Laser theory with finite atom-field interacting time

Deshui Yu and Jingbiao Chen*
Institute of Quantum Electronics, School of Electronics Engineering and Computer Science, Peking University, Beijing 100871, People's Republic of China
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We investigate the influence of atomic transit time τ on the laser linewidth by the quantum Langevin approach. With comparing the bandwidths of cavity mode κ, atomic polarization γ_{ab}, and atomic transit broadening τ^{-1}, we study the laser linewidth in different limits. We also discuss the spectrum of fluctuations of output field and the influence of pumping statistics on the output field.The influence of atomic transit time τ on laser field has not been carefully discussed before, to our knowledge. In particular, a laser operating in the region of $\gamma_{ab} \ll \tau^{-1} \ll \kappa/2$ appears not to have been analyzed in previous laser theories. Our work could be a useful complementarity to laser theory. It is also an important theoretical foundation for the recently proposed active optical atomic clock based on bad-cavity laser mechanism.

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I. INTRODUCTION

Laser theory has been worked out for almost 50 years. In 1958, Schawlow and Townes discussed the laser linewidth for high-Q cavity (the cavity loss rate κ is much smaller than the damping rate of atomic polarization γ_{ab}). In this case, the laser linewidth, which is usually called a Schawlow-Townes parameter, can be expressed as $\Delta\nu_{\mathrm{ST}}=\kappa/(2I_o)$ [1], where I_o is the intracavity intensity of the laser light in the unit of photon number.

References [2–6] have discussed the laser linewidth in the general case, both for good-Q cavity ($\kappa \ll \gamma_{ab}$) and bad-Q cavity ($\kappa \gg \gamma_{ab}$). They have shown that laser linewidth can be generally expressed as $\Delta\nu=\Delta\nu_{\mathrm{ST}}[\gamma_{ab}/(\kappa/2+\gamma_{ab})]^2$. One could see that if $\kappa \ll \gamma_{ab}$, we arrive at the usual Schawlow-Townes diffusion parameter. On the contrary, we have $\Delta\nu \ll \Delta\nu_{\mathrm{ST}}$ if $\kappa \gg \gamma_{ab}$. However, all of these theories [2–6] are developed under the assumption that the atomic transit time τ is much longer than the damping times of any atomic variables. In this case, one could assume τ is infinite. No one has carefully studied the opposite case, $\tau \ll \gamma_{ab}^{-1}$ [7,8].

In this paper we study the general laser theory for all of the cases by the well-known Heisenberg-Langevin approach. Since the influence of atomic transit time τ on the laser field has not been carefully considered, this paper could be a useful complement for laser theory. This work is originally triggered by the recently proposed active optical clock [9] based on bad-cavity laser mechanism, and it is the important theoretical foundation for active optical clock with atomic beam. On the other hand, it is also important to atomic beam maser [10], including micromaser [7,8] and microlaser [11] with atomic beam.

The notations used throughout the paper are the same as in Ref. [2]. Following the Heisenberg-Langevin approach, in Sec. II, we define the macroscopic atomic operators, list the corresponding dynamic equations, and convert them into c-number stochastic differential equations. We also discuss the quantum and c-number correlation functions between two macroscopic atomic operators. In Sec. III, we carefully discuss the laser linewidth in different limit cases, and in Sec. IV, we calculate the spectrum of fluctuations of the field outside the cavity. Finally, our conclusions are summarized in Sec. V. All of the correlation functions of quantum Langevin noise operators are shown in the Appendix.

II. HEISENBERG-LANGEVIN METHOD

Quantum Heisenberg-Langevin equations. Here we consider the case of a two-level atomic beam interacting with a single-mode cavity as shown in Fig. 1. Before entering the cavity, all atoms are pumped onto the upper lasing state a. The lower lasing level is b. The frequency difference between a and b is ω_{ab}, and ω_l is the frequency of cavity mode. All atoms have the same velocities v, thus what we consider here is a homogeneous laser system. The atomic transit time is $\tau=l/v$, where l is the width of laser cross section. The damping rates of the populations of the upper and lower

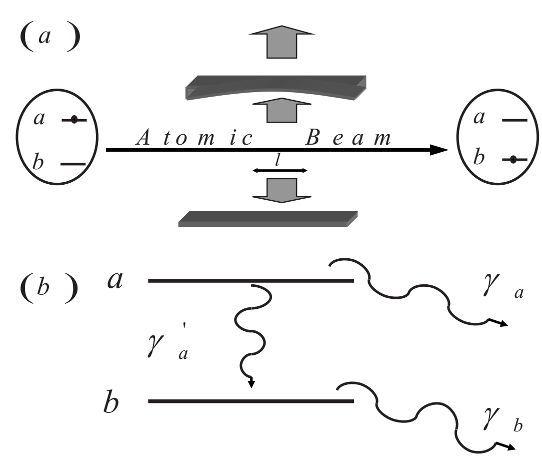

FIG. 1. The scheme of a laser system. (a) A two-level atomic beam passing through the single-mode cavity interacts with the laser field. Before entering the cavity, all atoms are prepared onto the upper level $|a\rangle$. The spontaneous decay rate of state $|a\rangle$ to state $|b\rangle$ is γ_a'. (b) The damping rates of states $|a\rangle$ and $|b\rangle$ to other states are γ_a and γ_b, respectively.

*jbchen@pku.edu.cn

levels to the other atomic levels are γ_a and γ_b, respectively. γ_a' is the spontaneous damping rate between two lasing levels, and γ_{ab} is the damping rate of the atomic polarization, which obeys the condition $2\gamma_{ab} > \gamma_a + \gamma_a' + \gamma_b$ [12]. Here we define two damping rates

$$\gamma_{\min} = \min(\gamma_a, \gamma_b, \gamma_a', \gamma_{ab}), \tag{1}$$

$$\gamma_{\max} = \max(\gamma_a, \gamma_b, \gamma_a', \gamma_{ab}). \tag{2}$$

$\gamma_{\min}$ denotes the minimum damping rate among γ_a, γ_b, γ_a', γ_{ab}, while $\gamma_{\max}$ denotes the maximum.

In order to denote the finite atom-field interacting time, we must introduce the rectangle function for the jth atom

$$\Gamma_j(t) = \Theta(t - t_j) - \Theta(t - t_j - \tau), \tag{3}$$

where $\Theta(t)$ is the unit step function $[\Theta(t)=1$ for $t>0$, $\Theta(t)=1/2$ for $t=0$, and $\Theta(t)=0$ for $t<0]$. The function $\Gamma_j(t)$ is used to describe the atomic transit time for the jth atom. One should note that if $\tau > \gamma_{\min}^{-1}$, the step functions turn on the atom-field interaction, while the damping of atomic variables closes it. If $\tau < \gamma_{\max}^{-1}$, the step functions will both turn on and shut off the atom-field interaction.

All the atomic and field operators and the quantum Heisenberg-Langevin equations for the single-atom operators have been derived in Ref. [2]. They are also applicable here, but the step function $\Theta(t-t_j)$ should be replaced by the rectangle function $\Gamma_j(t)$. The diffusion coefficients of correlation functions of two single-atom noise operators $d_{\alpha\beta}$ have been also derived in Ref. [2].

By adding up the individual atomic operators and taking into account of the corresponding injection times, one could define the macroscopic atomic operators, which can be expressed as $M(t)=-i\Sigma_j\Gamma_j(t)\sigma_-^j(t)$, $N_a(t)=\Sigma_j\Gamma_j(t)\sigma_{aa}^j(t)$, $N_b(t)=\Sigma_j\Gamma_j(t)\sigma_{bb}^j(t)$. Following the same approach introduced in Ref. [2], one could derive the dynamic equations for the field and macroscopic atomic operators,

$$\dot{a}(t) = -\frac{\kappa}{2}a(t) + gM(t) + F_\kappa(t), \tag{4}$$

$$\dot{N}_a(t) = R(1-A) - (\gamma_a + \gamma_a')N_a(t) - g[a^+(t)M(t) + M^+(t)a(t)] + F_a(t), \tag{5}$$

$$\dot{M}(t) = -RC - (\gamma_{ab} + i\Delta)M(t) + g[N_a(t) - N_b(t)]a(t) + F_M(t), \tag{6}$$

$$\dot{N}_b(t) = -RB - \gamma_b N_b(t) + \gamma_a' N_a(t) + g[a^+(t)M(t) + M^+(t)a(t)] + F_b(t), \tag{7}$$

where we have included the detuning $\Delta=\omega_l-\omega_{ab}$ and the macroscopic noise operators (Langevin forces) are defined as

$$F_a(t) = \sum_j \dot{\Gamma}_j(t)\sigma_{aa}^j(t) + \sum_j \Gamma_j(t) f_a^j(t) - R(1-A), \tag{8}$$

$$F_b(t) = \sum_j \dot{\Gamma}_j(t)\sigma_{bb}^j(t) + \sum_j \Gamma_j(t) f_b^j(t) + RB, \tag{9}$$

$$F_M(t) = -i\sum_j \dot{\Gamma}_j(t)\sigma_-^j(t) - i\sum_j \Gamma_j(t) f_-^j(t) + RC, \tag{10}$$

with $\langle\sigma_{aa}^j(t_j+\tau)\rangle_Q=A$, $\langle\sigma_{bb}^j(t_j+\tau)\rangle_Q=B$, and $-i\langle\sigma_-^j(t_j+\tau)\rangle_Q=C$ denoting the quantum average values of atomic operators for the jth atom when it exists from the cavity. R is the mean pumping rate, which is defined in Ref. [2]. The average values of the above Langevin forces are all zero, $\langle F_b(t)\rangle=0$, $\langle F_a(t)\rangle=0$, $\langle F_M(t)\rangle=0$.

Using the above definitions of noise operators, we find the correlation functions of macroscopic noise forces can be generally written in the form

$$\langle F_\alpha(t)F_\beta(t')\rangle = D_{\alpha\beta}^{(0)}\delta(t-t') + D_{\alpha\beta}^{(1)}\delta(t-t'-\tau) + D_{\alpha\beta}^{(2)}\delta(t-t'+\tau), \tag{11}$$

where $D_{\alpha\beta}^{(i)}$, $(\alpha,\beta=a,b,\ M,\ M^+;\ i=0,1,2)$ are the quantum diffusion coefficients. In the Appendix, we list all of the quantum correlation functions.

c-number correlation functions. Above, we have discussed the quantum Langevin equations for the macroscopic atomic variables. Following the same approach of Ref. [2], one could derive the c-number stochastic Langevin equations under the condition of choosing some particular ordering for products of atomic and field operators. All of the dynamic equations for c-number stochastic variables could be found in Ref. [2]. On the other hand, the quantum noise operators should also be converted into the c-number noise variables $\mathcal{F}_\alpha(t)$ $(\alpha=\kappa,\ \mathcal{M},\ a,$ and $b)$, whose correlation functions $\langle\mathcal{F}_\alpha(t)\mathcal{F}_\beta(t')\rangle$ could be expressed as

$$\langle\mathcal{F}_\alpha(t)\mathcal{F}_\beta(t')\rangle = \mathcal{D}_{\alpha\beta}^{(0)}\delta(t-t') + \mathcal{D}_{\alpha\beta}^{(1)}\delta(t-t'-\tau) + \mathcal{D}_{\alpha\beta}^{(2)}\delta(t-t'+\tau), \tag{12}$$

where $\mathcal{D}_{\alpha\beta}^{(i)}$ are the c-number Langevin diffusion coefficients. Following the same approach introduced in Ref. [6], one could get the relations between $D_{\alpha\beta}^{(i)}$ and $\mathcal{D}_{\alpha\beta}^{(i)}$,

$$\sum_i \mathcal{D}_{aa}^{(i)} = \sum_i D_{aa}^{(i)} - g[\langle\mathcal{M}^*(t)\mathcal{A}(t)\rangle + \langle\mathcal{A}^*(t)\mathcal{M}(t)\rangle], \tag{13}$$

$$\sum_i \mathcal{D}_{\mathcal{M}\mathcal{M}}^{(i)} = \sum_i D_{MM}^{(i)} + 2g\langle\mathcal{M}(t)\mathcal{A}(t)\rangle, \tag{14}$$

$$\sum_i \mathcal{D}_{bb}^{(i)} = \sum_i D_{bb}^{(i)} - g[\langle\mathcal{M}^*(t)\mathcal{A}(t)\rangle + \langle\mathcal{A}^*(t)\mathcal{M}(t)\rangle], \tag{15}$$

$$\sum_i \mathcal{D}_{\mathcal{M}^*\mathcal{M}^*}^{(i)} = \sum_i D_{M^+M^+}^{(i)} + 2g\langle\mathcal{A}^*(t)\mathcal{M}^*(t)\rangle, \tag{16}$$

$$\sum_i \mathcal{D}_{ab}^{(i)} = \sum_i D_{ab}^{(i)} + g[\langle\mathcal{M}^*(t)\mathcal{A}(t)\rangle + \langle\mathcal{A}^*(t)\mathcal{M}(t)\rangle], \tag{17}$$

$$\sum_i \mathcal{D}_{a\mathcal{M}}^{(i)} = \sum_i D_{aM}^{(i)}, \tag{18}$$

$$\sum_i \mathcal{D}_{b\mathcal{M}}^{(i)} = \sum_i D_{bM}^{(i)}, \tag{19}$$

$$\sum_i \mathcal{D}_{\mathcal{M}^*\mathcal{M}}^{(i)} = \sum_i D_{M^+M}^{(i)}, \tag{20}$$

$$\sum_i \mathcal{D}_{\mathcal{M}^*a}^{(i)} = \sum_i D_{M^+a}^{(i)}, \tag{21}$$

$$\sum_i \mathcal{D}_{\mathcal{M}^*b}^{(i)} = \sum_i D_{M^+b}^{(i)}, \tag{22}$$

where $i=0,1,2$. We can only get the relation between two summations $\Sigma_i \mathcal{D}_{\alpha\beta}^{(i)}$ and $\Sigma_i D_{\alpha\beta}^{(i)}$, not the expression of $\mathcal{D}_{\alpha\beta}^{(i)}$.

Steady-state solutions. The steady-state solutions for the mean values of the field and atomic variables for laser operation are obtained by dropping the noise terms of the c-number Langevin equations and setting the time derivatives equal to zero. The analytical solutions are very complex, and one could numerically solve the steady-state equations. Here, we discuss two limit cases:

(1) $\gamma_{\max}\tau \ll 1$. Since the atomic transit time τ is much shorter than the damping times of atomic variables, one could ignore the effect of atomic damping. It is easy to get the following steady-state values:

$$|\mathcal{A}_o|^2 = (R/\kappa)B, \tag{23}$$

$$\mathcal{N}_{ao} = \frac{R\tau}{2}\left(1 + \frac{\Delta^2}{\Omega_R^2} - \frac{\mathrm{Re}[C]}{\Omega_R \tau}\right), \tag{24}$$

$$\mathcal{N}_{bo} = \frac{\Omega^2}{\Omega_R^2}\frac{R\tau}{2}\left(1 + \frac{\Omega_R}{\Omega^2 \tau}\mathrm{Re}[C]\right). \tag{25}$$

The sum rule is given by $A+B=1$, $\mathcal{N}_{ao}+\mathcal{N}_{bo}=R\tau$, and the values of A, B, and C can be expressed as

$$A = \cos^2(\Omega_R\tau) + \frac{\Delta^2}{\Omega_R^2}\sin^2(\Omega_R\tau), \tag{26}$$

$$B = \frac{\Omega^2}{\Omega_R^2}\sin^2(\Omega_R\tau), \tag{27}$$

$$C = \frac{\Omega}{\Omega_R}\sin(\Omega_R\tau)\cos(\Omega_R\tau) - i\frac{\Delta\Omega}{\Omega_R^2}\sin^2(\Omega_R\tau), \tag{28}$$

where $\Omega_R \equiv \sqrt{\Omega^2+\Delta^2}$ is the total Rabi frequency. From the above equations one could obtain the steady-state value of photon number for different detuning Δ. We have the following condition:

$$\frac{(\tau\sqrt{g^2 I_0 + \Delta^2})^2}{(\tau\sqrt{g^2 R/\kappa})^2} = \sin^2 \tau\sqrt{g^2 I_0 + \Delta}, \tag{29}$$

from which the photon number can be determined.

In the resonant case ($\Delta=0$), we could assume that $\mathcal{A}_o$ and $\mathcal{M}_o$ are both real. If $g\tau\sqrt{I_0}=m\pi+\frac{\pi}{2}$, we have $\mathcal{N}_{ao}=\mathcal{N}_{bo}$, which denotes that the number of atoms on level a is equal to the number on level b in the atom-field interacting region. One could also obtain the results $A\approx 0$, $B\approx 1$, and $C\approx 0$. That is to say all atoms are on the lower lasing level $|b\rangle$ when they exit from the cavity, and each atom contributes its whole energy to the laser field.

(2) $\gamma_{\min}\tau \gg 1$. The atomic transit time τ is much larger than the damping time of atomic variables. Therefore, we have $A=0$, $B=0$, and $C=0$. The resonant case has been discussed in Ref. [2].

Amplitude and phase quadrature components. Following the same approach of Ref. [2], we could make a linearization to the c-number Langevin equations, and then reexpressing them in terms of the Fourier transformations. Solving the algebraic equations, we could express the amplitude and phase quadrature components $\delta X(\omega)$ and $\delta Y(\omega)$, which are defined in Ref. [2], as

$$\delta X(\omega) \approx -\frac{i\omega(\gamma_{ab}+\kappa/2-i\omega)}{g(\gamma_b - i\omega)\Sigma(\omega)}\Bigg(\frac{1}{2}[\mathcal{F}_{\mathcal{M}}(\omega) + \mathcal{F}_{\mathcal{M}}^*(-\omega)](\gamma_b - i\omega) + g|\mathcal{A}_o|\frac{\gamma_b - \gamma_a' - i\omega}{\gamma_b + \gamma_a' - i\omega}\mathcal{F}_a(\omega) - g|\mathcal{A}_o|\mathcal{F}_b(\omega)\Bigg), \tag{30}$$

and

$$\delta Y(\omega) \approx \left(1 + \frac{\Upsilon(\omega)}{\Sigma(\omega)}\right)\frac{g}{2\omega(\kappa/2+\gamma_{ab}-i\omega)}[\mathcal{F}_{\mathcal{M}}(\omega) - \mathcal{F}_{\mathcal{M}}^*(-\omega)], \tag{31}$$

where

$$\Upsilon(\omega) = \left(i\frac{\Delta}{g}(\kappa/2 - i\omega)\right)^2, \tag{32}$$

and

$$\Sigma(\omega) = -\left(\frac{\omega}{g}(\kappa/2+\gamma_{ab}-i\omega)\right)^2 - \left(\frac{\Delta}{g}(\kappa/2-i\omega)\right)^2 + 2ig|\mathcal{A}_o|^2\frac{\omega(\kappa/2+\gamma_{ab}-i\omega)(\kappa-i\omega)(\gamma_a+\gamma_b-2i\omega)}{g(\gamma_b-i\omega)(\gamma_a+\gamma_a'-i\omega)}. \tag{33}$$

In the resonant case ($\Delta=0$), $\delta X(\omega)$ and $\delta Y(\omega)$ are the same results of Ref. [2].

In the next section, we will discuss the laser linewidth and spectrum by using the quardrature components of the field fluctuations.

III. LASER LINEWIDTH AND SPECTRUM

From Eq. (31), one could derive the spectrum of phase quadrature component of field fluctuations inside the cavity, $(\delta Y^2)_\omega$, which is defined in Ref. [2], and could be expressed as

$$(\delta Y^2)_\omega = \eta(\omega)\frac{g^2}{4\omega^2[(\kappa/2+\gamma_{ab})^2+\omega^2]}\big(2\gamma_{ab}(\mathcal{N}_{ao}+\mathcal{N}_{bo}) + 2R(A+B) + R(C-C^*)^2 p - R2\,\mathrm{Re}\{[\mathcal{G}_{--}(\tau) + \mathcal{G}_{++}(\tau) + \mathcal{G}_{+-}(\tau) + \mathcal{G}_{-+}(\tau)]e^{i\omega\tau}\}\big). \tag{34}$$

Here we have defined $\eta(\omega)\equiv|1+\frac{\Upsilon(\omega)}{\Sigma(\omega)}|^2\geqslant 1$, and, in the resonant case, we have $\eta(\omega)\equiv 1$. In the usual case, we have $\mathcal{N}_{ao}\approx\mathcal{N}_{bo}$. One could see that in the limit of $(\kappa/2+\gamma_{\min})\tau \gg 1$, all of the correlation functions $\mathcal{G}_{\alpha\beta}(\pm\tau)$ are zero, and $A=B=C=0$. Thus $(\delta Y^2)_\omega$ is the same as the result of Ref. [2] in the resonant case.

Phase-diffusion coefficient and linewidth. Let us consider the spectrum of phase quadrature, Eq. (34), which yields the phase-diffusion coefficient and the laser line shape. For a small fluctuation of laser phase, the spectrum of phase fluctuations is simply related to the spectrum of the phase quadrature component of the field fluctuations, namely,

$$(\delta\varphi^2)_\omega = \frac{1}{I_o}(\delta Y^2)_\omega. \tag{35}$$

In the following, we discuss the laser phase noise in different limit cases.

A. $(\kappa/2+\gamma_{\max})\tau\ll 1$

In this case, the bandwidth τ^{-1} of atomic transit broadening is larger than that of the sum of loss rate of cavity and the maximum damping rate of atomic variables, $(\kappa/2+\gamma_{\max})$. Thus, when considering the atom-field interaction, one could ignore the influence of all damping of cavity and atomic variables. As we have discussed in Sec. II, in this case we have $A+B=1$, $\mathrm{Im}\,C=-\frac{\Delta\Omega}{\Omega_R^2}\sin^2(\Omega_R\tau)$, and the correlation functions are given by $\mathcal{G}_{--}(\tau)+\mathcal{G}_{+-}(\tau)=1$, $\mathcal{G}_{++}(\tau)=\mathcal{G}_{-+}(\tau)=0$. Finally, the phase quadrature component is given by

$$(\delta\varphi^2)_\omega = \frac{\eta(\omega)}{I_o\omega^2}\frac{(\kappa/2+\gamma_{ab})^2}{(\kappa/2+\gamma_{ab})^2+\omega^2}\frac{g^2}{4(\kappa/2+\gamma_{ab})^2}[4\gamma_{ab}\mathcal{N}_{ao} + 2R(1-\cos\omega\tau)+4\kappa Bp\Delta^2/g^2]. \tag{36}$$

As one could see, $\eta(\omega)$, which is caused by the field detuning, will increase the phase noise. Due to the low-pass factor $\frac{(\kappa/2+\gamma_{ab})^2}{(\kappa/2+\gamma_{ab})^2+\omega^2}$, we could use $(\kappa/2+\gamma_{ab})$ to replace the frequency in cosine function, that is, $\cos(\kappa/2+\gamma_{ab})\tau$. Using the fact $(\kappa/2+\gamma_{ab})\tau\ll 1$, we are left with

$$(\delta\varphi^2)_\omega \approx \frac{\eta(\omega)}{\omega^2}\frac{(\kappa/2+\gamma_{ab})^2}{(\kappa/2+\gamma_{ab})^2+\omega^2}\left[\mathcal{D}_{\mathrm{ST}}^{(1)}\left(\frac{\gamma_{ab}}{\kappa/2+\gamma_{ab}}\right)^2+\mathcal{D}_{\mathrm{ST}}^{(2)} + \mathcal{D}(\Delta)\right], \tag{37}$$

where $\mathcal{D}_{\mathrm{ST}}^{(1)}$ and $\mathcal{D}_{\mathrm{ST}}^{(2)}$ are the Schawlow-Townes diffusion coefficients

$$\mathcal{D}_{\mathrm{ST}}^{(1)} = \frac{g^2\mathcal{N}_{ao}}{I_o\gamma_{ab}}, \quad \mathcal{D}_{\mathrm{ST}}^{(2)} = \frac{Rg^2\tau^2+\kappa}{4I_o}, \tag{38}$$

and $\mathcal{D}(\Delta)=\kappa Bp(\frac{\Delta}{\kappa/2+\gamma_{ab}})^2$, which is completely caused by the field detuning. $\mathcal{D}_{\mathrm{ST}}^{(1)}$ has the form of the usual Schawlow-Townes parameter [2], which is caused by the damping of the laser system, $(\kappa/2+\gamma_{ab})$, while $\mathcal{D}_{\mathrm{ST}}^{(2)}$ has the form of the micromaser spectrum [8], which is caused by the finite interacting time τ. Here we have phenomenally written out κ in $\mathcal{D}_{\mathrm{ST}}^{(2)}$ [13]. However, we always have the relation of $Rg^2\tau^2 \gg \kappa$ due to the threshold condition. Thus we could leave out κ. As one could see, although the damping rate of atomic polarization γ_{ab} is much smaller than the atomic transit bandwidth τ^{-1}, they both contribute laser linewidth. One could not ignore either one.

In microlaser theory [7,8], which is discussed by the density operator approach (photon statistical approach), the authors have assumed the damping rate of atomic polarization is so small that one could ignore its influence, and the linewidth of microlaser is given by $\mathcal{D}_{\mathrm{ST}}^{(2)}$. Here we arrive at the same result by using the quantum Langevin approach. However, besides $\mathcal{D}_{\mathrm{ST}}^{(2)}$, the laser linewidth is also associated with $\mathcal{D}_{\mathrm{ST}}^{(1)}$. Thus one should not simply ignore the influence of the damping of atomic polarization on the linewidth of laser field as done in Ref. [8].

We define the final laser linwidth is $\mathcal{D}$. Therefore, in this case, we have

$$\mathcal{D} = \mathcal{D}_{\mathrm{ST}}^{(1)}\left(\frac{\gamma_{ab}}{\kappa/2+\gamma_{ab}}\right)^2+\mathcal{D}_{\mathrm{ST}}^{(2)}+\mathcal{D}(\Delta). \tag{39}$$

We are not interested in $\mathcal{D}(\Delta)$, and only consider the resonant case. Now we compare $\mathcal{D}_{\mathrm{ST}}^{(1)}$ and $\mathcal{D}_{\mathrm{ST}}^{(2)}$. In the case of maximal photon number (shown in Sec. II), the steady-state value of $\mathcal{N}_{ao}$ is given by $R\tau/2$. Thus we are left with $\mathcal{D}_{ST}^{(1)} \gg \mathcal{D}_{\mathrm{ST}}^{(2)}$ due to $\gamma_{ab}^{-1}\gg\tau$. Further, if $\gamma_{ab}\gg\kappa$, we have

$$\mathcal{D}_{\mathrm{ST}}^{(1)}\left(\frac{\gamma_{ab}}{\kappa/2+\gamma_{ab}}\right)^2 \gg \mathcal{D}_{\mathrm{ST}}^{(2)}, \tag{40}$$

and the final linewidth is

$$\mathcal{D}\approx\mathcal{D}_{\mathrm{ST}}^{(1)}. \tag{41}$$

On the other hand, if $(\gamma_{ab}/\kappa)\ll\kappa\tau\ll 1$, we have

$$\mathcal{D}_{\mathrm{ST}}^{(1)}\left(\frac{\gamma_{ab}}{\kappa/2+\gamma_{ab}}\right)^2 \ll \mathcal{D}_{\mathrm{ST}}^{(2)}. \tag{42}$$

Thus the laser linewidth is almost determined by $\mathcal{D}_{\mathrm{ST}}^{(2)}$,

$$\mathcal{D}\approx\mathcal{D}_{\mathrm{ST}}^{(2)}. \tag{43}$$

It coincides with the result of Ref. [8]. Here, we have shown more conditions for applying this result.

B. $\gamma_{\max}\ll\tau^{-1}\ll\kappa/2$

In this case, as in the recently proposed active optical clock [9] with atomic beam, we can assume $\cos(\kappa/2+\gamma_{ab})\tau$ to be zero, since $\kappa\tau/2\gg 1$ and the cosine function has oscillated for many times. Consequently, the average is zero. Therefore, the phase quadrature component of the field fluctuations can be expressed as

$$(\delta\varphi^2)_\omega \approx \frac{\eta(\omega)}{\omega^2}\frac{(\kappa/2+\gamma_{ab})^2}{(\kappa/2+\gamma_{ab})^2+\omega^2}\left(\mathcal{D}_{\mathrm{ST}}^{(1)}\frac{\gamma_{ab}^2}{\kappa^2/4}+\mathcal{D}_{\mathrm{ST}}^{(2)}\frac{\tau^{-2}}{\kappa^2/4} + \mathcal{D}(\Delta)\right), \tag{44}$$

In the resonant case, we have

$$\mathcal{D}_{\mathrm{ST}}^{(2)}\frac{\tau^{-2}}{\kappa^2/4}=\frac{Rg^2}{4I_o}(\kappa/2)^{-2}. \tag{45}$$

One could see that if the bandwidth of cavity mode κ is much larger than that of the atomic transit broadening τ^{-1}, laser linewidth is unaffected by the transit time τ. It could be easily understood. The storage time for a photon in cavity is about κ^{-1}. The atom-photon interacting time is no longer limited by τ, but κ^{-1}. Thus, the atomic transit time τ is unimportant in the laser system.

From the condition $\gamma_{ab}\tau \ll 1$ we also know that

$$\mathcal{D}_{\mathrm{ST}}^{(1)}\frac{\gamma_{ab}^2}{\kappa^2/4}\ll \mathcal{D}_{\mathrm{ST}}^{(2)}\frac{\tau^{-2}}{\kappa^2/4}. \tag{46}$$

Therefore, the final laser linewidth is given by

$$\mathcal{D}\approx g^2/\kappa, \tag{47}$$

where we have used the fact $I_o=R/\kappa$.

This case has never been pointed out in any previous laser theories. The potential applications of the result include the most recently proposed active optical clock with atomic beam [9]. Moreover, the well-know ammonia maser [10] also follows in this case. However, one should note that the thermal photon in the cavity will play a very important role at the microwave regime [13].

C. $\tau^{-1}\ll\gamma_{\min}$

As discussed in Ref. [2], the threshold condition is $\gamma_b > \gamma_a'$. In this case, when atoms left the atom-field interaction region, no atoms are on the states a and b. Therefore, we have $A=0$, $B=0$, and $C=0$; and all of the correlation functions $\mathcal{G}_{\alpha\beta}(\pm\tau)$ are zero. The laser linewidth is given by

$$(\delta\varphi^2)_\omega \approx \eta(\omega)\frac{\mathcal{D}_{\mathrm{ST}}^{(1)}}{\omega^2}\frac{(\kappa/2+\gamma_{ab})^2}{(\kappa/2+\gamma_{ab})^2+\omega^2}\left(\frac{\gamma_{ab}}{\kappa/2+\gamma_{ab}}\right)^2. \tag{48}$$

From Eq. (48), one could see that the laser linewidth does not depend on the atomic transit time τ. It could be understood that, due to $\gamma_{ab}^{-1}\ll\tau$, the atom-field interacting time is limited by the damping time of atomic polarization. In this case, the final linewidth is given by

$$\mathcal{D}=\mathcal{D}_{\mathrm{ST}}^{(1)}\left(\frac{\gamma_{ab}}{\kappa/2+\gamma_{ab}}\right)^2. \tag{49}$$

This formulation has been experimentally demonstrated in Ref. [14]

Above, we have discussed the laser linewidth in different limit cases. As one could see that the factor $\frac{(\gamma_{ab}+\kappa/2)^2}{(\gamma_{ab}+\kappa/2)^2+\omega^2}$ is present in all of the above limit cases. This factor origins from the atomic memory effect [15,16]. During the memory time $(\gamma_{ab}+\kappa/2)$, the spontaneous-emission events are correlated. Here we do not discuss it again.

Laser spectrum. After getting the laser linewidth, one could derive the power spectrum inside cavity, $\mathcal{S}_{\mathcal{A}}(\omega)$. Following the same process as in Ref. [15], we have

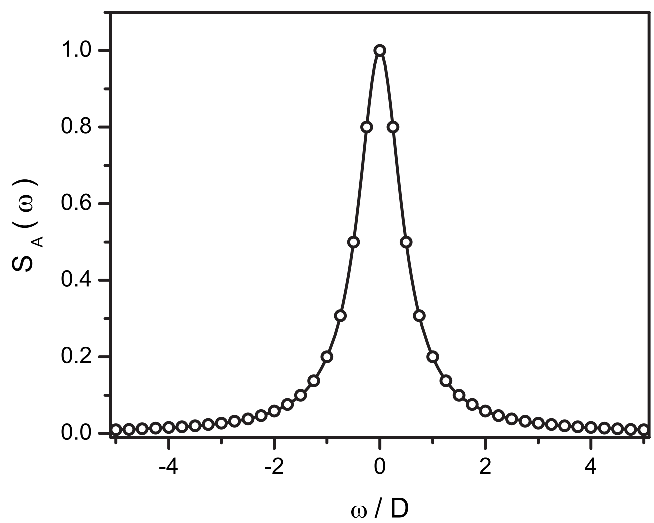

FIG. 2. The comparison between Eq. (50) (solid line) and Eq. (51) (dotted line) with $\mathcal{D}/\kappa=10^{-5}$ and $2\gamma/\kappa=10$.

$$\mathcal{S}_{\mathcal{A}}(\omega)\propto \exp\left(\frac{\mathcal{D}}{2(\kappa/2+\gamma_{ab})}\right)\sum_{n=0}^{\infty}\left[\frac{1}{n!}\left(-\frac{\mathcal{D}}{2(\kappa/2+\gamma_{ab})}\right)^n \times\frac{\mathcal{D}/2+n(\kappa/2+\gamma_{ab})}{(\omega-\omega_{ab})^2+[\mathcal{D}/2+n(\kappa/2+\gamma_{ab})]^2}\right]. \tag{50}$$

Usually, it is sufficiently to only consider the first term $n=0$,

$$\mathcal{S}_{\mathcal{A}}(\omega)\propto\frac{\mathcal{D}/2}{(\omega-\omega_{ab})^2+(\mathcal{D}/2)^2}, \tag{51}$$

which has the usual Lorentzian form. In Fig. 2, we show the comparison between Eqs. (50) and (51). As one could see, they are almost the same. Thus we can assume that the laser field has the form of $e^{-\mathcal{D}t}$, where $\mathcal{D}$ is the field bandwidth.

In the theory of microlaser spectrum, the authors of Ref. [8] have assumed that the laser field has a form of $e^{-\mathcal{D}t}$. That is to say the author has assumed that the microlaser spectrum that is a Lorentz line shape. However, it has not been proved before. Here, we have shown that this assumption is correct.

IV. SPECTRUM OF THE OUTPUT FIELD

Above we have discussed the laser field inside the cavity. On the other hand, one is interested in the output field. The relation between fields inside and outside the cavity has been established in Refs. [17–20]. In this section we investigate the spectrum of fluctuations for the field transmitted through the cavity port. From Ref. [2], the spectrum of the output field can be expressed as $V_{\mathcal{A}}=1+4\kappa(\delta X^2)_\omega$, where $(\delta X^2)_\omega$ is the spectrum of amplitude quadrature component and could be derived from Eq. (30) following the same approach as Ref. [2]. The first term on the right-hand side corresponds to the shot-noise contribution. For a coherent state, we have $V_{\mathcal{A}}=1$. Therefore, $V_{\mathcal{A}}<1$ means we have found squeezing in a quadrature component, and $V_{\mathcal{A}}=0$ corresponds the complete squeezing at some frequency ω [3]. The spectrum defined in this way corresponds to a homodyne detection. Actually, this spectrum corresponds to the normalized photocurrent obtained in a homodyne measurement of the field quadrature component.

Now we carefully consider the spectrum of the amplitude fluctuations of the output field. We introduce the dimensionless parameters

$$r \equiv R/\kappa, \quad a \equiv \gamma_a/\kappa, \quad a' \equiv \gamma_a'/\kappa,$$

$$b \equiv \gamma_b/\kappa, \quad c \equiv \gamma_{ab}/\kappa, \quad e \equiv g/\kappa,$$

$$t \equiv \tau\kappa, \quad x \equiv \omega/\kappa,$$

where r denotes the number of atoms entering the cavity during time κ^{-1}, and only consider the resonant case for simplicity. In the following, we consider the spectrum in two limit cases.

A. $\gamma_{\max} \ll \tau^{-1}$

For simplicity, we only consider the case of maximal photon number. As we have discussed before, the atomic states are given by $A=0$, $B=1$, and $C=0$ when atom exiting cavity, and we also have the relation $I_0=r$ and $g\tau\sqrt{I_0}=\frac{\pi}{2}$. (The spectrum is independent of m, and here we choose $m=0$.) In this case, by using the above spectral parameters, the spectrum of amplitude fluctuations is given by

$$V_{\mathcal{A}}(x) = 1 + 4\frac{(a+a')^2+x^2}{e^2[(a+b)^2+4x^2](1+x^2)|D(x)+2r|^2}\Bigg\{-rb^2 + r^2e^2(a't-p) + r(b^2+x^2)\left(1+(c-a-a')\frac{t}{2} + \frac{1}{2}\cos xt\right) + r^2e^2\frac{(b-a)^2+x^2}{(a+a')^2+x^2}[(a+a')t-p] - 2r^2e^2\left[\frac{(b-a')(a+a')+x^2}{(a+a')^2+x^2}\left(1+\frac{1}{2}bt-a't-(1-p)\cos xt\right) - x\frac{b-a-2a'}{(b+a')^2+x^2}(1-p)\sin xt\right]\Bigg\}, \quad (52)$$

with the following shorthand:

$$D(x) = \frac{-ix(1/2+c-ix)(b-ix)(a+a'-ix)}{e^2(a+b-i2x)(1-ix)}. \quad (53)$$

The condition $\gamma_{\max}\tau \ll 1$ requires that $at \ll 1$, $bt \ll 1$, $ct \ll 1$, and $a't \ll 1$. In Fig. 3, we show $V_{\mathcal{A}}(\omega)$ as a function of ω for different statistical parameters p. For regular pumping $p=1$, one can obtain complete noise quieting at zero frequency. For larger frequency, $V_{\mathcal{A}}(\omega)$ increases to the shot-noise level, and is saturated. For Poissonian pumping, $p=0$, $V_{\mathcal{A}}(\omega)$ is always on the shot-noise level. In Fig. 4, we show $V_{\mathcal{A}}(\omega)$ changing with ω for different pumping parameter r. As one could see, in the case of maximal photon number, the pumping rate does not affect the spectrum fluctuations of the output field.

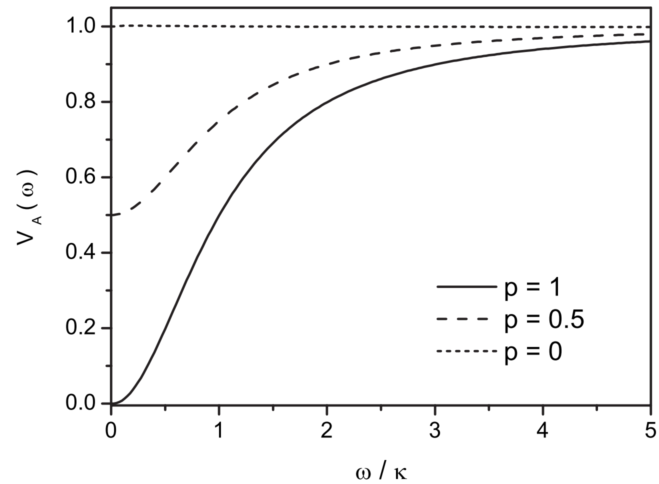

FIG. 3. Normalized spectrum of amplitude fluctuations for different statistical parameters p. Solid line, $p=1$ (regular pimping); dashed line, $p=0.5$ (intermediate case); dotted line, $p=0$ (Poissonian pumping). For all curves $a'=0$, $a=0.001$, $b=0.1$, $c=0.5$, $r=10^3$, $t=0.1$.

B. $\gamma_{\min} \gg \tau^{-1}$

In this case, the atom-field interaction is turn down by the damping of the atomic polarization. We have $A=0$, $B=0$, and $C=0$, and all of the correlation functions $\mathcal{G}_{\alpha\beta}(\pm\tau)$ are zero. Therefore, the form of $V_{\mathcal{A}}(\omega)$ is the same as the result in Ref. [2]. Here we do not discuss this case again.

V. SUMMARY

In summary, we have analyzed the influence of atomic transit time τ on the laser linewidth by the quantum Lange-

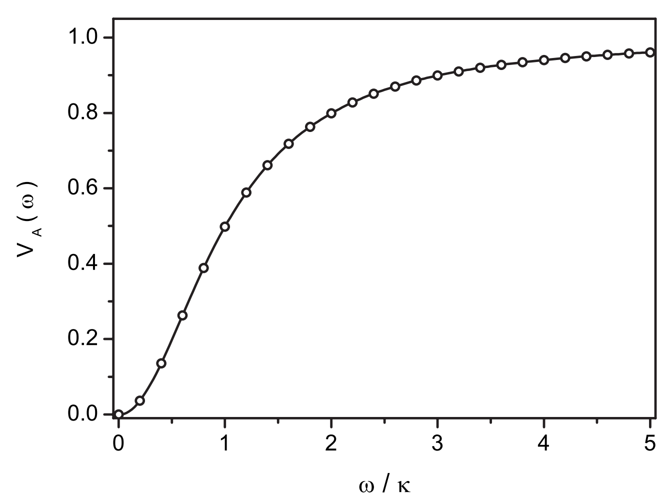

FIG. 4. Normalized spectrum of amplitude fluctuations for different pumping parameters r. Solid line, $r=10^2$; dotted line, 10^4. For both curves $a'=0$, $a=0.001$, $b=0.1$, $c=0.5$, $t=0.1$, $p=1$.

vin approach. As simplifying assumptions, the active medium considered here is homogenously broadened, and we have only dealt with a zero-temperature case. With comparing the bandwidths of cavity mode κ, atomic polarization γ_{ab}, and atomic transit broadening τ^{-1}, we discuss the laser linewidth in different limits. The laser theory of $\gamma_{ab} \gg \tau^{-1}$ has been carefully discussed by Ref. [2]. However, the contrary case has not been considered before. Thus our work could be a complementarity to the laser theory. It is also important to the new active optical clock proposed in Ref. [9]. Besides the field inside cavity, we also consider the spectrum of fluctuations of the output field and the influence of the role of pumping statistics on the output spectrum.

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APPENDIX: QUANTUM CORRELATION FUNCTIONS

Below we list all of the quantum correlation functions for the quantum macroscopic noise operators. These functions could be calculated by following the approach introduced in Ref. [21], and using the fact that all atoms are on the upper level when they enter the cavity,

$$\langle F_a(t)F_a(t')\rangle = [(\gamma_a+\gamma_a')\langle N_a(t)\rangle + R(1-p+A-A^2P)] \times \delta(t-t') + R(Ap-\mathcal{G}_{aa}(\tau))\delta(t-t'-\tau) + R(Ap-\mathcal{G}_{aa}(-\tau))\delta(t-t'+\tau), \tag{A1}$$

$$\langle F_b(t)F_b(t')\rangle = [\gamma_b\langle N_b(t)\rangle + \gamma_a'\langle N_a(t)\rangle + RB(1-Bp)]\delta(t-t') - R\mathcal{G}_{bb}(\tau)\delta(t-t'-\tau) - R\mathcal{G}_{bb}(-\tau)\delta(t-t'+\tau), \tag{A2}$$

$$\langle F_M(t)F_M(t')\rangle = -RC^2p\,\delta(t-t') + R\mathcal{G}_{--}(\tau)\delta(t-t'-\tau) + R\mathcal{G}_{--}(-\tau)\delta(t-t'+\tau), \tag{A3}$$

$$\langle F_M^+(t)F_M^+(t')\rangle = -RC^{*2}p\,\delta(t-t') + R\mathcal{G}_{++}(\tau)\delta(t-t'-\tau) + R\mathcal{G}_{++}(-\tau)\delta(t-t'+\tau), \tag{A4}$$

$$\langle F_M(t)F_M^+(t')\rangle = [R(B-C^*Cp) + (2\gamma_{ab}-\gamma_b)\langle N_b(t)\rangle + \gamma_a'\langle N_a(t)\rangle]\delta(t-t') - R\mathcal{G}_{-+}(\tau)\delta(t-t'-\tau) - R\mathcal{G}_{-+}(-\tau)\delta(t-t'+\tau), \tag{A5}$$

$$\langle F_M^+(t)F_M(t')\rangle = [R(1+A-C^*Cp) + (2\gamma_{ab}-\gamma_a-\gamma_a') \times\langle N_a(t)\rangle]\delta(t-t') - R\mathcal{G}_{+-}(\tau)\delta(t-t'-\tau) - R\mathcal{G}_{+-}(-\tau)\delta(t-t'+\tau), \tag{A6}$$

$$\langle F_a(t)F_b(t')\rangle = -[RABp + \gamma_a'\langle N_a(t)\rangle]\delta(t-t') - R\mathcal{G}_{ab}(\tau)\delta(t-t'-\tau) - R[\mathcal{G}_{ab}(-\tau) - Bp]\delta(t-t'+\tau), \tag{A7}$$

$$\langle F_b(t)F_a(t')\rangle = -[RABp + \gamma_a'\langle N_a(t)\rangle]\delta(t-t') - R[\mathcal{G}_{ba}(\tau) - Bp]\delta(t-t'-\tau) - R\mathcal{G}_{ba}(-\tau)\delta(t-t'+\tau), \tag{A8}$$

$$\langle F_a(t)F_M(t')\rangle = -RACp\,\delta(t-t') + iR\mathcal{G}_{a-}(\tau)\delta(t-t'-\tau) + R[i\mathcal{G}_{a-}(-\tau) + Cp]\delta(t-t'+\tau), \tag{A9}$$

$$\langle F_M(t)F_a(t')\rangle = [RC(1-Ap) + (\gamma_a+\gamma_a')\langle M(t)\rangle]\delta(t-t') + R[Cp + i\mathcal{G}_{-a}(\tau)]\delta(t-t'-\tau) + iR\mathcal{G}_{-a}(-\tau)\delta(t-t'+\tau), \tag{A10}$$

$$\langle F_a(t)F_M^+(t')\rangle = [RC^*(1-Ap) + (\gamma_a+\gamma_a')\langle M(t)\rangle]\delta(t-t') - iR\mathcal{G}_{a+}(\tau)\delta(t-t'-\tau) + R[C^*p - i\mathcal{G}_{a+}(-\tau)]\delta(t-t'+\tau), \tag{A11}$$

$$\langle F_M^+(t)F_a(t')\rangle = -RAC^*p\,\delta(t-t') + R[C^*p - i\mathcal{G}_{+a}(\tau)]\delta(t-t'-\tau) - iR\mathcal{G}_{+a}(-\tau)\delta(t-t'+\tau), \tag{A12}$$

$$\langle F_b(t)F_M(t')\rangle = [RC(1-Bp) + \gamma_b\langle M(t)\rangle]\delta(t-t') + iR\mathcal{G}_{b-}(\tau)\delta(t-t'-\tau) + iR\mathcal{G}_{b-}(-\tau)\delta(t-t'+\tau), \tag{A13}$$

$$\langle F_M(t)F_b(t')\rangle = -[RBCp + \gamma_a'\langle M(t)\rangle]\delta(t-t') + iR\mathcal{G}_{-b}(\tau)\delta(t-t'-\tau) + iR\mathcal{G}_{-b}(-\tau)\delta(t-t'+\tau), \tag{A14}$$

$$\langle F_b(t)F_M^+(t')\rangle = [RBC^*p + \gamma_a'\langle M(t)\rangle]\delta(t-t') - iR\mathcal{G}_{b+}(\tau)\delta(t-t'-\tau) - iR\mathcal{G}_{b+}(-\tau)\delta(t-t'+\tau), \tag{A15}$$

$$\langle F_M^+(t)F_b(t')\rangle = [RC^*(1-Bp) + \gamma_b\langle M^+(t)\rangle]\delta(t-t') - iR\mathcal{G}_{+b}(\tau)\delta(t-t'-\tau) - iR\mathcal{G}_{+b}(-\tau)\delta(t-t'+\tau). \tag{A16}$$

p is a parameter, which characterizes the pumping statistics: A Poissonian excitation statistics corresponds to $p=0$, and for a regular statistics we have $p=1$. The correlation function $\mathcal{G}_{\alpha\beta}(\tau)$ is defined as

$$\mathcal{G}_{\alpha\beta}(\tau) = \langle\sigma_\alpha^j(\tau)\sigma_\beta^j(0)\rangle_Q = \langle a|\mathcal{U}^{-1}(\tau)\sigma_\alpha\mathcal{U}(\tau)\sigma_\beta|a\rangle, \tag{A17}$$

and $\mathcal{G}_{\alpha\beta}(-\tau)$ is expressed as

$$\mathcal{G}_{\alpha\beta}(-\tau) = \langle \sigma_\alpha^j(0)\sigma_\beta^j(\tau)\rangle_Q = \langle a|\sigma_\alpha \mathcal{U}^{-1}(\tau)\sigma_\beta \mathcal{U}(\tau)|a\rangle. \tag{A18}$$

Here we have used the fact that all atoms are initially in the excited state $|a\rangle$, and $\mathcal{U}(\tau)=\exp(-iHt/\hbar)$ is the unitary time-evolution operator, which describes the evolution of the laser system. In the case of the interaction of an atom with a plane field, which could be viewed as the zero-order approximation of the atom-laser interaction, $\mathcal{U}(\tau)$ could be expressed as

$$\begin{aligned}\mathcal{U}(\tau) &= \cos(g\tau\sqrt{a^+a+1})\sigma_{aa} + \cos(g\tau\sqrt{a^+a})\sigma_{bb} \\ &\quad - i\frac{\sin(g\tau\sqrt{a^+a+1})}{\sqrt{a^+a+1}}a\sigma_{ab}e^{-i\Delta\tau} \\ &\quad - ia^+\frac{\sin(g\tau\sqrt{a^+a+1})}{\sqrt{a^+a+1}}\sigma_{ba}e^{i\Delta\tau}.\end{aligned} \tag{A19}$$

It is enough to calculate all of the correlation functions $\mathcal{G}_{\alpha\beta}(\pm\tau)$ to the zero-order approximation here.

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