$$

fixed by the set of subcollections $\{\rho_\alpha\}_{\alpha=1,\ldots,n}$ is normalized according to

$$\mathrm{Tr}\rho=\sum_{\alpha=1}^{n}\mathrm{Tr}_{\mathcal{H}}\rho_\alpha=1. \tag{3}$$

The set of trace class operators that are block diagonal is a closed subalgebra of the set of all trace class operators, whose dual space is given by the closed subalgebra of bounded operators also having a block diagonal structure.

Equivalently, one can say that this subclass of statistical operators provides information only on the expectation values of observables diagonal with respect to the label α, which thus correspond to the only relevant variables for the system under consideration. Considering a statistical operator in the subalgebra of block diagonal trace class operators $\rho \in \mathcal{TC}_{\text{diag}}(\mathcal{H}\otimes\mathbb{C}^n)$ and an observable given by a block diagonal bounded operator $B \in \mathcal{B}_{\text{diag}}(\mathcal{H}\otimes\mathbb{C}^n)$

$$B = (\mathsf{B}_1, \ldots, \mathsf{B}_\alpha, \ldots, \mathsf{B}_n),$$

with $\mathsf{B}_\alpha \in \mathcal{B}(\mathcal{H})$, the duality relation is given by

$$\langle B, \varrho \rangle = \sum_{\alpha=1}^{n} \mathrm{Tr}_{\mathcal{H}}(\mathsf{B}_\alpha \rho_\alpha). \tag{4}$$

It is now of interest to consider the expression of the generator of a quantum dynamical semigroup acting on this bipartite space when applied to such block diagonal states or observables, with the further constraint that the time evolved state or observable still preserves this simple block diagonal structure, thus defining a dynamics which remains within the spaces $\mathcal{TC}_{\text{diag}}(\mathcal{H}\otimes\mathbb{C}^n)$ or $\mathcal{B}_{\text{diag}}(\mathcal{H}\otimes\mathbb{C}^n)$, respectively. In the Schrödinger picture this generalized Lindblad structure can be written in terms of coupled equations for the different subcollections ρ_α according to [8,10]

$$\frac{d}{dt}\rho_\alpha = -\frac{i}{\hbar}[\mathsf{H}^\alpha, \rho_\alpha] + \sum_\lambda \sum_{\beta=1}^{n} \left(\mathsf{R}_\lambda^{\alpha\beta} \rho_\beta \mathsf{R}_\lambda^{\alpha\beta\dagger} - \frac{1}{2}\{\mathsf{R}_\lambda^{\beta\alpha\dagger}\mathsf{R}_\lambda^{\beta\alpha}, \rho_\alpha\} \right), \tag{5}$$

leading due to the duality relation Eq. (4) to the following equations in the Heisenberg picture for the components B_α of a block diagonal observable:

$$\frac{d}{dt}\mathsf{B}_\alpha = +\frac{i}{\hbar}[\mathsf{H}^\alpha, \mathsf{B}_\alpha] + \sum_\lambda \sum_{\beta=1}^{n} \left(\mathsf{R}_\lambda^{\beta\alpha\dagger} \mathsf{B}_\beta \mathsf{R}_\lambda^{\beta\alpha} - \frac{1}{2}\{\mathsf{R}_\lambda^{\beta\alpha\dagger}\mathsf{R}_\lambda^{\beta\alpha}, \mathsf{B}_\alpha\} \right). \tag{6}$$

In Eqs. (5) and (6) the index α runs from 1 to n, the operators H^α are self-adjoint on $\mathcal{H}$, and $\mathsf{R}_\lambda^{\beta\alpha}$ are operators on $\mathcal{H}$, with λ a further index labeling the various Lindblad operators.

Introducing the mapping

$$\mathcal{L}\varrho = \left(\frac{d}{dt}\rho_1, \ldots, \frac{d}{dt}\rho_\alpha, \ldots, \frac{d}{dt}\rho_n \right),$$

one can therefore write for the time evolution in the Schrödinger picture

$$\varrho(t) = (\rho_1(t), \ldots, \rho_\alpha(t), \ldots, \rho_n(t)) = e^{t\mathcal{L}}\varrho(0) = e^{t\mathcal{L}}(\rho_1(0), \ldots, \rho_\alpha(0), \ldots, \rho_n(0)),$$

and similarly for the Heisenberg picture using the mapping $\mathcal{L}'$ dual to $\mathcal{L}$ according to the relation Eq. (4). Now Eq. (5) provides a Markovian set of equations for the statistical operator $\varrho(t) = (\rho_1(t), \ldots, \rho_\alpha(t), \ldots, \rho_n(t))$ on $\mathcal{H}\otimes\mathbb{C}^n$, but a non-Markovian dynamics for the statistical operator

$$w(t) = \sum_{\alpha=1}^{n} \rho_\alpha(t), \tag{7}$$

which is a statistical operator on the Hilbert space $\mathcal{H}$ only. In particular it is not possible to define a mapping from $w(0)$ to $w(t)$ according to the noncommutativity of the following diagram:

$$\begin{array}{ccc} \varrho(0) = (\rho_1(0), \ldots, \rho_n(0)) & \xrightarrow{\exp(t\mathcal{L})} & \varrho(t) = (\rho_1(t), \ldots, \rho_n(t)) \\ \downarrow & & \downarrow \\ w(0) = \sum_{\alpha=1}^{n} \rho_\alpha(0) & \not\longrightarrow & w(t) = \sum_{\alpha=1}^{n} \rho_\alpha(t) \end{array}$$

arising because of the loss of information in going from $\varrho(t)$ to $w(t)$. The set of equations given by Eq. (5) thus provides a non-Markovian dynamics for the statistical operator $w(t)$ supposed to be expressible at any time as a mixture of subcollections $\rho_\alpha(t)$, or equivalently as a convex combination with weights $p_\alpha(t) = \mathrm{Tr}_{\mathcal{H}}\rho_\alpha(t)$ of statistical operators given by $w_\alpha(t) = \rho_\alpha(t)/\mathrm{Tr}_{\mathcal{H}}\rho_\alpha(t)$. This last standpoint stresses the appearance of the classical probability distribution $\{p_\alpha(t)\}_{\alpha=1,\ldots,n}$, which justifies the name random Lindblad equations or Lindblad rate equations [14,16], also given to equations falling within the class given by Eq. (5).

The statistical operator $w(t)$ can arise in a twofold way: Either by taking the trace of a block diagonal $\varrho(t)$ with respect to $\mathbb{C}^n$, corresponding to a situation in which one considers a reduced dynamics of the bipartite system with respect to the degrees of freedom which behave effectively in a classical way, or by assuming that the state of the system under study living in the Hilbert space $\mathcal{H}$ is specified at the initial time as a convex combination of n statistical operators with suitable weights, and retains this form throughout the dynamics. The first type of realization makes it intuitively clear why Eq. (5) encompasses non-Markovian situations. By looking at the time evolution of $w(t)$ only, one is considering a restricted set of variables with respect to the full collection $\{\rho_\alpha(t)\}_{\alpha=1,\ldots,n}$, for which the time evolution law would be Markovian. The set of relevant physical variables then determines whether or not the dynamics is Markovian. Statistical operators of the form given by Eq. (7) naturally appear in connection with a structured reservoir, the label α being then connected to a characterization of the reservoir itself, e.g., labeling different energy bands. This is the case both when considering a projection operator technique assuming classical correlated states between system and reservoir [7], and more simply in the so-called generalized Born-Markov approximation [14], using a classically correlated state in the derivation of the master equation to second order in the perturbation. As we shall argue below, statistical operators of this form also appear when considering a bipartite system interacting with a reservoir, when due to decoherence one of the two kinds of degrees of freedom behaves classically, and despite characterizing the initial preparation and being relevant for the interaction with the environment, cannot later be resolved by the measurement apparatus. Needless to say, the formal scheme developed above can also be implemented when the label α runs over a countable set or even in the continuum, corresponding to a Hilbert space construction as depicted in Eq. (2).

III. DERIVATION OF QUANTUM BLOCH-BOLTZMANN EQUATION

We now address the issue of the derivation of a master equation describing the dynamics of both internal and center of mass degrees of freedom of a quantum test particle interacting through collisions with a reservoir such as a background gas. Here we will outline a heuristic derivation, leaving a more detailed and microscopic one for a later presentation, in particular we will concentrate only on the incoherent terms corresponding to gain and loss terms in the classical case, leaving aside the Hamiltonian contributions due to the kinetic term and forward scattering. To do this we will build on the known results for the quantum linear Boltzmann equation [19–22], describing in a quantum framework the center of mass degrees of freedom, and on the so called Bloch-Boltzmann equation [11,23–26], which accounts for a semiclassical description of both internal and center of mass degrees of freedom. As a starting point we take the equations of motion considered in [11] for the collection $\{\rho(\boldsymbol{P})\}_{\boldsymbol{P}\in\mathbb{R}^3}$ of trace class operators on the space $\mathbb{C}^n$ of internal degrees of freedom, labeled with the continuous index $\boldsymbol{P}$ now characterizing the classical momentum of the test particle. This result extends the work in [23,26] by considering also the dynamics of the center of mass degrees of freedom of the test particle, which in [23,26] is supposed to have infinite mass and therefore to be at rest. It is most convenient to write the equation putting into evidence the constraints due to momentum and energy conservation in the single interaction events, thus obtaining

$$\begin{aligned}\frac{d}{dt}\rho(\boldsymbol{P}) = {}& \frac{n_{\text{gas}}}{m_*^2}\sum_{\substack{ijkl\\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}}\int d\boldsymbol{P}'\int d\boldsymbol{p}'\int d\boldsymbol{p}\;\mu_\beta(\boldsymbol{p}')\,\delta\!\left(\frac{P'^2}{2M}+\frac{p'^2}{2m}+\hbar\omega_j-\frac{P^2}{2M}-\frac{p^2}{2m}-\hbar\omega_i\right)\\ &\times\delta^3(\boldsymbol{P}'+\boldsymbol{P}'-\boldsymbol{p}-\boldsymbol{p})f_{ij}(\mathrm{rel}(\boldsymbol{p},\boldsymbol{P}),\mathrm{rel}(\boldsymbol{p}',\boldsymbol{P}'))f_{kl}^*(\mathrm{rel}(\boldsymbol{p}',\boldsymbol{P}'),\mathrm{rel}(\boldsymbol{p},\boldsymbol{P}))E_{ij}\rho(\boldsymbol{P})E_{kl}^\dagger\\ &-\frac{1}{2}\frac{n_{\text{gas}}}{m_*^2}\sum_{\substack{ijkl\\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}}\int d\boldsymbol{P}'\int d\boldsymbol{p}'\int d\boldsymbol{p}\;\mu_\beta(\boldsymbol{p})\,\delta\!\left(\frac{P^2}{2M}+\frac{p^2}{2m}+\hbar\omega_j-\frac{P'^2}{2M}-\frac{p'^2}{2m}-\hbar\omega_i\right)\\ &\times\delta^3(\boldsymbol{P}+\boldsymbol{p}-\boldsymbol{P}-\boldsymbol{P}')f_{ij}(\mathrm{rel}(\boldsymbol{p}',\boldsymbol{P}'),\mathrm{rel}(\boldsymbol{p}',\boldsymbol{P}'))f_{kl}^*(\mathrm{rel}(\boldsymbol{p}',\boldsymbol{P}'),\mathrm{rel}(\boldsymbol{p},\boldsymbol{P}))\{E_{kl}^\dagger E_{ij},\rho(\boldsymbol{P})\}.\end{aligned} \tag{8}$$

In this equation M and m denote the masses of test particle and gas particles, respectively, m_* is the reduced mass, $\mu_\beta(\boldsymbol{p})$ is the Maxwell-Boltzmann distribution of the gas,

$$\mu_\beta(\boldsymbol{p}) = \frac{1}{\pi^{3/2}p_\beta^3}\exp\left(-\frac{\boldsymbol{p}^2}{p_\beta^2}\right), \tag{9}$$

with $p_\beta=\sqrt{2m/\beta}$ the most probable momentum at temperature $T=1/(k_B\beta)$, relative momenta denoted as

$$\mathrm{rel}(\boldsymbol{p},\boldsymbol{P}) \equiv \frac{m_*}{m}\boldsymbol{p}-\frac{m_*}{M}\boldsymbol{P}, \tag{10}$$

and

$$f_{ij}(\boldsymbol{p},\boldsymbol{p}') \equiv f(\boldsymbol{p}',j\to\boldsymbol{p},i) \tag{11}$$

indicating the complex scattering amplitude for a transition from an *in* state with labels $\boldsymbol{p}',j$ to an *out* state with labels $\boldsymbol{p},i$. Finally, the matrices E_{ij} correspond to the mappings between energy eigenstates, providing a basis of operators in $\mathbb{C}^n$, according to

$$E_{ij} = |i\rangle\langle j|, \tag{12}$$

while $\hbar\omega_j$ is the energy of the j level and $\mathcal{E}_{ij}=\hbar\omega_i-\hbar\omega_j$ denote the possible transition energies. Using a δ-function momentum conservation Eq. (8) can also be expressed using as a variable the momentum transfer $\boldsymbol{Q}=\boldsymbol{P}'-\boldsymbol{P}$ in single collisions, thus coming to

$$\begin{aligned}\frac{d}{dt}\rho(\boldsymbol{P}) = {}& \frac{n_{\text{gas}}}{m_*^2}\sum_{\substack{ijkl\\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}}\int d\boldsymbol{Q}\int d\boldsymbol{p}\;\mu_\beta(\boldsymbol{p}-\boldsymbol{Q})\,\delta\!\left(\frac{(\boldsymbol{P}+\boldsymbol{Q})^2}{2M}+\frac{(\boldsymbol{p}-\boldsymbol{Q})^2}{2m}-\frac{P^2}{2M}-\frac{p^2}{2m}+\mathcal{E}_{ij}\right)f_{ij}(\mathrm{rel}(\boldsymbol{p},\boldsymbol{P}),\mathrm{rel}(\boldsymbol{p},\boldsymbol{P})-\boldsymbol{Q})\\ &\times f_{kl}^*(\mathrm{rel}(\boldsymbol{p},\boldsymbol{P}),\mathrm{rel}(\boldsymbol{p},\boldsymbol{P})-\boldsymbol{Q})E_{ij}\rho(\boldsymbol{P}+\boldsymbol{Q})E_{kl}^\dagger-\frac{1}{2}\frac{n_{\text{gas}}}{m_*^2}\sum_{\substack{ijkl\\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}}\int d\boldsymbol{Q}\int d\boldsymbol{p}\;\mu_\beta(\boldsymbol{p})\,\delta\!\left(\frac{(\boldsymbol{P}+\boldsymbol{Q})^2}{2M}+\frac{(\boldsymbol{p}-\boldsymbol{Q})^2}{2m}-\frac{P^2}{2M}-\frac{p^2}{2m}+\mathcal{E}_{ij}\right)\end{aligned}$$

$$\times f_{ij}(\mathrm{rel}(\boldsymbol{p},\boldsymbol{P})-\boldsymbol{Q},\mathrm{rel}(\boldsymbol{p},\boldsymbol{P}))f^*_{kl}(\mathrm{rel}(\boldsymbol{p},\boldsymbol{P})-\boldsymbol{Q},\mathrm{rel}(\boldsymbol{p},\boldsymbol{P}))\{E^\dagger_{kl}E_{ij},\rho(\boldsymbol{P})\}.$$

Introducing now by the notation $\parallel\boldsymbol{Q}$ and $\perp\boldsymbol{Q}$ the component of a vector parallel and perpendicular to the momentum transfer $\boldsymbol{Q}$, so that $\boldsymbol{P}_{\parallel\boldsymbol{Q}}=(\boldsymbol{P}\cdot\boldsymbol{Q})\boldsymbol{Q}/Q^2$ and $\boldsymbol{P}_{\perp\boldsymbol{Q}}=\boldsymbol{P}-\boldsymbol{P}_{\parallel\boldsymbol{Q}}$, respectively, one has using Eq. (10) in the δ function of energy conservation

$$\delta\left(\frac{(\boldsymbol{P}+\boldsymbol{Q})^2}{2M}+\frac{(\boldsymbol{p}-\boldsymbol{Q})^2}{2m}-\frac{P^2}{2M}-\frac{p^2}{2m}+\mathcal{E}_{ij}\right)=\delta\left(\frac{Q^2}{2m_*}-\frac{\boldsymbol{Q}}{m_*}\cdot\mathrm{rel}(\boldsymbol{p}_{\parallel\boldsymbol{Q}},\boldsymbol{P}_{\parallel\boldsymbol{Q}})+\mathcal{E}_{ij}\right),\tag{13}$$

so that in the integral one can use the replacement

$$\mathrm{rel}(\boldsymbol{p}_{\parallel\boldsymbol{Q}},\boldsymbol{P}_{\parallel\boldsymbol{Q}})=\frac{1}{2}\left(1+\frac{\mathcal{E}_{ji}}{Q^2/(2m_*)}\right)\boldsymbol{Q},\tag{14}$$

and therefore

$$\begin{aligned}\frac{d}{dt}\rho(\boldsymbol{P})=\;&\frac{n_{\mathrm{gas}}}{m_*^2}\sum_{\substack{ijkl\\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}}\int d\boldsymbol{Q}\int d\boldsymbol{p}\;\mu_\beta(\boldsymbol{p}-\boldsymbol{Q})\,\delta\left(\frac{Q^2}{2m_*}-\frac{\boldsymbol{Q}}{m_*}\cdot\mathrm{rel}(\boldsymbol{p}_{\parallel\boldsymbol{Q}},\boldsymbol{P}_{\parallel\boldsymbol{Q}})+\mathcal{E}_{ij}\right)\\
&\times f_{ij}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ji}}{Q^2/m_*}\boldsymbol{Q},\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ji}}{Q^2/m_*}\boldsymbol{Q}\right)\\
&\times f^*_{kl}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{lk}}{Q^2/m_*}\boldsymbol{Q},\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{lk}}{Q^2/m_*}\boldsymbol{Q}\right)E_{ij}\rho(\boldsymbol{P}+\boldsymbol{Q})E^\dagger_{kl}\\
&-\frac{1}{2}\frac{n_{\mathrm{gas}}}{m_*^2}\sum_{\substack{ijkl\\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}}\int d\boldsymbol{Q}\int d\boldsymbol{p}\;\mu_\beta(\boldsymbol{p})\,\delta\left(\frac{Q^2}{2m_*}-\frac{\boldsymbol{Q}}{m_*}\cdot\mathrm{rel}(\boldsymbol{p}_{\parallel\boldsymbol{Q}},\boldsymbol{P}_{\parallel\boldsymbol{Q}})+\mathcal{E}_{ij}\right)\\
&\times f_{ij}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q},\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}\right)\\
&\times f^*_{kl}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q},\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}\right)\{E^\dagger_{kl}E_{ij},\rho(\boldsymbol{P})\}.\end{aligned}$$

One can now perform the translation $\boldsymbol{p}\to\boldsymbol{p}+m\boldsymbol{Q}/(2m_*)+m\boldsymbol{P}_{\parallel\boldsymbol{Q}}/M+m\mathcal{E}_{ji}\boldsymbol{Q}/Q^2$ in the gain term, and similarly for the loss one, which does not affect the argument of the scattering amplitudes, thus obtaining

$$\begin{aligned}\frac{d}{dt}\rho(\boldsymbol{P})=\;&\frac{n_{\mathrm{gas}}m}{m_*^2}\sum_{\substack{ijkl\\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}}\int d\boldsymbol{Q}\int d\boldsymbol{p}\;\mu_\beta\left(\boldsymbol{p}+\frac{m}{m_*}\frac{\boldsymbol{Q}}{2}+\frac{m}{M}(\boldsymbol{P}_{\parallel\boldsymbol{Q}}-\boldsymbol{Q})+\frac{\mathcal{E}_{ij}}{Q^2/m}\boldsymbol{Q}\right)\delta(\boldsymbol{Q}\cdot\boldsymbol{p})\\
&\times f_{ij}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q},\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}\right)\\
&\times f^*_{kl}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q},\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}\right)E_{ij}\rho(\boldsymbol{P}-\boldsymbol{Q})E^\dagger_{kl}\\
&-\frac{1}{2}\frac{n_{\mathrm{gas}}m}{m_*^2}\sum_{\substack{ijkl\\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}}\int d\boldsymbol{Q}\int d\boldsymbol{p}\;\mu_\beta\left(\boldsymbol{p}+\frac{m}{m_*}\frac{\boldsymbol{Q}}{2}+\frac{m}{M}\boldsymbol{P}_{\parallel\boldsymbol{Q}}+\frac{\mathcal{E}_{ij}}{Q^2/m}\boldsymbol{Q}\right)\delta(\boldsymbol{Q}\cdot\boldsymbol{p})\\
&\times f_{ij}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q},\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}\right)\\
&\times f^*_{kl}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q},\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}\right)\{E^\dagger_{kl}E_{ij},\rho(\boldsymbol{P})\},\end{aligned}$$

where we also performed the change of variables $\boldsymbol{Q}\to-\boldsymbol{Q}$ in the gain term and used the simple relation $m/(2m_*)-m/M=1-m/(2m_*)$. Noting that

$$\int d\boldsymbol{p}\, g(\boldsymbol{p})\delta(\boldsymbol{Q}\cdot\boldsymbol{p}) = \frac{1}{Q}\int_{Q^{\perp}} d\boldsymbol{p}\, g(\boldsymbol{p}_{\perp\boldsymbol{Q}}), \tag{15}$$

where the integration on the right-hand side is restricted to momenta of the gas particle perpendicular to the momentum transfer, we obtain the equation

$$\begin{aligned}
\frac{d}{dt}\rho(\boldsymbol{P}) = \frac{n_{\text{gas}}m}{m_*^2}\sum_{\substack{ijkl\\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}}&\int\frac{d\boldsymbol{Q}}{Q}\int_{Q^{\perp}}d\boldsymbol{p}\,\mu_\beta\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}+\frac{m}{m_*}\frac{\boldsymbol{Q}}{2}+\frac{m}{M}(\boldsymbol{P}_{\parallel\boldsymbol{Q}}-\boldsymbol{Q})+\frac{\mathcal{E}_{ij}}{Q^2/m}\boldsymbol{Q}\right)\\
&\times f_{ij}\left(\text{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q},\text{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}\right)\\
&\times f_{kl}^*\left(\text{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q},\text{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}\right)E_{ij}\rho(\boldsymbol{P}-\boldsymbol{Q})E_{kl}^\dagger\\
&-\frac{1}{2}\frac{n_{\text{gas}}m}{m_*^2}\sum_{\substack{ijkl\\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}}\int d\boldsymbol{Q}\int d\boldsymbol{p}\,\mu_\beta\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}+\frac{m}{m_*}\frac{\boldsymbol{Q}}{2}+\frac{m}{M}\boldsymbol{P}_{\parallel\boldsymbol{Q}}+\frac{\mathcal{E}_{ij}}{Q^2/m}\boldsymbol{Q}\right)\\
&\times f_{ij}\left(\text{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q},\text{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}\right)\\
&\times f_{kl}^*\left(\text{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})-\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q},\text{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}})+\frac{\boldsymbol{Q}}{2}+\frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}\right)\{E_{kl}^\dagger E_{ij},\rho(\boldsymbol{P})\}.
\end{aligned} \tag{16}$$

Let us now recall the expression of the dynamic structure factor for a gas of free particles obeying Maxwell-Boltzmann statistics, which is given by

$$S_{\text{MB}}(\boldsymbol{Q},E) = \sqrt{\frac{\beta m}{2\pi}}\frac{1}{Q}\exp\left(-\frac{\beta}{8m}\frac{(Q^2+2mE)^2}{Q^2}\right), \tag{17}$$

where the variables $\boldsymbol{Q}$ and E denote energy transfer and momentum transfer in a scattering event. The dynamic structure factor is a two-point correlation function appearing in the expression of the scattering cross section of a probe scattering off a macroscopic sample, in the present case the gas, expressed in terms of momentum and energy transferred in the collision [28,29]. Its general expression is given by the Fourier transform with respect to energy and momentum transfer of the density-density correlation function of the medium, and for the case of a sample of noninteracting particles can be analytically evaluated to give Eq. (17). The physical meaning of the dynamic structure factor for the characterization of scattering of a test particle off a gas explains its natural appearance in the expression of the quantum linear Boltzmann equation, as already recognized in [20,22,30,31]. As we are now going to show, the dynamic structure factor also appears when considering internal degrees of freedom, the energy transfer being now also related to the energy absorbed or released as a consequence of internal transitions. Exploiting Eq. (15), we observe in fact the identity

$$\begin{aligned}
&\mu_\beta\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}+\frac{m}{m_*}\frac{\boldsymbol{Q}}{2}+\frac{m}{M}\boldsymbol{P}_{\parallel\boldsymbol{Q}}+\frac{\mathcal{E}_{ij}}{Q^2/m}\boldsymbol{Q}\right)\\
&\quad=\mu_\beta\left[\boldsymbol{p}_{\perp\boldsymbol{Q}}+\left(\frac{Q^2+2m[E(\boldsymbol{Q},\boldsymbol{P})+\mathcal{E}_{ij}]}{Q^2}\right)\frac{\boldsymbol{Q}}{2}\right],
\end{aligned}$$

leading via Eqs. (9) and (17) to

$$\begin{aligned}
&\frac{m}{Q}\mu_\beta\left(\boldsymbol{p}_{\perp\boldsymbol{Q}}+\frac{m}{m_*}\frac{\boldsymbol{Q}}{2}+\frac{m}{M}\boldsymbol{P}_{\parallel\boldsymbol{Q}}+\frac{\mathcal{E}_{ij}}{Q^2/m}\boldsymbol{Q}\right)\\
&\quad=\mu_\beta(\boldsymbol{p}_{\perp\boldsymbol{Q}})S_{\text{MB}}(\boldsymbol{Q},E(\boldsymbol{Q},\boldsymbol{P})+\mathcal{E}_{ij}),
\end{aligned} \tag{18}$$

where $\mu_\beta(\boldsymbol{p}_{\perp\boldsymbol{Q}})$ denotes the Maxwell-Boltzmann distribution in two dimensions. The quantity

$$E(\boldsymbol{Q},\boldsymbol{P}) = \frac{(\boldsymbol{P}+\boldsymbol{Q})^2}{2M}-\frac{P^2}{2M}=\frac{Q^2}{2M}+\frac{\boldsymbol{Q}\cdot\boldsymbol{P}}{M}, \tag{19}$$

actually only depending on $\boldsymbol{P}_{\parallel\boldsymbol{Q}}$, is the energy transferred to the center of mass in a collision in which the momentum of the test particle changes from $\boldsymbol{P}$ to $\boldsymbol{P}+\boldsymbol{Q}$. Relying on Eqs. (18) and (15) we can therefore finally write

$$
\begin{aligned}
\frac{d}{dt}\rho(\boldsymbol{P}) = \frac{n_{\rm gas}}{m_*^2} \sum_{\substack{ijkl \\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}} & \int d\boldsymbol{Q} \int_{\boldsymbol{Q}^\perp} d\boldsymbol{p}\ \mu_\beta(\boldsymbol{p}_{\perp\boldsymbol{Q}}) S_{\rm MB}(\boldsymbol{Q}, E(\boldsymbol{Q},\boldsymbol{P}-\boldsymbol{Q}) + \mathcal{E}_{ij}) \\
&\times f_{ij}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q} \right) \\
&\times f^*_{kl}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q} \right) E_{ij}\rho(\boldsymbol{P}-\boldsymbol{Q})E^\dagger_{kl} \\
&- \frac{1}{2}\frac{n_{\rm gas}}{m_*^2} \sum_{\substack{ijkl \\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}} \int d\boldsymbol{Q} \int_{\boldsymbol{Q}^\perp} d\boldsymbol{p}\ \mu_\beta(\boldsymbol{p}_{\perp\boldsymbol{Q}}) S_{\rm MB}(\boldsymbol{Q}, E(\boldsymbol{Q},\boldsymbol{P}) + \mathcal{E}_{ij}) \\
&\times f_{ij}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q} \right) \\
&\times f^*_{kl}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q} \right) \{E^\dagger_{kl}E_{ij}, \rho(\boldsymbol{P})\},
\end{aligned}
\tag{20}
$$

where because of the δ function of energy conservation Eq. (13) exploited in coming to this final expression the transition energy $\mathcal{E}_{ij}$ must be equal to zero whenever $\boldsymbol{Q}$ is equal to zero, due to the fact that we consider the gas particles as structureless. We stress the dependence on $\boldsymbol{P}-\boldsymbol{Q}$ in the gain term with respect to $\boldsymbol{P}$ in the loss term. Note also the very natural appearance of the argument $E(\boldsymbol{Q},\boldsymbol{P})+\mathcal{E}_{ij}$ in the dynamic structure factor, corresponding to the energy transfer in the interaction events, due to both the momentum exchange and the internal transition, whenever the scattering is not elastic.

Equation (20) is equivalent to Eq. (8), but it is written in a more convenient way for the sake of considering a quantum description of the center of mass degrees of freedom. The quantum master equation for the dynamics of both internal and center of mass degrees of freedom has to be of Lindblad form and to coincide with the semiclassical expression (20) when the diagonal matrix elements are considered in the momentum representation. Moreover, due to the homogeneity of the gas, the equation has to reflect the physical invariance under translations, which is expressed at the level of the master equation by the property of covariance under translations, corresponding to the fact that the generator of the master equation commutes with the generator of translations. This property has been considered at a formal level in [32–35], leading to a general mathematical characterization of Lindblad structures complying with translational invariance, and discussed in a physical framework in [20,31]. In view of these requirements, the quantum master equation is simply obtained by making operator valued the relevant physical expressions appearing in the equation and depending on the momentum of the test particle, such as the dynamic structure factor and scattering amplitude. In this way one obtains the following master equation for a statistical operator ϱ on the space $L^2(\mathbb{R}^3)\otimes\mathbb{C}^n$, which is manifestly in Lindblad form:

$$
\begin{aligned}
\frac{d}{dt}\varrho = \frac{n_{\rm gas}}{m_*^2} \sum_{\substack{ijkl \\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}} & \int d\boldsymbol{Q} \int_{\boldsymbol{Q}^\perp} d\boldsymbol{p}\ \mu_\beta(\boldsymbol{p}_{\perp\boldsymbol{Q}}) \Bigg[f_{ij}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q} \right) \\
&\times e^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \sqrt{S_{\rm MB}(\boldsymbol{Q},E(\boldsymbol{Q},\mathsf{P})+\mathcal{E}_{ij})} E_{ij}\varrho E^\dagger_{kl} \sqrt{S_{\rm MB}(\boldsymbol{Q},E(\boldsymbol{Q},\mathsf{P})+\mathcal{E}_{kl})} e^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} \\
&\times f^\dagger_{kl}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q} \right) \Bigg] \\
&- \frac{1}{2}\frac{n_{\rm gas}}{m_*^2} \sum_{\substack{ijkl \\ \mathcal{E}_{ij}=\mathcal{E}_{kl}}} \int d\boldsymbol{Q} \int_{\boldsymbol{Q}^\perp} d\boldsymbol{p}\ \mu_\beta(\boldsymbol{p}_{\perp\boldsymbol{Q}}) \Bigg\{ f^\dagger_{kl}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q} \right) \\
&\times f_{ij}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q} \right) S_{\rm MB}(\boldsymbol{Q},E(\boldsymbol{Q},\mathsf{P})+\mathcal{E}_{ij}) E^\dagger_{kl}E_{ij}, \varrho \Bigg\},
\end{aligned}
\tag{21}
$$

where X and P denote the position and momentum operators of the massive test particle, and the scattering amplitudes f_{ij} appearing operator valued describe inelastic scattering with a momentum transfer $\boldsymbol{Q}$ between two channels differing in energy by $\mathcal{E}_{ij}$. One immediately checks that the diagonal matrix elements in the momentum representation of Eq. (21) do coincide with Eq. (20), and furthermore that, if the internal degrees of freedom are neglected, one comes back to the quantum linear Boltzmann equation [22], which, together with the correct behavior under translations, granted by the very operator structure of the equation, provides a further argument for the assessment of the off-diagonal matrix elements. The step leading from Eq. (20) to Eq. (21), which corresponds to promoting the classical momentum to the corresponding operator, similarly to what happens in standard quantization procedures, relies on the specific structure of Eq. (20), and can also be applied when the internal degrees of freedom are neglected, in which case it leads to the correct version of the quantum linear Boltzmann equation, as confirmed by independent derivations [19,21,22,30]. Of course the ultimate justification for Eq. (21) relies on a microscopic derivation, which can be obtained similarly but with much lengthier calculations than in [22]. Due to the quite complicated expression, it is worth introducing a more compact notation by defining the Lindblad operators

$$\mathsf{L}_{\boldsymbol{Q},\boldsymbol{p},\mathcal{E}} = e^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar}L(\boldsymbol{p},\mathsf{P};\boldsymbol{Q},\mathcal{E}), \tag{22}$$

where

$$L(\boldsymbol{p},\mathsf{P};\boldsymbol{Q},\mathcal{E}) = \sum_{\substack{ij\\ \mathcal{E}_{ij}=\mathcal{E}}} f_{ij}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}\right) \times \sqrt{\frac{n_{\mathrm{gas}}}{m_*^2}\mu_\beta(\boldsymbol{p}_{\perp\boldsymbol{Q}})}\sqrt{S_{\mathrm{MB}}(\boldsymbol{Q},E(\boldsymbol{Q},\mathsf{P})+\mathcal{E}_{ij})}E_{ij}, \tag{23}$$

thus writing Eq. (21) in the compact and manifestly Lindblad form

$$\frac{d}{dt}\varrho = \sum_{\mathcal{E}} \int d\boldsymbol{Q}\int_{\boldsymbol{Q}^\perp} d\boldsymbol{p}\left(\mathsf{L}_{\boldsymbol{Q},\boldsymbol{p},\mathcal{E}}\varrho\mathsf{L}^\dagger_{\boldsymbol{Q},\boldsymbol{p},\mathcal{E}} - \frac{1}{2}\{\mathsf{L}^\dagger_{\boldsymbol{Q},\boldsymbol{p},\mathcal{E}}\mathsf{L}_{\boldsymbol{Q},\boldsymbol{p},\mathcal{E}},\varrho\}\right). \tag{24}$$

We will refer to Eq. (24) or equivalently Eq. (21) as the quantum Bloch-Boltzmann equation.

IV. REDUCED NON-MARKOVIAN DYNAMICS

We now want to point out two different situations of physical relevance in which by relying on Eq. (24) one can obtain a description of non-Markovian behaviors, typically showing up in nonexponential decay, e.g., of coherences of the system under study. Despite focusing on a concrete class of physical systems, our analysis generally applies to the case of a bipartite quantum system interacting with a reservoir, provided all degrees of freedom of the bipartite system are involved in the interaction mechanism between system and reservoir, thus generating the entanglement which accounts for the memory effects. This provides a realization and clarification of the scheme envisaged in [15,16], calling for extra fictitious unobserved degrees of freedom in order to lead to a Lindblad rate equation realizing in the Born approximation a generalized Lindblad structure. In particular, our result goes beyond the Born approximation and displays the full-fledged generalized Lindblad structure Eq. (5), allowing for truly coupled equations for the different subcollections ρ_α and considering both the case of a discrete and a continuous label α.

As we discussed in Sec. I the non-Markovian behavior described via Eq. (5) arises when one goes over from Eq. (24) to a semiclassical description, and a classical label characterizing the initial state cannot be resolved or accounted for in the final measurement. This semiclassical picture of the dynamics holds if the initial state of the system is prepared so that one of the two degrees of freedom is in a classical state, or if decoherence affects the two kind of degrees of freedom of the bipartite state on different time scales, so that, e.g., the motional dynamics can be treated classically while the internal degrees of freedom still require a full quantum treatment. In this framework knowledge about the way in which the system is prepared usually naturally provides information about both parts of the bipartite system, while the final detection scheme is not necessarily fine enough to fully characterize the outgoing state. In different contexts it might also possibly arise as a consequence of superselection rules.

A. Description of center of mass degrees of freedom

Let us consider first a situation in which we put our test particle, or equivalently a sufficiently dilute collection of such test particles so that they can be considered as noninteracting, in a dense inert gas. The test particle will undergo many collisions quickly, leading to a classical characterization of the motion of its center of mass, so that only the diagonal matrix elements in the momentum representation of Eq. (24) are left on a time scale set by the collisional decoherence mechanism, which leads us back to Eq. (20), which is also called the Bloch-Boltzmann equation. Of course, due to the complexity of the master equation (24) such a behavior, though naturally expected on physical grounds and usually invoked in the literature on decoherence [36], cannot be easily demonstrated in realistic situations. It has, however, been confirmed by means of Monte Carlo simulations, which also allow for estimates of the decoherence rates [37]. It is convenient to write the equation in a more compact way introducing the following $\mathbb{C}$-number rate operators:

$$M^{jl}_{ik}(\boldsymbol{P}+\boldsymbol{Q};\boldsymbol{Q}) = \delta_{\mathcal{E}_{ij},\mathcal{E}_{kl}}\frac{n_{\mathrm{gas}}}{m_*^2}\int_{\boldsymbol{Q}^\perp} d\boldsymbol{p}\ \mu_\beta(\boldsymbol{p}_{\perp\boldsymbol{Q}})S_{\mathrm{MB}}(\boldsymbol{Q},E(\boldsymbol{Q},\boldsymbol{P}) + \mathcal{E}_{ij})f_{ij}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\boldsymbol{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_*}\boldsymbol{Q}\right)$$

$$\times f^*_{kl}\left(\mathrm{rel}(\boldsymbol{p}_{\perp Q},\boldsymbol{P}_{\perp Q}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp Q},\boldsymbol{P}_{\perp Q}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{kl}}{Q^2/m_*}\boldsymbol{Q}\right), \quad (25)$$

which provide the rates for scattering from $\boldsymbol{P}$ to $\boldsymbol{P}+\boldsymbol{Q}$, including the dependence on the indices for internal degrees of freedom and the transition energy $\mathcal{E}_{ij}$. In the limit of an infinitely massive test particle, these rate coefficients can be checked to go over to those derived in [26] for the case of an immobile system. Exploiting the expression Eq. (25) for the rate operators, we can write Eq. (20) more compactly as

$$\frac{d}{dt}\rho(\boldsymbol{P}) = \sum_{ijkl}\int d\boldsymbol{Q}\left(M^{jl}_{ik}(\boldsymbol{P};\boldsymbol{Q})E_{ij}\rho(\boldsymbol{P}-\boldsymbol{Q})E^\dagger_{kl} - \frac{1}{2}M^{jl}_{ik}(\boldsymbol{P}+\boldsymbol{Q};\boldsymbol{Q})\{E^\dagger_{kl}E_{ij},\rho(\boldsymbol{P})\}\right). \quad (26)$$

Despite the enormous complexity of this integro-differential operator equation one can consider some simplified situations, allowing to put into evidence non-Markovian behaviors arising for the reduced statistical operator $\rho=\int d\boldsymbol{P}\,\rho(\boldsymbol{P})$, only describing the internal degrees of freedom, once the dynamics of the various subcollections $\{\rho(\boldsymbol{P})\}_{\boldsymbol{P}\in\mathbb{R}^3}$ is given by Eq. (26). Let us consider the most simple conceivable situation, taking an internal $\mathbb{C}^2$ space and only allowing for elastic scattering. We thus have $M^{jl}_{ik}\propto\delta_{ij}\delta_{kl}$, and further restricting to ourselves forward scattering we can write

$$M^{jl}_{ik}(\boldsymbol{P}+\boldsymbol{Q};\boldsymbol{Q}) = \delta_{ij}\delta_{kl}\delta^3(\boldsymbol{Q})\xi_{ik}(\boldsymbol{P}),$$

parametrizing the rate operators by means of the functions $\xi_{ik}(\boldsymbol{P})$. According to Eq. (12) we denote by $E_{ii}=|i\rangle\langle i|$ the maps between the same energy eigenstates, corresponding to the projectors on the two one-dimensional subspaces of $\mathbb{C}^2$, so that Eq. (26) now simplifies to

$$\frac{d}{dt}\rho(\boldsymbol{P}) = \sum_{ik}\xi_{ik}(\boldsymbol{P})E_{ii}\rho(\boldsymbol{P})E_{kk} - \frac{1}{2}\left\{\sum_i \xi_{ii}(\boldsymbol{P})E_{ii},\rho(\boldsymbol{P})\right\}. \quad (27)$$

The dynamics of the single subcollections $\rho(\boldsymbol{P})$ can now be easily studied. Setting

$$\rho_{ij}(\boldsymbol{P}) = \langle i|\rho(\boldsymbol{P})|j\rangle$$

for the matrix elements of the collection $\{\rho(\boldsymbol{P})\}_{\boldsymbol{P}\in\mathbb{R}^3}$ of matrices in $\mathbb{C}^2$, one immediately sees that there is no dynamics for the populations, in that $\dot{\rho}_{ii}(\boldsymbol{P})=0$, for $i=1,2$, so that in particular integrating over the possible momentum dependence also $\dot{\rho}_{ii}=0$, for $i=1,2$. The coherences of the subcollections $\rho_{12}(\boldsymbol{P})=\rho^*_{21}(\boldsymbol{P})$ are instead described by the equation

$$\frac{d}{dt}\rho_{12}(\boldsymbol{P}) = \left(-\frac{1}{2}\xi_{11}(\boldsymbol{P}) - \frac{1}{2}\xi_{22}(\boldsymbol{P}) + \xi_{12}(\boldsymbol{P})\right)\rho_{12}(\boldsymbol{P}), \quad (28)$$

which, on introducing what we might call, in the absence of better names, a momentum-dependent friction coefficient

$$\Xi(\boldsymbol{P}) = \frac{1}{2}\xi_{11}(\boldsymbol{P}) + \frac{1}{2}\xi_{22}(\boldsymbol{P}) - \xi_{12}(\boldsymbol{P}), \quad (29)$$

which in view of Eq. (25) has a positive real part proportional to the averaged modulus of the difference of forward scattering amplitudes [26], is easily solved by

$$\rho_{12}(\boldsymbol{P},t) = e^{-\Xi(\boldsymbol{P})t}\rho_{12}(\boldsymbol{P},0).$$

Let us now consider an initial state of the form $\rho_{12}(\boldsymbol{P},0)=\rho_{12}(0)\mu_\beta(\boldsymbol{P})$, corresponding to a preparation in which the test particle, or equivalently the dilute ensemble of noninteracting test particles, is in a classical thermal state as far as the center of mass is concerned, and has a nonvanishing initial value for the coherences of the internal degrees of freedom. The behavior in time of the off-diagonal matrix elements observed for the internal degrees of freedom only, not resolving the momentum of the considered test particle, is then given by

$$\rho_{12}(t) = \Lambda(t)\rho_{12}(0),$$

with

$$\Lambda(t) = \int d\boldsymbol{P}\, e^{-\Xi(\boldsymbol{P})t}\mu_\beta(\boldsymbol{P}). \quad (30)$$

It is now immediately evident that behaviors utterly different from the usual Markovian exponential decay in time appear depending on the actual expression of the momentum-dependent $\Xi(\boldsymbol{P})$, the Markovian case obviously corresponding to a constant friction coefficient $\Xi(\boldsymbol{P})=\eta$.

For the simple case $\Xi(\boldsymbol{P})=a\boldsymbol{P}^2$ one immediately obtains a power law decay of the form

$$\Lambda(t) = \frac{1}{(1+t/\tau)^{3/2}}, \quad (31)$$

where we have set $\tau=1/(aP^2_\beta)$, indicating a natural reference time. Another simple expression of the friction coefficient leads instead to a stretched exponential. Considering in fact $\Xi(\boldsymbol{P})=b/\boldsymbol{P}^2$ one has to evaluate

$$\Lambda(t) = \frac{1}{\pi^{3/2}P^3_\beta}\int d\boldsymbol{P}\, e^{-\Xi(\boldsymbol{P})t}e^{-\boldsymbol{P}^2/P^2_\beta}, \quad (32)$$

so that by exploiting the result [38]

$$\int_0^\infty dx\, x^2 e^{-a/x^2}e^{-bx^2} = \sqrt{\frac{\pi}{16b^3}}(1+2\sqrt{ab})e^{-2\sqrt{ab}},$$

we obtain

$$\Lambda(t) = [1+(t/\tau)^{1/2}]e^{-(t/\tau)^{1/2}}, \quad (33)$$

describing a stretched exponential decay in time with a square root correction, where the reference time is now set by $\tau = P_\beta^2/(4b)$. These two simple choices for the friction coefficient $\Xi(\boldsymbol{P})$, amenable to an analytical treatment, have clearly shown the appearance of strongly non-Markovian behaviors for the operator $\rho(t)$. The considered example is obviously quite simplified and does not describe in a realistic way all possible aspects of the dynamics, e.g., the redistribution of population in the internal degrees of freedom. It allows us, however, to easily grasp some non-Markovian aspects of the generalized Lindblad structure given by Eq. (5), of which Eq. (26) provides a simple example, even though with a continuous index.

B. Description of internal degrees of freedom

We now focus on a quite different situation, in which we study the dynamics of our test particle when flying through an interferometer for massive particles, e.g., of the Mach-Zender type, as recently realized also for the quantitative study of decoherence [39]. In such a case the initial preparation is engineered so as to ensure a coherent superposition of states of the motional degrees of freedom of the system while, in the absence of a further selection in the prepared state, the internal degrees of freedom can be described by a classical distribution, corresponding to a partially diagonal statistical operator. The diagonal matrix elements of Eq. (24) with respect to the internal degrees of freedom naturally lead to coupled master equations for the collection $\{\rho_r\}_r$ of trace class operators in $L^2(\mathbb{R}^3)$, defined according to $\rho_r = \langle r|\varrho|r\rangle$. In order to keep a compact notation we introduce the rate operators

$$\mathsf{R}^{rj}(\boldsymbol{p},\boldsymbol{Q}) = f_{rj}\left(\mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) - \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{rj}}{Q^2/m_*}\boldsymbol{Q}, \mathrm{rel}(\boldsymbol{p}_{\perp\boldsymbol{Q}},\mathsf{P}_{\perp\boldsymbol{Q}}) + \frac{\boldsymbol{Q}}{2} + \frac{\mathcal{E}_{rj}}{Q^2/m_*}\boldsymbol{Q}\right)\sqrt{\frac{n_{\mathrm{gas}}}{m_*^2}\mu_\beta(\boldsymbol{p}_{\perp\boldsymbol{Q}})} \times e^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar}\sqrt{S_{\mathrm{MB}}(\boldsymbol{Q},E(\boldsymbol{Q},\mathsf{P}) + \mathcal{E}_{rj})}, \tag{34}$$

operator valued on $L^2(\mathbb{R}^3)$, so that the master equations for the subcollections $\{\rho_r\}_r$ explicitly exhibit the generalized Lindblad structure given by Eq. (5), with the additional appearance of integrals over continuous indices

$$\frac{d}{dt}\rho_r = \sum_j \int d\boldsymbol{Q}\int_{\boldsymbol{Q}^\perp} d\boldsymbol{p}\left(\mathsf{R}^{rj}(\boldsymbol{p},\boldsymbol{Q})\rho_j\mathsf{R}^{rj}(\boldsymbol{p},\boldsymbol{Q})^\dagger - \frac{1}{2}\{\mathsf{R}^{jr}(\boldsymbol{p},\boldsymbol{Q})^\dagger\mathsf{R}^{jr}(\boldsymbol{p},\boldsymbol{Q}),\rho_r\}\right). \tag{35}$$

The set of master equations given by Eq. (35) do provide a non-Markovian dynamics for the statistical operator $\rho = \sum_{r=1}^n \rho_r$ observed at the outcome of the experiment, e.g., to determine the visibility of the interference fringes, when the detection scheme cannot resolve the state of the internal degrees of freedom.

In a typical experimental situation for the study of collisional decoherence one can safely neglect the dependence on the momentum operator P in the rate operators defined by Eq. (34), replacing it by the classical value of the momentum of the incoming test particle, due to the fact that on the decoherence time scale the dissipative dynamics of the momentum does not play a role [22,40,41]. This brings in an important simplification in Eq. (35), which can now be written using the unitary operators $e^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar}$, describing the momentum kicks causing decoherence of the center of mass, and $\mathbb{C}$-number positive collision rates $\lambda_{jr}(\boldsymbol{Q})$, which depend on the internal state of the test particle, thus obtaining

$$\frac{d}{dt}\rho_r = \sum_j \int d\boldsymbol{Q}[\lambda_{rj}(\boldsymbol{Q})e^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar}\rho_j e^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} - \lambda_{jr}(\boldsymbol{Q})\rho_r]. \tag{36}$$

For the case in which the collisions do not lead to transitions between different internal states, so that $\lambda_{jr}(\boldsymbol{Q}) = \delta_{jr}\lambda_r(\boldsymbol{Q})$, one comes to the following master equation describing a dynamics determined by momentum kicks of amount $\boldsymbol{Q}$, taking place with a probability density $\mathcal{P}_r(\boldsymbol{Q})$ which depends on the internal state of the test particle:

$$\frac{d}{dt}\rho_r = \Lambda_r \int d\boldsymbol{Q}\, \mathcal{P}_r(\boldsymbol{Q})(e^{i\boldsymbol{Q}\cdot\mathsf{X}/\hbar}\rho_r e^{-i\boldsymbol{Q}\cdot\mathsf{X}/\hbar} - \rho_r). \tag{37}$$

In Eq. (37) the probability density $\mathcal{P}_r(\boldsymbol{Q})$ is defined according to

$$\mathcal{P}_r(\boldsymbol{Q}) = \frac{\lambda_r(\boldsymbol{Q})}{\Lambda_r}, \tag{38}$$

with $\Lambda_r = \int d\boldsymbol{Q}\,\lambda_r(\boldsymbol{Q})$ the total scattering rate for particles with internal state r. The master equations (37) are easily solved in the position representation according to

$$\langle \boldsymbol{x}|\rho_r(t)|\boldsymbol{y}\rangle = e^{-\Lambda_r[1-\phi_r(\boldsymbol{x}-\boldsymbol{y})]t}\langle \boldsymbol{x}|\rho_r(0)|\boldsymbol{y}\rangle, \tag{39}$$

with

$$\phi_r(\boldsymbol{x}-\boldsymbol{y}) = \int d\boldsymbol{Q}\, \mathcal{P}_r(\boldsymbol{Q})e^{i\boldsymbol{Q}\cdot(\boldsymbol{x}-\boldsymbol{y})/\hbar}$$

the Fourier transform of the probability density given by Eq. (38), that is to say its characteristic function [42]. In particular, the prefactor is the characteristic function of a compound Poisson process, composed according to the probability density $\mathcal{P}_r(\boldsymbol{Q})$. This equation describes a quite general physical situation in which one has a sequence of interaction events between system and environment, distributed in time according to a Poisson distribution, each one characterized by a random momentum transfer, drawn according to a certain probability density fixed by the microphysical interaction mechanism [43,44]. At variance with a simple Poisson process the momentum transfer is not deterministically fixed to be the same in each collision, but is a random variable depending on the details of the collision.

We now look at the dynamics of the matrix elements of the whole statistical operator $\rho = \sum_r \rho_r$, responsible for the description of the measurement outcomes at the output of the interferometer. To do this we consider an initial state of the form $\rho_r(0) = p_r\rho(0)$, with $p_r \geqslant 0$ and $\sum_r p_r = 1$, corresponding to a preparation in which the populations in the internal states are distributed according to certain weights p_r, while the statistical operator $\rho(0)$ in $L^2(\mathbb{R}^3)$ characterizes the quantum state of the center of mass. We thus come to the following expression for the solution of the non-Markovian set of master equations given by Eq. (37):

$$\langle \boldsymbol{x}|\rho(t)|\boldsymbol{y}\rangle = \sum_r p_r e^{-\Lambda_r[1-\phi_r(\boldsymbol{x}-\boldsymbol{y})]t}\langle \boldsymbol{x}|\rho(0)|\boldsymbol{y}\rangle, \qquad (40)$$

where the multiplicative prefactor determines the weight of the matrix elements in the position representation, both with elapsing time, and as a function of the distance $|\boldsymbol{x}-\boldsymbol{y}|$. Such an expression provides information on the loss of coherence responsible for reduction in the visibility of the interference fringes. If only one of the weights p_r is different from zero, and therefore equal to 1, one falls back to the usual Markovian exponential decay in time, possibly with a modulation in the spatial dependence. In particular the coherence of the quantum state over spatially separated points depends on the details of the functions $\phi_r(\boldsymbol{x}-\boldsymbol{y})$, given by the Fourier transform of the probability density of momentum kicks in the scattering events. In a typical situation such functions quickly go to zero in their dependence on the distance $|\boldsymbol{x}-\boldsymbol{y}|$, so that, e.g., in the Markovian case one is simply left with a constant exponential loss of visibility [39]. With the present more general initial state even for $\phi_r(\boldsymbol{x}-\boldsymbol{y})\simeq 0$ one has a nontrivial structure describing a decay of coherence other than exponential

$$\langle \boldsymbol{x}|\rho(t)|\boldsymbol{y}\rangle \simeq \Psi(t)\langle \boldsymbol{x}|\rho(0)|\boldsymbol{y}\rangle, \qquad (41)$$

where the function

$$\Psi(t) = \sum_r p_r e^{-\Lambda_r t} \qquad (42)$$

is the survival probability of a multiexponential distribution, i.e., the probability to have no event up to a time t for such a distribution [42]. Depending on the weights p_r and the rates Λ_r, not only simple deviations from the exponential law can appear, but also utterly different behaviors. To clarify this point let us consider for the sake of example a geometric distribution of weights, with ratio $p_0=\mathrm{e}^{-a}$, $a\in\mathbb{R}_+$, so that

$$p_r = (1-p_0)p_0^r,$$

and a geometric progression of rates

$$\Lambda_r = \Lambda_0\gamma_0^r,$$

with ratio $\gamma_0=\mathrm{e}^{-b}$, $b\in\mathbb{R}_+$, and the reference rate Λ_0 as scale factor. The survival probability then reads

$$\Psi(t) = (1-p_0)\sum_r p_0^r e^{-\gamma_0^r\Lambda_0 t},$$

which, due to the relation [45]

$$\Psi(\Lambda_0 t) = \frac{1}{p_0}[\Psi(t)-(1-p_0)e^{-\Lambda_0 t}],$$

exhibits at long times a power law decay

$$\Psi(\Lambda_0 t) \simeq \frac{1}{(\Lambda_0 t)^{a/b}}, \quad t \gg 1. \qquad (43)$$

This simplest example allowing for an analytical treatment already shows the rich variety of non-Markovian behaviors which might arise when one uses the generalized Lindblad structure given by Eq. (35), which provides a further example of the general result Eq. (5) for the case of a sum over a discrete index. In particular Eq. (35) describes how the dependence of the scattering events on the internal structure of the test particle affects the loss of coherence in position space, which in turn determines the reduction of visibility in an interferometric experiment. As it appears from Eq. (43), this can lead to very strong deviations from the exponential decay, such as power law behaviors.

V. CONCLUSIONS

In this paper we have considered a class of non-Markovian behaviors, arising when dealing with a bipartite quantum system interacting with a reservoir. The concrete bipartite system considered was given by a massive test particle, for which both internal and center of mass degrees of freedom have been taken into account. The reservoir was assumed as a structureless gas, affecting our test particle through collisions whose microscopic characterization depends on both its motional and internal state. As a starting point we have derived in Sec. III a quantum master equation describing such dynamics in a nonperturbative way, expressed by Eq. (24), which can also be termed the quantum Bloch-Boltzmann equation in that it describes at the quantum level both kinds of degrees of freedom. When due to decoherence or features of the initial preparation, one of the two degrees of freedom is to be described classically, one obtains from the quantum Bloch-Boltzmann equation two examples of a generalized Lindblad structure recently considered for the description of non-Markovian dynamics [7,8]. Such a generalized Lindblad expression has been outlined in Sec. II, clarifying its mathematical structure and physical motivation. For the case at hand non-Markovian effects, leading to decay behaviors of coherences of the system given by stretched exponentials or power laws instead of simple exponentials, appear when the degrees of freedom allowing for a classical description are not resolved in the final measurement, only focusing on the quantum degrees of freedom. This provides a concrete realization of a proposed mechanism for the appearance of such generalized Lindblad structures [15,16], further clarifying the origin of the non-Markovian behaviors. These behaviors have been spelled out in Sec. IV, focusing on the dynamics of the internal state of a test particle interacting with an inert gas, as well as on loss of coherence of a massive particle flying through an interferometer where it interacts with a background gas. It is to be stressed that in the physical examples considered in Secs. IV A and IV B one has to deal with decoupled subcollections of statistical operators, so that the non-Markovian features arise from the average over the classical index in the initial condition. The generalized Lindblad structure given by Eq. (5) also allows one to consider coupled equations for the different subcollections, and for these situations one naturally expects a much more complicated non-Markovian dynamics.

It immediately appears that the outlined scheme leading to a class of non-Markovian evolutions generally applies in the presence of the interaction of a bipartite quantum system with a quantum environment, when one of the quantum labels of the system becomes classical and can be averaged over. More generally, such a class of non-Markovian evolutions appear in the presence of a classical degree of freedom, described by means of some discrete or continuous label, which is involved in the characterization of the interaction between two quantum degrees of freedom, and is averaged over in order to give the relevant dynamics. This classical label might as well appear on the side of the environment, corresponding to so-called structured reservoirs, or on the side of the system, as in the case of a bipartite system. The present work can naturally be extended to include an internal structure in the gas particles, which could also influence the scattering amplitude, introducing new channels. In particular, a detailed analysis of the rate operators based on microphysical informations could pave the way to new interferometric experiments for the quantitative study of decoherence, exhibiting more general behaviors than exponential decay of visibility with elapsed interaction time.

ACKNOWLEDGMENTS

The author is grateful to Heinz-Peter Breuer, Klaus Hornberger, and Ludovico Lanz for helpful discussions and reading of the manuscript.

[1] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2007).
[2] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
[3] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[4] A. A. Budini, Phys. Rev. A **69**, 042107 (2004).
[5] A. Shabani and D. A. Lidar, Phys. Rev. A **71**, 020101 (2005).
[6] C. A. Rodriguez-Rosario and E. C. G. Sudarshan, e-print arXiv:0803.1183.
[7] H.-P. Breuer, J. Gemmer, and M. Michel, Phys. Rev. E **73**, 016139 (2006).
[8] H.-P. Breuer, Phys. Rev. A **75**, 022103 (2007).
[9] L. Lanz, B. Vacchini, and O. Melsheimer, J. Phys. A **40**, 3123 (2007).
[10] P. Blanchard and A. Jadczyk, Ann. Phys. **4**, 583 (1995).
[11] R. Alicki and S. Kryszewski, Phys. Rev. A **68**, 013809 (2003).
[12] J. Fischer and H.-P. Breuer, Phys. Rev. A **76**, 052119 (2007).
[13] H.-P. Breuer, e-print arXiv:0707.0172.
[14] A. A. Budini, Phys. Rev. E **72**, 056106 (2005).
[15] A. A. Budini and H. Schomerus, J. Phys. A **38**, 9251 (2005).
[16] A. A. Budini, Phys. Rev. A **74**, 053815 (2006).
[17] M. Esposito and P. Gaspard, Phys. Rev. E **68**, 066112 (2003).
[18] M. Esposito and P. Gaspard, Phys. Rev. E **76**, 041134 (2007).
[19] B. Vacchini, Phys. Rev. Lett. **84**, 1374 (2000).
[20] B. Vacchini, J. Math. Phys. **42**, 4291 (2001).
[21] K. Hornberger, Phys. Rev. Lett. **97**, 060601 (2006).
[22] K. Hornberger and B. Vacchini, Phys. Rev. A **77**, 022112 (2008).
[23] R. Dümcke, Commun. Math. Phys. **97**, 331 (1985).
[24] R. F. Snider, Int. Rev. Phys. Chem. **17**, 185 (1998).
[25] S. Kryszewski and J. Czechowska, Phys. Rev. A **74**, 022719 (2006).
[26] K. Hornberger, EPL **77**, 50007 (2007).
[27] R. Alicki and M. Fannes, *Quantum Dynamical Systems* (Oxford University Press, Oxford, 2001).
[28] F. Schwabl, *Advanced Quantum Mechanics*, 2nd ed. (Springer, New York, 2003).
[29] L. Pitaevskii and S. Stringari, *Bose-Einstein condensation* (Oxford University Press, Oxford, 2003.
[30] B. Vacchini, Phys. Rev. E **63**, 066115 (2001).
[31] F. Petruccione and B. Vacchini, Phys. Rev. E **71**, 046134 (2005).
[32] A. S. Holevo, Rep. Math. Phys. **32**, 211 (1993).
[33] A. S. Holevo, Rep. Math. Phys. **33**, 95 (1993).
[34] A. S. Holevo, Izvestiya Mathematics **59**, 427 (1995).
[35] A. S. Holevo, in *Irreversibility and Causality (Goslar, 1996)*, Lecture Notes in Physics Vol. 504 (Springer, Berlin, 1998), pp. 67–81.
[36] E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, and I.-O. Stamatescu, *Decoherence and the Appearance of a Classical World in Quantum Theory*, 2nd ed. (Springer, Berlin, 2003).
[37] H.-P. Breuer and B. Vacchini, Phys. Rev. E **76**, 036706 (2007).
[38] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1965).
[39] K. Hornberger, S. Uttenthaler, B. Brezger, L. Hackermüller, M. Arndt, and A. Zeilinger, Phys. Rev. Lett. **90**, 160401 (2003).
[40] B. Vacchini, J. Mod. Opt. **51**, 1025 (2004).
[41] K. Hornberger, J. E. Sipe, and M. Arndt, Phys. Rev. A **70**, 053608 (2004).
[42] W. Feller, *An Introduction to Probability Theory and Its Applications* (John Wiley & Sons, New York, 1971), Vol. II.
[43] B. Vacchini, Phys. Rev. Lett. **95**, 230402 (2005).
[44] B. Vacchini, J. Phys.: Conf. Ser. **67**, 012040 (2007).
[45] P. A. Alemany, J. Phys. A **30**, 6587 (1997).