Chirped chiral solitons in the nonlinear Schrödinger equation with self-steepening and self-frequency shift

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We find exact solutions to the nonlinear Schrödinger equation (NLSE) in the presence of self-steepening and a self-frequency shift. These include periodic solutions and localized solutions of dark-bright type which can be chiral, the chirality being controlled by the sign of the self-steepening term. A form of self-phase-modulation that can be tuned by higher-order nonlinearities as well as by the initial conditions, distinct from the nonlinear Schrödinger equation, characterizes these solutions. In certain nontrivial parameter domains, solutions are found to satisfy the linear Schrödinger equation, indicating the possibility of linear superposition in this nonlinear system. Dark and bright solitons exist in both the anomalous and normal dispersion regimes, and a duality between the dark-bright type of solution and kinematic higher-order chirping is also seen. Localized kink solutions similar to NLSE solitons, but with very different self-phase-modulation, are identified.

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The nonlinear Schrödinger equation (NLSE)

$$i\psi_x + a_1\psi_{tt} + a_2|\psi|^2\psi = 0 \qquad (1)$$

governs the dynamics of picosecond pulse propagation in optical fibers [1], where a_1 is the group velocity dispersion (GVD) parameter and a_2 specifies the strength of Kerr nonlinearity. As predicted by Hasegawa and Tappert [2] and experimentally observed by Mollenauer *et al.* [3], this system supports stable soliton solutions owing their existence to complete integrability [4]. With the advent of high-intensity laser beams, it has become possible to generate optical pulses with width of the order of 10 fs. Higher-order effects like third-order dispersion, self-steepening of the pulse due to the dependence of the slowly varying part of the nonlinear polarization on time, and the self-frequency shift arising from the delayed Raman response become important in the study of the propagation of these pulses. In order to account for them, Kodama [5] and Kodama and Hasegawa [6] proposed a higher-order nonlinear Schrödinger equation as a generalization of the NLSE:

$$i\psi_x + a_1\psi_{tt} + a_2|\psi|^2\psi + i[a_3\psi_{ttt} + a_4(|\psi|^2\psi)_t + a_5\psi(|\psi|^2)_t] = 0, \qquad (2)$$

where a third-order dispersion with coefficient a_3, a self-steepening term with coefficient a_4, and a self-frequency shift effect with coefficient a_5 have been added. This model, unlike the NLSE, is not integrable in general. A few integrable cases have been identified: (i) the Sasa-Satsuma case $[a_3:a_4:(a_4+a_5)=1:6:3]$ [7], (ii) the Hirota case $[a_3:a_4:(a_4$

$+a_5)=1:6:0]$ [8], and (iii) derivative NLSEs of types I and II [9]. Many restrictive special solutions of bright and dark type have been obtained [10–12].

The effect of third-order dispersion is significant for femtosecond pulses when the GVD is close to zero. It is negligible for optical pulses whose width is of the order of 100 fs or more, having power of the order of 1 W and GVD far away from zero. However, in this case self-steepening as well as self-frequency shift terms are still dominant and should be retained. The effects of these higher-order terms on pulse propagation have been extensively studied numerically [1,13], and some special solutions to this system are also known [14].

In this Rapid Communication, we report the existence of a different class of localized as well as periodic solutions for the NLSE in the presence of self-steepening and self-frequency shift. The localized solutions include dark, bright, and kink-type solitons. These complex solitons are generally chirped and show a different form of self-phase-modulation. Unlike the NLSE, where chirping is of kinematic origin (controlled by initial conditions), the chirping in the present case has both kinematic and dynamic origin. The former varies as the reciprocal of the intensity, whereas the latter is directly proportional to the intensity and depends upon higher-order nonlinearities only. It is evident that kinematic chirping plays a significant role for dark solitons, whereas the dynamical chirping would be important for bright solitons. We found that for certain special values of parameters the intensity profiles of these solitons are similar to those of NLSE solitons, while retaining the nontrivial phase structure due to higher-order nonlinear terms. It is also seen that, in the same regime, kink solitons exist which otherwise are not allowed in this system. For certain parameter values, this model mimics the NLSE; however, the presence of higher-

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order nonlinearities crucially affects the dynamics. And as a consequence one sees that the fundamental bright soliton of the NLSE is no longer a valid solution. Further, trivial phase dark solitons exist in the anomalous dispersion regime and are found to be chiral, with the direction of propagation set by the self-steepening term. Chirped dark and bright solitons are found to exist in the normal as well as the anomalous dispersion regime. Chirped dark solitons in the anomalous regime, whereas chirped bright solitons in the normal dispersion regime, exhibit chirality, which is controlled by the self-steepening term. Observation of chiral solitons is one of the main results of this Rapid Communication. Very interestingly, it is seen that for some nontrivial choice of parameters, this system behaves like the free-particle Schrödinger equation with appropriate constants, making this system amenable to linear superposition, which is otherwise forbidden in this nonlinear system. This system is found to be Painlevé integrable, thereby establishing the existence of regular solutions.

Modulo a trivial kinematic phase, the complex envelope traveling wave solutions can be generally represented as

$$\psi(x,t) = \rho(\xi)e^{i\chi(\xi)}, \tag{3}$$

where $\xi = \alpha(t - ux)$ is the traveling coordinate, and ρ and χ are real functions of ξ. Here, α is a scale parameter and $u = 1/v$ with v being the group velocity of the wave packet. The ansatz solution leads to the compatibility conditions

$$-\alpha u\rho' + 2\alpha^2 a_1\chi'\rho' + \alpha^2 a_1\chi''\rho + 3\alpha a_4\rho^2\rho' + 2\alpha a_5\rho^2\rho' = 0, \tag{4}$$

$$\alpha u\chi'\rho + \alpha^2 a_1\rho'' - \alpha^2 a_1\chi'^2\rho + a_2\rho^3 - \alpha a_4\chi'\rho^3 = 0. \tag{5}$$

Equation (4) can be exactly integrated to yield

$$\chi' = \frac{u}{2\alpha a_1} + \frac{c}{\alpha a_1\rho^2} - \frac{(3a_4 + 2a_5)}{4\alpha a_1}\rho^2, \tag{6}$$

where c is to be determined by the initial conditions. It is to be noted that the phase has a nontrivial form and has two intensity-dependent chirping terms, apart from the kinematic first term. As is evident, the second term is of kinematic origin and is common to the Schrödinger equation as well. The last term is due to higher nonlinearities and leads to chirping that is exactly inverse to that of the former. This is a form of self-phase-modulation that is controlled by interaction. The amplitude equation (5) reduces to

$$\theta_1\rho'' + \theta_2\rho + \theta_3\rho^3 + \theta_4\rho^5 = \frac{c^2}{\rho^3} \tag{7}$$

with $\theta_1 = \alpha^2 a_1^2$, $\theta_2 = (u^2 - ca_4 + 2ca_5)/4$, $\theta_3 = (2a_1a_2 - ua_4)/2$, and $\theta_4 = (4a_4 - 1)(3a_4 + 2a_5)/16$. We note that the nontrivial contribution from higher-order nonlinear terms is through θ_4, which is zero for $a_4 = 1/4$ or $a_4 : a_5 = -2:3$ (assuming $a_4 \neq 0$ and $a_5 \neq 0$), and as we shall soon show, this results in interesting physical consequences. In the case when $a_4 : a_5 = -2:3$, both the intensity as well as the phase will not have any new features due to higher-order nonlinearities and the solutions will exactly resemble NLSE solutions. However, unlike in the NLSE, both dark and bright solitons exist, in

both the normal and anomalous dispersion regimes. When $c = 0$, the existence of fundamental bright solitons with $\rho = A\,\mathrm{sech}(\xi)$ is forbidden, and only dark solitons with $\rho = A\tanh(\xi)$ exist, with $\alpha^2 = u^2/2a_1^2$. Furthermore, these dark solitons in the anomalous dispersion regime, i.e., $a_1 > 0$, respect the inequality $a_4 u \geq 2|a_1||a_2|$, which restricts them to travel only along one direction, given by the sign of a_4. This is an example of a chiral soliton which is absent in the NLSE. When $c \neq 0$, both bright and dark solitons exist, and satisfy $2a_1a_2 > ua_4$ and $2a_1a_2 < ua_4$, respectively, showing that both have mutually exclusive velocity space. As a consequence, in the anomalous dispersion regime, dark solitons obey the inequality $2|a_1||a_2| < ua_4$ and hence are chiral, whereas in the normal dispersion regime bright solitons satisfy $-2|a_1||a_2| > ua_4$ and hence are also chiral. Notice that the directionality of these solitons is due to the presence of higher-order terms; the sign of a_4 decides the direction in which solitons are allowed to propagate. For $a_4 = 1/4$, the intensity profile will be the same as for the NLSE, whereas the phase will still show nontrivial chirping. In this case also both dark and bright solitons can be chiral, and can exist in the normal and anomalous dispersion regimes, which is in sharp contrast to the NLSE.

It is very intriguing to see that, when $a_4 = -2a_5/3$ and $u = -3a_1a_2/a_5$, Eq. (7) combined with Eq. (6) reduces to the free-particle Schrödinger equation in ψ. So for this choice of parameters, in the presence of both Kerr and higher-order nonlinearities, the effective evolution equation for ψ is linear, and one would expect to see phenomena like interference, which is forbidden otherwise in this system.

Equation (7) can be cast into a convenient form using $\rho = \sqrt{\sigma}$:

$$\frac{\theta_1}{2}\sigma'' + 2\theta_2\sigma + \frac{3\theta_3}{2}\sigma^2 + \frac{4\theta_4}{3}\sigma^3 = k, \tag{8}$$

where k is a constant fixed by initial conditions. Solutions for this equation, with $\theta_4 \neq 0$, can be found by a conformal Möbius transformation:

$$\sigma = \frac{A + Bf}{C + Df}, \tag{9}$$

which for some suitable A, B, C, and D connects σ to the elliptic function f. These elliptic functions, as is known, are generalizations of trigonometric and hyperbolic functions and appear in the solutions of many nonlinear equations.

Considering the importance of localized solutions, we set $f(\xi) = \mathrm{sech}(\xi)$, and look for allowed values of A, B, C, and D for which (9) is a solution of Eq. (8). The consistency condition leads to $A = 8\theta_4\widetilde{A} - 3\theta_3$, $B = 8\theta_4\widetilde{B} - 3\theta_3\widetilde{D}$, $C = 8\theta_4$, and $D = 8\theta_4\widetilde{D}$, where $\widetilde{A}$, $\widetilde{B}$, $\widetilde{D}$, and α are given by

$$(1024\theta_4^4)\widetilde{A}^3 + (1536\theta_2\theta_4^3 + 432\theta_3^3\theta_4^2 - 864\theta_3^2\theta_4^2)\widetilde{A}$$
$$+ (-3\theta_3^4 + 162\theta_3^3\theta_4 - 576\theta_2\theta_3\theta_4^2 - 768k\theta_4^2) = 0,$$

$$\widetilde{B} = -[\widetilde{D}(-54\theta_3^2 + 27\theta_3^3 + 96\theta_2\theta_4 + 128\theta_4^4\widetilde{A}^2)]/64\widetilde{A}\theta_4^2,$$
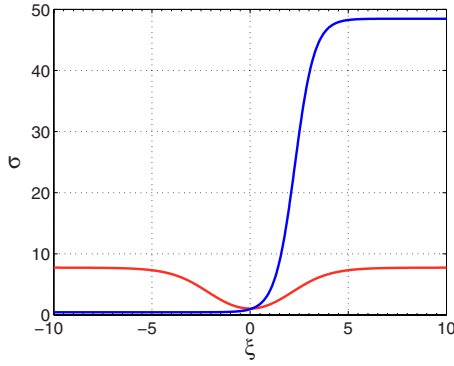
FIG. 1. (Color online) Intensity profile of few solutions: (i) Dark soliton (in red color) for $a_1=1.6001$, $a_2=-2.6885$, $a_3=-2.8302$, $a_4=0.308\,14$, $a_5=0.766\,04$, $u=4.1185$, $c=-3.1186$, $k=-77.965$, $A=35.36$, $\alpha=4.7421$, $B=-11.912$, $C=4.5702$, and $D=17.855$; (ii) kink soliton (in blue color) for $a_1=-11.197$, $a_2=44.778$, $a_3=6.219$, $a_4=19.066$, $a_5=37.301$, $u=884.36$, $c=-13\,810$, $k=4360.2$, $A=0.940\,14$, $\alpha=8.6548$, $B=0.067\,45$, and $D=-0.979\,21$.

$$\widetilde{D} = \pm\, 8\sqrt{2}\widetilde{A}\theta_4/\sqrt{54\theta_3^2 - 27\theta_3^3 - 96\theta_2\theta_4 - 64\theta_4^2\widetilde{A}^2},$$

and

$$\alpha^2 = (18\theta_3^2 - 9\theta_3^2 - 32\theta_2\theta_4 - 64\widetilde{A}\theta_4^2)/8a_1^2\theta_4.$$

Since the exact closed form solution is known, a simple maxima-minima analysis is sufficient to distinguish parameter regimes supporting dark and bright solitary waves [15]. In this case, when $AD > BC$ one gets a bright soliton, whereas if $AD < BC$ then a dark soliton exists. Figure 1 depicts the intensity profile of a typical dark soliton. It is interesting to note that for dark solitons, from Eq. (6), the kinematic chirping is dominant at the center of the pulse, whereas the higher-order chirping is dominant away from the center (see Fig. 2). However, exactly the opposite is true for bright solitons where the center is dominated by higher-order chirping and kinematic chirping is important away from the center. This shows that there is duality between dark soliton–kinematic chirping and bright soliton–higher-order chirping.
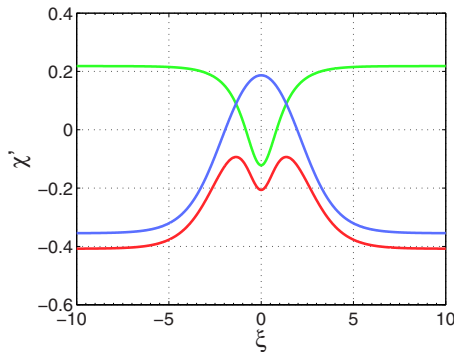

FIG. 2. (Color online) Phase profile of the dark soliton plotted in Fig. 1 (in red color). The blue curve shows the contribution from kinematic chirp whereas the green curve shows the contribution from higher-order chirp. The chirp reversal is clearly seen as peaks in the red curve.

A mutual cancellation will occur at some point(s) when both kinematic chirping and higher-order chirping are comparable and have opposite signs, and will result in chirp reversal at the point(s) of cancellation. Chirp reversal plays a significant role in fiber optics, and has attracted considerable attention recently in the context of pulse retrieval in dispersion-nonlinearity-managed optical fibers [16–19]. Chirp reversal occurs at $\xi_r = \pm\cosh^{-1}[(D\sigma_c - B)/(A - \sigma_c C)]$, provided $-4c/(3a_4 + 2a_5) > 0$ and $(D\sigma_c - B)/(A - \sigma_c C) \geqslant 1$, where $\sigma_c = \sqrt{-4c/(3a_4 + 2a_5)}$. We have plotted χ' against ξ in Fig. 2 where the chirp reversal is clearly seen as two maxima in the profile.

It should be noted that Eq. (8) with $\theta_4 \neq 0$ has no kink solutions, which are of the type

$$\sigma = \frac{A + B\tanh(\xi)}{C + D\tanh(\xi)}. \tag{10}$$

However, for $a_4 = 1/4$, θ_4 is zero, allowing the existence of this kind of solution. In this case, $k = -(\theta_1^2 + 2\theta_2^2)/3\theta_3$, $A = -[2(\theta_1 + \theta_2) \pm \sqrt{12(\theta_1^2 - \theta_2^2) - 18\theta_2 k}]/3\theta_3$, $B = \Gamma D$ where $\Gamma = -(2\theta_1 A + 3\theta_3 A^2 + 8\theta_2 A - 6k)/(6\theta_3 A + 4\theta_2 - 2\theta_1)$, $C = 1$, and $D = \pm\sqrt{(2k - 4\theta_2 A - 3\theta_3 A^2)/2\theta_1(A - \Gamma)}$. Figure 1 shows the intensity profile of a typical kink solution. These solutions, being asymmetric around $\xi = 0$, have an interesting phase profile which shows chirp reversal only once along the profile. The condition for existence of this reversal is given by $-4c/(3a_4 + 2a_5) > 0$ and $-1 \leqslant (A - \sigma_c C)/(D\sigma_c - B) \leqslant 1$, and the point of reversal is $\xi_r = \tanh^{-1}[(A - \sigma_c C)/(D\sigma_c - B)]$.

Solutions to Eq. (8) via (9) are not restricted to localized ones alone; periodic solutions also exist. In fact, it is easy to show that

$$\sigma = \frac{-3\theta_3}{8\theta_4} \pm \frac{C\cos(\xi)}{\sqrt{2} \pm \cos(\xi)} \tag{11}$$

is a periodic solution of Eq. (8), provided $\theta_1^3 + 2\theta_1^2(2\theta_2 - \frac{9\theta_3^2}{8\theta_4} + \frac{9\theta_3^2}{16\theta_4}) + \frac{8\theta_4}{3}(\frac{-3\theta_2\theta_3}{2\theta_4} + \frac{27\theta_3^3}{64\theta^2} - \frac{\theta_3^4}{128\theta_4^3})^2 = 0$ and $C = -2\theta_1^2(2\theta_2 - \frac{9\theta_3^2}{8\theta_4} + \frac{9\theta_3^2}{16\theta_4})$.

Apart from the solutions discussed above, the amplitude equation (7), albeit with different parameters, has been carefully studied in the context of the cubic quintic nonlinear Schrödinger equation [20,21]. It has been shown that this equation possesses a rich solution space, where the solutions are expressible in terms of Weierstrass functions, and the nature of the solution crucially depends upon initial conditions. A similar analysis for this system would be relevant, and will shed light on the structure of the solution space.

A natural question arises whether the model is integrable in this regime or not. Following the Ablowitz-Ramani-Segur algorithm, we investigate the singularity structure of the ordinary differential equation (8), which is obtained from an exact reduction of the original partial differential equation (2) [22,23]. Interestingly, we found that the ordinary differential equation represented by (8) possesses poles as the only movable singularities, which implies that this system indeed has the Painlevé (P) property [22,23]. Hence, we see that this

system passes the Painlevé test, and is Painlevé integrable, which guarantees the existence of regular solutions in general.

In conclusion, we have found a class of exact solutions to the NLSE system in the presence of self-steepening and self-frequency shift terms. These include localized solutions of dark-bright type, kink solutions, and periodic solutions. These solutions have nontrivial phase chirping which varies as a function of intensity and are different from that in Ref. [10] where the solutions had a trivial phase. A nontrivial connection of this system with the linear Schrödinger equation in appropriate limits is pointed out. A duality is seen between the dark-bright type of solution and kinematic higher-order chirping. A form of self-phase-modulation has been observed in this case that shows chirp reversal across the pulse profile. It is known that prechirping of pulses often leads to a better quality of pulse; in particular it is quite effective with a distributed GVD and nonlinearity [16,19,24,25]. In this context, the solutions having chirping due to initial conditions as well as dynamical conditions will provide a better control. It is noted that for some parameter values the intensity and phase of these solitons will exactly be the same as NLSE solitons, and are found to be chiral, with the direction of propagation controlled by self-steepening term. Both dark and bright solitons are found to exist in both the normal and anomalous dispersion regimes. It is seen in some cases that the intensity of these solitons will be like NLSE solitons and only the phase structure will be different. Kink solutions are found to exist in this system for special choices of parameters. The system is seen to possess the P property and hence is Painlevé integrable.

[1] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, Boston, 2001).
[2] A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 142 (1973).
[3] L. F. Mollenauer, R. H. Stolen, and J. P. Gordon, Phys. Rev. Lett. **45**, 1095 (1980).
[4] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972).
[5] Y. Kodama, J. Stat. Phys. **39**, 597 (1985).
[6] Y. Kodama and A. Hasegawa, IEEE J. Quantum Electron. **23**, 510 (1987).
[7] N. Sasa and J. Satsuma, J. Phys. Soc. Jpn. **60**, 409 (1991).
[8] R. Hirota, J. Math. Phys. **14**, 805 (1973).
[9] D. Anderson and M. Lisak, Phys. Rev. A **27**, 1393 (1983).
[10] S. L. Palacios, A. Guinea, J. M. Fernandez-Diaz, and R. D. Crespo, Phys. Rev. E **60**, R45 (1999).
[11] Z. Li, L. Li, H. Tian, and G. Zhou, Phys. Rev. Lett. **84**, 4096 (2000).
[12] C. N. Kumar and P. Durganandini, Pramana, J. Phys. **53**, 271 (1999).
[13] M. Trippenbach and Y. B. Band, Phys. Rev. A **57**, 4791 (1998).
[14] J. R. de Oliveira, M. A. de Moura, J. M. Hickmann, and A. S. L. Gomes, J. Opt. Soc. Am. B **9**, 2025 (1992).
[15] V. M. Vyas, T. S. Raju, C. N. Kumar, and P. K. Panigrahi, J. Phys. A **39**, 9151 (2006).
[16] J. D. Moores, Opt. Lett. **21**, 555 (1996).
[17] V. I. Kruglov, A. C. Peacock, and J. D. Harvey, Phys. Rev. Lett. **90**, 113902 (2003).
[18] S. Kumar and A. Hasegawa, Opt. Lett. **22**, 372 (1997).
[19] R. Atre and P. K. Panigrahi, Phys. Rev. A **76**, 043838 (2007).
[20] L. Gagnon and P. Winternitz, J. Phys. A **21**, 1493 (1988).
[21] H. W. Schurmann, Phys. Rev. E **54**, 4312 (1996).
[22] M. J. Ablowitz and A. S. Fokas, *Complex Variables: Introduction and Applications* (Cambridge University Press, Cambridge, UK, 2003).
[23] B. Grammaticos and A. Ramani, in *Integrability of Nonlinear Systems*, edited by Y. Kosmann-Schwarzbach, B. Grammaticos, and K. M. Tamizhmani, Lecture Notes in Physics Vol. 638 (Springer-Verlag, Berlin, 2004), and references therein.
[24] I. Gabitov and S. K. Turitsyn, JETP Lett. **63**, 861 (1996).
[25] I. R. Gabitov and S. K. Turitsyn, Opt. Lett. **21**, 327 (1996).