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Collective oscillations in ultracold atomic gas

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Using both fluid and kinetic descriptions, with repulsive forces between nearby atoms included, we discuss the basic oscillations and waves of a cloud of ultracold atoms confined in a magneto-optical trap. The existence of a hybrid mode, with properties similar to both plasma and acoustic waves, is described in detail. Tonks-Dattner resonances for confined hybrid modes in a spherical cloud are discussed, and the prediction of a nonlinear coupling between the dipole resonance and the hybrid modes is considered. Landau damping processes and quasilinear diffusion in velocity space are also discussed.

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I. INTRODUCTION

In recent years, and mainly due to the emergence of laser cooling techniques [1], there has been an increasing interest in the physics of ultracold atoms. This interest has been mainly concentrated on the study of Bose Einstein condensates, and on the theoretical and experimental understanding of their properties [2,3]. However, attention has also been given recently to the study of collective oscillations of noncondensed atomic molasses in magneto-optical traps [4–6]. From this work, collective behavior similar to that observed in plasma physics has emerged, leading to the discovery of an equivalent electric charge for neutral atoms, and of electrostatic types of interaction between nearby atoms [7], and to Coulomb-like explosions of the atomic cloud when the magnetic confinement is switched off [8]. The theoretical modeling of such collective processes is still not well established, and this is the main motivation of the present work. Here we propose to apply the well-known methods of waves and oscillations in plasmas to such processes in a neutral gas.

In this work we consider the collective behavior of an ultracold atomic gas, in order to identify the basic mechanisms of oscillation, to establish its frequencies, and to derive dispersion relations for its basic modes of propagation. We will use both fluid and kinetic descriptions where the main forces associated with the laser cooling processes are retained.

This paper is organized as follows. In Sec. II, we define some essential parameters to describe the features of a cold atomic cloud confined in a magneto-optical trap. The basic set of fluid equations is also established. In Sec. III, we report the existence of a hybrid mode for short wavelength scales, somewhere between plasma waves and acoustic waves. In Sec. IV, we address the oscillations for the long wavelength scales, where we discover modified Tonks-Dattner resonances, which correspond to confined hybrid oscillations inside the atomic cloud, formally similar to those in Ref. [9]. Nonlinear hybrid resonances driven by the dipolar oscillations are predicted in Sec. V. We also use a wave kinetic approach, in order to refine the fluid description and describe additional phenomena. Hence, in Sec. VI we describe the system via a Fokker-Planck equation that can be directly derived from the Schrödinger equation for the collective field of the atoms in the radiation and trapping fields. It is shown that the waves and oscillations in the gas can be damped by a resonant atomic interaction with the collective oscillations, which is a manifestation of Landau damping. We will also show that a spectrum of collective oscillations will lead to diffusion in the atomic velocity space, thus preventing the laser cooling process from proceeding further. This effect is added to other diffusion processes already known in the literature. Finally, some conclusions are stated.

II. BASIC EQUATIONS AND FLUID DESCRIPTION

The simplest model to describe a gas of cold atoms in a magneto-optical trap is based on Doppler cooling [10,11] and on the spatial confinement due to the presence of a magnetic field gradient. The relevant average forces acting on a single atom are based on the quasiresonant radiation pressure force and can be written (for a convenient choice of the relative polarizations of the laser beams and the atomic transition) for each direction r_i ($r_1=x$, $r_2=y$, $r_3=z$) as

$$F_i(r_i,v_i)=\frac{\hbar k_L\Gamma}{2}s_{\mathrm{inc}}\left(\frac{\Gamma^2}{\Gamma^2+4(\Delta-\mu_i r_i-k_L v_i)^2}-\frac{\Gamma^2}{\Gamma^2+4(\Delta+\mu_i r_i+k_L v_i)^2}\right). \tag{1}$$

This expression relies on the low-intensity Doppler model for the magneto-optical force (incident on-resonance saturation parameter per beam $s_{\mathrm{inc}}=I_0/I_{\mathrm{sat}}\ll 1$). The Zeeman shifts (described by $\mu_i r_i$) and Doppler shifts ($k_L v_i$) are responsible for trapping and cooling, respectively. Here I_0 is the laser intensity of laser beams incident along the six directions, Γ the natural linewidth of the transition used in the cooling process, and Δ the frequency detuning between the laser frequency $\omega_L=k_L c$ and the atomic transition frequency ω_{at}. Assuming symmetric forces ($\mu_i=\mu$) along each of the three directions Ox, Oy, and Oz one can write to first order in $\vec{r}$ and $\vec{v}$

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$$\vec{F}_{\mathrm{MOT}} = -\kappa\vec{r} - \alpha\vec{v}, \tag{2}$$

where κ is the spring constant of the trap and α the friction coefficient,

$$\alpha = -8\hbar k_L^2 s_{\mathrm{inc}} \frac{\Delta/\Gamma}{(1+4\Delta^2/\Gamma^2)^2}, \quad \kappa = \alpha\mu/k_L, \tag{3}$$

which is related to the dipole frequency

$$\omega_d = \sqrt{\frac{\kappa}{M}}, \tag{4}$$

where M represents the mass of a single atom. Even though most magneto-optical traps using, for instance, alkali-metal atoms (such as Na, Li, K, Rb, or Cs) are not completely described by such a simple Doppler model [1], the forces above constitute a good first approach to describe the dynamics of single atoms in a magneto-optical trap.

This description of a magneto-optical trap is known to be limited to only a moderate number of atoms (typically 10^5). For larger atom numbers, additional forces need to be taken into consideration. A second force to be considered is the shadow force, or absorption force $\vec{F}_A$, which was first discussed by Dalibard [12]. This is associated with the gradient of the incident laser intensity due to laser absorption by the atomic cloud. It is an attractive force which can be determined by a Poisson type of equation as

$$\vec{\nabla}\cdot\vec{F}_A = -\frac{\sigma_L^2 I_0}{c} n(\vec{r}) \tag{5}$$

where $n(\vec{r})$ is the atom density and σ_L the laser absorption cross section. Finally, a third force, $\vec{F}_R$, can be called the repulsive force or radiation trapping force, and was first considered by Sesko *et al.* [13]. It describes atomic repulsion, due to the radiation pressure of scattered photons on nearby atoms, and can also be determined by a Poisson type of equation,

$$\vec{\nabla}\cdot\vec{F}_R = \frac{\sigma_R\sigma_L I_0}{c} n(\vec{r}), \tag{6}$$

where σ_R is the atom scattering cross section. A detailed discussion of these forces and explicit expressions for the cross sections σ_R and σ_L can be found, for instance, in [8,13]. These expressions for the forces acting on the atomic clouds and due to the laser cooling beams correspond to the simplest possible description of the laser cloud interaction, and can be used in a first approximation to model the fluid dynamics of the ultracold gas, which can be derived by computing the zeroth and the first moments of the Fokker-Planck equation, neglecting the diffusion term [8]. The basic set of equations can then be written as

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{v}) = 0, \tag{7}$$

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{\vec{\nabla}P}{Mn} + \frac{F_T}{M}, \tag{8}$$

where n and $\vec{v}$ are the density and velocity field of the gas respectively, $\vec{F}_T = \vec{F}_{\mathrm{MOT}} + \vec{F}_c$, and P is the gas pressure. Here we have also defined the collective force $\vec{F}_c = \vec{F}_A + \vec{F}_R$, which is determined by the Poisson equation resulting from Eqs. (5) and (6),

$$\vec{\nabla}\cdot\vec{F}_c = Qn, \quad Q = (\sigma_R - \sigma_L)\sigma_L I_0/c. \tag{9}$$

The system can then be regarded as a one-component plasma where the electrostatic attractive force due to ions can be formally replaced by the confining force, in such a way that, for the unperturbed state $n=n_0$, we may represent the total force through a Laplace-like equation

$$\vec{\nabla}\cdot\vec{F}_T(n=n_0) = 0. \tag{10}$$

In typical experimental conditions, the repulsive forces largely dominate over the shadow effect, and the quantity Q is positive [4,7,8], which allows us to define the typical frequency

$$\omega_P = \sqrt{\frac{Qn_0}{M}}, \tag{11}$$

which is a straightforward generalization of the well-known electron plasma frequency. Comparing this expression with the usual definition of the plasma frequency ω_{pe} in an ionized medium, we conclude that neutral atoms behave as if they had an equivalent electric charge, as first noticed in [7], with the value $q_{\mathrm{eff}} = \sqrt{\epsilon_0 Q}$, where ϵ_0 is the vacuum electric permitivity. The experimental value observed for this effective atomic charge is $q_{\mathrm{eff}} \sim 10^{-4} - 10^{-6}$ times the electron charge. In a typical MOT experiment, we expect to have $n_0 \approx 10^{10}$ cm^{-3}, $M \approx 10^{-25}$ kg, which yields a plasma frequency in the range of $\omega_P/2\pi = q_{\mathrm{eff}}/2\pi\sqrt{m_e/M}\omega_{pe} \approx 100$ Hz. It is clear from Eq. (11) that plasmalike oscillations are possible only for $Q>0$. Therefore, they cannot occur when the shadow force (5) dominates over the repulsive scattering force (6).

In order to conclude the analogy between the fundamental parameters for the plasma and the cold atoms, one remark should be made concerning the motion of the center of mass

$$\vec{R} = \sum_{i=1}^{N} \vec{r}_i m_i/M, \tag{12}$$

where $M = \sum_{i=1}^{N} m_i$. Therefore, for a typical neutral plasma with spherical geometry, it is a well-known result that the center of mass oscillates at the so-called Mie frequency $\omega_M = \omega_{pe}/\sqrt{3}$, where the factor $\sqrt{3}$ arises from the spherical symmetry [14] and is an essential parameter in the description of resonances in clusters [15]. However, for the case of a spherical cloud of cold atoms, it is a simple task to verify that the center of mass obeys the following equation of motion:

$$\frac{d^2\vec{R}}{dt^2} + \omega_d^2\vec{R} = \vec{0}, \tag{13}$$

which states that $\vec{R}$ oscillates exactly at the dipole frequency ω_d corresponding to the generalized plasma frequency ω_P in the unperturbed state $n=n_0$, as given by Eq. (11). The reason for this difference lies in the fact that, in the case of spherical plasmas, the restoring force depends on the charge balance between ions and electrons n_i-n_e, which depends on the shape of the cloud. In the case of cold atoms, the potential is unequivocally determined by the laser and the magnetic field (say trapping) parameters, and therefore the shape of the atomic cloud plays no role.

In what follows, we make use of the set of fluid equations derived in this section for a quasicontinuum medium and describe the main collective modes. We start from the infinite medium approximation, to emphasize the nature of the oscillations, and then introduce in Sec. IV the effect of the finite size of the cloud.

III. PLASMA HYBRID WAVES

We first assume oscillations that can be excited in the cold gas with a wavelength much smaller than its radius. The medium can therefore be assumed as infinite. We then assume that the equilibrium state of the gas is perturbed by oscillations with frequency ω and wave vector $\vec{k}$. In the sense of linear response theory, we linearize the above fluid and Poisson equations, by defining perturbations around the equilibrium quantities

$$n = n_0 + \tilde{n}, \quad \vec{F} = F_0 + \delta\vec{F}, \quad \vec{v} = v_0 + \delta\vec{v}. \tag{14}$$

Since the trapping force $\vec{F}_{\mathrm{MOT}}$ defines only the equilibrium quantities and plays no role in the modes we are about to describe, we drop the subscript c for the perturbation in the collective force $\delta\vec{F}_c$ in (14) for the sake of simplicity. For the closure of the system of fluid equations, one equation of state for the hydrodynamical pressure must be given. In this paper, we assume that P is given by an adiabatic equation of the form

$$P(n) \sim n^{\gamma}, \tag{15}$$

where γ represents the adiabatic constant. The implications of this assumption will be stated in Sec. V, in the context of a kinetic description. Using the latter together with Eqs. (7)–(9), we can easily obtain

$$\left[\frac{\partial}{\partial t}\left(\alpha + \frac{\partial}{\partial t}\right) + \omega_P^2 - u_S^2\nabla^2\right]\tilde{n} = \left(\frac{u_S^2\vec{\nabla}\tilde{n}}{n_0} - \frac{\delta\vec{F}}{M}\right)\cdot\vec{\nabla}n_0 \tag{16}$$

where u_S can be identified with the sound speed,

$$u_S^2 = \gamma\frac{P_0}{Mn_0}, \tag{17}$$

and P_0 is the equilibrium gas pressure.

We will now assume that the atomic equilibrium density n_0 is uniform, in consistence with the harmonic nature of the trapping potential, therefore neglecting the right-hand side of Eq. (16). The influence of boundary conditions and inhomogeneities on the collective oscillations of the gas will be discussed later. Assuming a space-time dependence of the perturbations $\tilde{n}$ and $\delta\vec{F}$ of the form $\exp(i\vec{k}\cdot\vec{r}-i\omega t)$, with a complex frequency $\omega=\omega_r+i\omega_i$, we obtain for the dispersion relation and for the corresponding damping rate the values

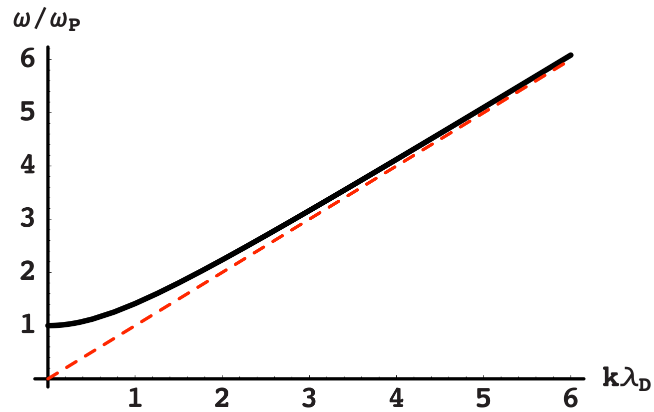

FIG. 1. (Color online) Normalized dispersion relation ω/ω_P plotted against $k\lambda_D$. The red dashed line represents the acoustic asymptotic behavior of the hybrid waves.

$$\omega_r^2 = \omega_P^2 + k^2u_S^2 + \frac{3}{4}\alpha^2, \quad \omega_i = \frac{\alpha}{2} \tag{18}$$

(see Fig. 1). In the limit of very small viscosity $\alpha \ll \omega_P$, the latter dispersion relation reduces to $\omega^2=\omega_P^2+k^2u_S^2$, which is formally identical to the dispersion relation of electron or plasma waves in ionized media (also known as Langmuir waves) but where the electron thermal velocity $v_{\mathrm{the}} = \sqrt{k_BT_e/m_e}$, ($T_e$ and m_e are the electron temperature and mass) is replaced by the sound velocity divided by a numerical factor, $u_S/\sqrt{3}$. According to the experimental parameters mentioned in the previous section, we estimate the sound velocity to be $u_S \propto \sqrt{k_BT/M} \approx 20$ cm/s. This shows that the wave mode described by Eq. (18) contains elements of both electron plasma waves and acoustic waves. It possesses a lower cutoff, given by $\omega_r=\sqrt{\omega_P^2+3\alpha^2/4}$, which is typical of an electron plasma wave, but its phase velocity tends to the sound velocity u_S and becomes weakly dispersive like an acoustic wave. Its corresponding quasiparticles can therefore be seen as hybrid entities, somewhere between plasmons and phonons. The excitation of such modes in a typical MOT setup could be driven by modulating one of the six trapping laser beams. Such a modulation should be nearly resonant with ω_P, and hence one may excite hybrid waves by sweeping the modulation frequency around 100 Hz. Associated with the propagation of such waves, we expect to observe a periodic variation of the intensity of the luminosity. By measuring the period of this oscillation, one can identify the excited mode. The existence of such a hybrid mode is one of the main results of this paper.

IV. MODIFIED TONKS-DATTNER RESONANCES

The hybrid mode discussed above is meaningful only in infinite and homogeneous media. In physical terms, its dispersion relation can only be applied to waves that propagate locally, with wavelength scales much smaller than the inhomogeneity scale and the cloud dimensions. Let us consider now oscillations with a wavelength that is comparable with the size of the atomic cloud. In this case we can no longer neglect the boundary conditions. Going back to Eq. (16), we assume that the density perturbations oscillate at a frequency ω as previously, but the corresponding spatial structure will be determined by the expressions

$$[\nabla^2 + k^2(\vec{r})]\tilde{n} = \frac{\delta\vec{F}}{Mu_S^2}\cdot\vec{\nabla}n_0 + \vec{\nabla}\ln n_0\cdot\vec{\nabla}\tilde{n},$$

$$\vec{\nabla}\cdot\delta\vec{F} = Q\tilde{n} \tag{19}$$

where the space-dependent wave number $k(\vec{r})$ is defined by

$$k^2(\vec{r}) = [\omega^2 - \omega_P^2(\vec{r})]/u_S^2. \tag{20}$$

Before going into a more complex model, it is useful to consider the simple one-dimensional problem [16]. In the case of a uniform slab of cold gas, we have $\vec{\nabla}n_0=\vec{0}$, except at the boundaries $x=0$ and $x=L$. Equations (19) and (20) then reduce to a simple one-dimensional equation

$$\frac{d^2\tilde{n}}{dx^2} + \frac{1}{u_S^2}[\omega^2 - \omega_P^2(x)]\tilde{n} = 0. \tag{21}$$

Taking the boundary conditions $\tilde{n}(0)=\tilde{n}(L)=0$, we obtain the following dispersion relation:

$$\omega_\nu^2 = \omega_P^2\left[1 + \left(\pi\nu\frac{\lambda_D}{L}\right)^2\right], \tag{22}$$

where the quantum number ν can take the values $0,1,2,3,\ldots$, and the quantity $\lambda_D=u_S/\omega_P$ is the Debye length for a cold neutral gas, in analogy with the plasma definition (where, however, the sound speed u_S is replaced by $\sqrt{3}v_{\rm the}$, as mentioned before). This defines a natural length above which plasma effects should be expected. Following the previously estimated values for ω_P and u_S, we expect to observe a Debye length of the order of $\lambda_D \approx 100\ \mu$m. In a typical MOT experiment, the radius of the cloud varies in the range $a \approx 1-5$ mm, yielding the relation $\lambda_D/a \ll 1$. Hence, plasma-like effects are expected to occur in a cloud containing a moderate number of atoms. As a remark, we should stress that there is no experimental evidence so far of any equation of state $P(n)$, which may compromise the definition of the sound speed u_S.

The relation (22) shows that the finite dimensions of the slab imply the existence of a series of resonant modes with an integer number of half wavelengths. The cylindrical geometry was considered, for the plasma case, in a famous paper by Parker, Nickel, and Gould in 1964 [9], but it is more natural here to consider a spherical geometry for the ultracold gas which, to our knowledge, was not derived for a plasma. We expect to find an infinite series of resonances, similar to Eq. (22), known as the Tonks-Dattner resonances [17,18]. For this purpose, we consider the internal oscillations in a spherical cloud with radius a in the homogeneous case, where $\vec{\nabla}n_0(r)=\vec{0}$, for $0 \geq r < a$, for which analytical solutions can be found. These results remain qualitatively valid even for a more realistic density profile, as shown in Ref. [9], for the case of an inhomogeneous cylindrical plasma. The case of a nonuniform spherical plasma will be addressed in future work.

Performing a separation of variables, we can obtain solutions of the form

$$\tilde{n}(\vec{r}) = R(r)Y(\theta,\phi), \tag{23}$$

where (r,θ,ϕ) are spherical coordinates. After separation of variables, we get the usual spherical harmonics for the angular part of the density perturbation,

$$Y(\theta,\phi) = P_l^m(\cos\theta)\exp(im\phi), \tag{24}$$

where $P_l^m(\cos\theta)$ are the associated Legendre polynomials, l is a positive integer or zero, and $|m| < l$. The radial equation resulting from (19) and (23) can be written as

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + [k^2r^2 - l(l+1)]R = 0. \tag{25}$$

By using a simple transformation of variables, $x=kr$, and $S(kr)=\sqrt{kr}R(r)$ this equation can be reduced to a Bessel equation

$$\frac{d^2S}{dx^2} + \frac{1}{x}\frac{dS}{dx} + \left(1 - \frac{(l+1/2)^2}{x^2}\right)S = 0. \tag{26}$$

The solutions with regular behavior at the origin $x=0$ are therefore given by Bessel functions of the first kind, $J_{l+1/2}(x)$. From this we conclude that the Tonks-Dattner modes in a spherical homogeneous cold atom cloud are determined by

$$\tilde{n}(\vec{r},t) = \sum_{l,m}\tilde{n}_l(t)\frac{J_{l+1/2}(kr)}{\sqrt{kr}}P_l^m(\cos\theta)\exp(im\phi), \tag{27}$$

where $\tilde{n}_l(t)$ have small amplitudes such that $|\tilde{n}_l| \ll n_0$. The mode frequencies can be obtained by remarking that $\tilde{n}$ should vanish at the border $r=a$. This implies that the allowed values for k have to obey the condition $k=z_{\nu,l}/a$, where $z_{\nu,l}$ represents the νth zero of the Bessel function of order $(l+1/2)$. We are then led to the mode frequencies

$$\omega_{\nu,l}^2 = \omega_P^2\left[1 + \left(z_{\nu,l}\frac{\lambda_D}{a}\right)^2\right]. \tag{28}$$

Comparing with the rectangular case of Eq. (22), we see that the allowed eigenfrequencies for a spherical cloud now depend on two quantum numbers ν and l. But, in contrast with the similar quantum mechanical solutions for hydrogenlike atoms, we have no hierarchical relation between these quantum numbers. The normalized radial profiles for the lowest-order solutions are illustrated in Fig. 2, which is formally similar to the spectrum that was presented in the past by Dattner for the case of a plasma cylinder [18]. Also, a recent experimental result on ultracold plasma was published show-

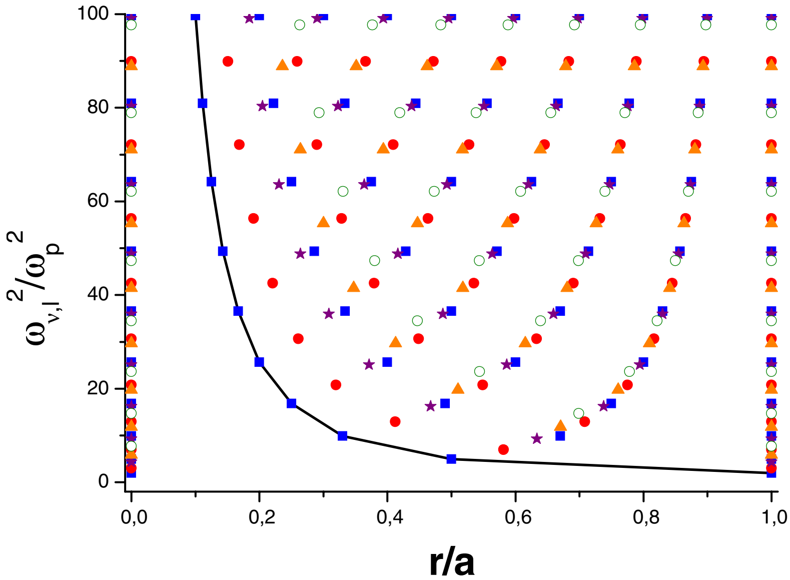

FIG. 2. (Color online) Normalized modes $\omega_{\nu,l}^2/\omega_p^2$ plotted against the nodes $r_{\nu,l}/a$ of the radial solution for the density perturbation $\tilde{n}$, for $\lambda_D/a=0.1$ and $1<\nu<10$. Blue squares ($l=0$), red circles ($l=1$), violet stars ($l=2$), yellow triangles ($l=3$), and green open circles ($l=4$). The full line is plotted at $l=0$ and scales as $1/\nu^2$.

ing the evidence of Tonks-Dattner modes excited by a radio-frequency electric field [19].

In a more realistic description, the present rigid (Dirichlet) boundaries will eventually have to be replaced by soft boundaries and a generic density profile $n_0(r)$ must be assumed, for which numerical solutions have to be found [20].

V. NONLINEAR OSCILLATIONS DRIVEN BY DIPOLAR RESONANCES

Due to the intrinsic nonlinearity of the fluid description and the collective behavior of the medium, it is possible to couple dipole oscillations of the cloud with the hybrid modes. Going back to the fluid equations, and assuming an oscillating mean velocity of the form $\vec{v}_0 \sin(\omega_d t+\phi)$ for the center of mass, we obtain

$$\frac{\partial^2\tilde{n}}{\partial t^2} + n_0\vec{\nabla}\cdot\frac{\partial\vec{v}}{\partial t} + \vec{v}_0\cdot\vec{\nabla}\left(\cos(\omega_d t+\phi)\vec{\nabla}\tilde{n} + \sin(\omega_d t+\phi)\frac{\partial\tilde{n}}{\partial t}\right) = 0. \tag{29}$$

Using the factorization $\tilde{n}(\vec{r},t)=\tilde{A}(t)\tilde{n}(\vec{r})$, the coupling between the center of mass and the hybrid modes can be approximately described by the canonical Mathieu equation

$$\frac{\partial^2\tilde{A}}{\partial\tau^2} + [\Delta + 2\epsilon\cos(2\tau)]\tilde{A} = 0, \tag{30}$$

where we have used

$$\tau = \frac{\omega_d}{2}t, \quad \Delta = \frac{4}{\omega_d^2}(\omega^2 - k^2u_S^2), \quad \epsilon = \frac{2}{\omega_d}\vec{v}_0\cdot\vec{\nabla}\ln\tilde{n}. \tag{31}$$

It is well known that such an equation has unstable regions. For $|\epsilon|\ll 1$, the first transition to the instability zone occurs for $\Delta\simeq 1+\epsilon$. We therefore expect to observe an instability of the hybrid and Tonks-Dattner modes, driven by dipole oscillations for

$$\omega^2 \simeq \frac{\omega_d}{2}\left(\frac{\omega_d}{2} + \vec{v}_0\cdot\vec{\nabla}\ln\tilde{n}\right). \tag{32}$$

This simple discussion demonstrates the existence of very interesting nonlinear collective phenomena in a cold atom gas. But in the present work we are mainly focused on the linear properties of the medium. A more rigorous and detailed study of nonlinear coupling between dipole and Tonks-Dattner oscillations will be left to future work.

VI. KINETIC DISPERSION RELATIONS

The analysis of collective oscillations in a cold gas can be refined by using a kinetic description based on the classical probability distribution function $W(\vec{r},\vec{q},t)$. This description will allow us to include resonant kinetic processes, which enhance the energy exchanges between part of the atomic population and the hybrid modes excited in the medium. Moreover, a kinetic approach avoids the use of an equation of state for the hydrodynamical pressure P. It is rather determined by the statistics, as we will see in the following discussion. In the presence of diffusion, the probability distribution function $W(\vec{r},\vec{q},t)$ obeys the Fokker-Planck equation [21]

$$\left(\frac{\partial}{\partial t} + \frac{\hbar\vec{q}}{M}\cdot\vec{\nabla}\right)W = -\frac{\partial}{\partial\vec{k}}(\vec{F}_{\text{tot}}W) + \sum_{ij}D_{ij}\frac{\partial^2 W}{\partial q_i\,\partial q_j}, \tag{33}$$

where the total force $\vec{F}_{\text{tot}}$ includes the radiative and the damping forces, and the diffusive tensor D_{ij} is due to the fluctuations of the radiative force and spontaneous emission, as discussed by several authors [21,22]. Here, for simplicity, and because we want to focus on the oscillating modes, we neglect the diffusion term. Diffusion effects will not be completely ignored, rather they will reappear later in a different context. We are then led to a kinetic equation of the Vlasov type, with a damping correction, as given by

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \frac{\vec{F}}{M}\cdot\frac{\partial}{\partial \vec{v}}\right)W = -\alpha(\vec{v})W, \quad (34)$$

where $\vec{v}=\hbar\vec{q}/M$ is the atom velocity, and the collective (shadow minus repulsive) force $\vec{F}_c$ is determined by a new Poisson equation, which can be written as

$$\vec{\nabla}\cdot\vec{F}_c = Q\int W(\vec{r},\vec{v},t)d\vec{v}. \quad (35)$$

This equation is obviously identical to (9), because the integral is nothing but the density $n(\vec{r},t)$, the normalization of the distribution function. On the other hand, by taking the average of the first two momenta of the kinetic equation (34), we will be able to derive fluid equations such as (9) and (10). Notice that the parameter α appearing in the fluid equations is an averaged value of the quantity $\alpha(\vec{v})$ appearing in the wave kinetic equation (34). In order to focus our attention on the purely kinetic processes, we will assume that $\alpha=0$ but, at the end of this section, we will discuss the influence of a finite value for this parameter. We now consider some equilibrium state $W_0(\vec{v})$ and assume a sinusoidal perturbation, such that $\delta\vec{F}$ and $\tilde{W}$ evolve in space and time as $\exp(i\vec{k}\cdot\vec{r}-i\omega t)$. In what follows, we drop the subscript c in the collective force, in consistence with the previous calculations. After linearization, the two previous equations reduce to

$$\tilde{W} = -\frac{i}{M}\frac{\delta\vec{F}\cdot\partial W_0/\partial\vec{v}}{(\omega-\vec{k}\cdot\vec{v})}, \quad i\vec{k}\cdot\delta\vec{F} = Q\int \tilde{W}(\vec{v})d\vec{v}. \quad (36)$$

From here we get the dispersion relation for collective cold atom oscillations with frequency ω and wave vector $\vec{k}$

$$1 + \frac{Q}{Mk^2}\int\frac{\vec{k}\cdot\partial W_0/\partial\vec{v}}{(\omega-\vec{k}\cdot\vec{v})} = 0. \quad (37)$$

This is similar to that of electrostatic waves in unmagnetized plasmas, and can be rewritten as $1+\chi(\omega,\vec{k})=0$, where the quantity $\chi(\omega,\vec{k})$ is the susceptibility. In order to understand the physical implications of such a dispersion relation, let us consider first a simple monokinetic atomic distribution, of the form $W_0(\vec{v})=n_0\delta(\vec{v}-\vec{v}_0)$, corresponding to a beam of atoms with density n_0 and velocity $\vec{v}_0$. In this case, Eq. (36) reduces to

$$1 - \frac{Qn_0}{M(\omega-\vec{k}\cdot\vec{v}_0)^2} = 0. \quad (38)$$

This is nothing but the Doppler-shifted plasma oscillations discussed above. For $\vec{v}_0=\vec{0}$, this reduces to $\omega=\omega_P \equiv (Qn_0/M)^{1/2}$. Second, we consider the case of two atomic populations, one at rest with density n_0, and another moving with velocity $\vec{v}_1$ with density $n_1 \ll n_0$. This is described by $W_0(\vec{v})=n_0\delta(\vec{v})+n_1\delta(\vec{v}-\vec{v}_1)$. The corresponding dispersion relation is given by

$$1 - \frac{Qn_0}{M\omega^2} - \frac{Qn_1}{M(\omega-\vec{k}\cdot\vec{v}_1)^2} = 0. \quad (39)$$

The interest of this dispersion relation is that it leads to an instability, which means that the cold atoms can oscillate spontaneously at a frequency close to ω_P. The energy source driving such oscillations is provided by the atomic beam with density n_1. This can be seen by assuming a complex frequency $\omega=\omega_r+i\omega_i$. The maximum growth rate predicted by Eq. (39) corresponds to the resonant case where $\omega_r=\vec{k}\cdot\vec{v}_1$, and is determined by

$$\omega_i = \frac{\sqrt{3}}{2}\omega_P\left(\frac{n_1}{2n_0}\right)^{1/3}. \quad (40)$$

This is very similar to an electrostatic beam–plasma instability, but can also be seen as the dynamical analog of the collective atom recoil laser [23], where the incident laser beam is replaced by the atomic beam, and where the resulting stimulated emission is not made of photons, but of hybrid quasiparticles. The critical region where unstable modes may show up is given by

$$K < (1+N^{2/3})^{3/2}, \quad (41)$$

where we define the dimensionless quantities $K=\vec{k}\cdot\vec{v}_1/\omega_p$, $\Omega=\omega/\omega_p$, and $N=n_1/n_0$. In Fig. 3, we plot the growth rate $\Gamma(K)=\omega_i/\omega_p$ for the unstable region given above and the real part of the spectrum $\Omega_r=\omega_r/\omega_p$, both given by the roots of the dispersion relation (39). We observe that the stable modes bifurcate exactly at the vanishing point of the unstable ones.

Finally, we consider a generic equilibrium quasidistribution $W_0(\vec{v})$. The atomic susceptibility can be split into its real and imaginary parts, $\chi=\chi_r+i\chi_i$. By using the Kramers-Kroenig relation between χ_r and χ_i, and assuming that most of the atoms have velocity smaller than the phase velocity ω/k, which is a very plausible assumption for a gas of ultracold atoms, we obtain

$$\chi_r(\omega,\vec{k}) = -\frac{1}{\omega^2}(\omega_P^2 + k^2u_S^2),$$

$$\chi_i(\omega,\vec{k}) = i\pi\frac{Q}{Mk^2}\left(\frac{\partial G_0}{\partial v}\right)_{\omega/k}, \quad (42)$$

where v is the parallel component of the atom velocity, and we have identified the sound speed with the integral

$$u_S^2 = \frac{1}{n_0}\int W_0(\vec{v})v^2 d\vec{v} = \frac{1}{n_0}\int G_0(v)v^2 dv. \quad (43)$$

The quantity $G_0(v)$ introduced here is the average of the distribution function over the perpendicular velocities. The latter result avoids the postulation of the equation of state (15). However, since the hydrodynamical pressure

$$P(\vec{r},\vec{v},t) = \frac{1}{3}Mn(\vec{r},\vec{v},t)(\langle v^2\rangle - \langle v\rangle^2) \quad (44)$$

is a statistical variable, we can make use of Eqs. (43) and (17) to write

$$\gamma = 3. \tag{45}$$

This means that the hybrid modes are essentially a one-dimensional process, if we recall that $\gamma=(2+d)/d$, which is a well-known result for the electron plasma waves. This justifies *a posteriori* the hybrid character of these oscillations. Going back to Eq. (39), we can easily obtain the dispersion relation by using $1+\chi_r(\omega_r,\vec{k})=0$, which coincides with Eq. (18), and the wave damping defined by the expression

$$\omega_i = -\frac{\chi_i(\omega_r,\vec{k})}{(\partial\chi_r/\partial\omega)_{\omega=\omega_r}} = \frac{\pi}{\omega}\frac{Q}{Mk^2}\left(\frac{\partial G_0}{\partial v}\right)_{\omega/k}. \tag{46}$$

This is a nondissipative wave damping, which is not related with any increase of the entropy of the physical system; it is the so-called Landau damping. It describes the resonant interactions between the wave and the atomic population which has a parallel velocity nearly equal to the wave phase velocity. Usually, for a thermal equilibrium distribution $W_0(\vec{v})$, this quantity is negative, and corresponds to wave damping. But the sign of ω_i can change for a nonthermal distribution, eventually leading to wave instability and wave growth.

We can take a step further in the kinetic description of the collective oscillations in the cold atom cloud, and consider a broad spectrum of fluctuations, described by the total wave intensity

$$I(t) = \int I(\vec{k},t)\frac{d\vec{k}}{(2\pi)^3}, \tag{47}$$

where the spectral intensity is defined by $I(\vec{k},t) = \tilde{W}^*(\vec{k},t)\tilde{W}(\vec{k},t)$. Following the usual steps of the plasma quasilinear theory and adapting it to the present context, we can say that each spectral component behaves in accordance with the above description, and evolves in time according to the equation

$$\frac{d}{dt}I(\vec{k},t) = 2\omega_i(\vec{k},t)I(\vec{k},t) + S(\vec{k},t), \tag{48}$$

where $S(\vec{k},t)$ is any given source term, and the total damping rate $\gamma_k(t)$ slowly evolves in time due to the slow time evolution of the equilibrium (or quasiequilibrium) distribution $W_0(\vec{v},t)$, which can be considered constant only on a short time scale. The temporal evolution of $W_0(\vec{v},t)$ under the influence of the fluctuation spectrum is determined by a diffusion equation of the form

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \frac{\partial}{\partial\vec{v}}\cdot\mathbf{D}\cdot\frac{\partial}{\partial\vec{v}}\right)W_0(\vec{v},t) = 0, \tag{49}$$

where the diffusion tensor $\mathbf{D}$ associated with the collective oscillations is determined by

$$\mathbf{D}(\vec{v},t) = \frac{\omega_P^2}{n_0^2}\int I(\vec{k},t)\frac{\vec{k}\otimes\vec{k}}{(\omega-\vec{k}\cdot\vec{k})}\frac{d\vec{k}}{(2\pi)^3}. \tag{50}$$

Comparing this with our previous kinetic equation (33) it can be seen that the existence of a collective spectrum of oscillations introduces an additional diffusion effect in atomic velocity space, which tends to prevent the atomic cooling process. As we have noticed, these results are valid only in the limit of a negligible viscosity parameter, $\alpha\to 0$. A finite value of α will have two distinct consequences. First, it will lead to the damping coefficient already stated by Eq. (18), adding to the purely kinetic Landau damping. Second, it will broaden the Landau resonance appearing in the dispersion relation (37) and in the diffusion coefficient (50), therefore reducing the efficiency of the resonant atom–collective wave interactions associated with the Landau resonance and with the quasilinear diffusion. It will therefore compete with the kinetic effects described in this section. The combined influence of viscous and kinetic damping is outside the scope of the present work and would deserve a separate investigation.

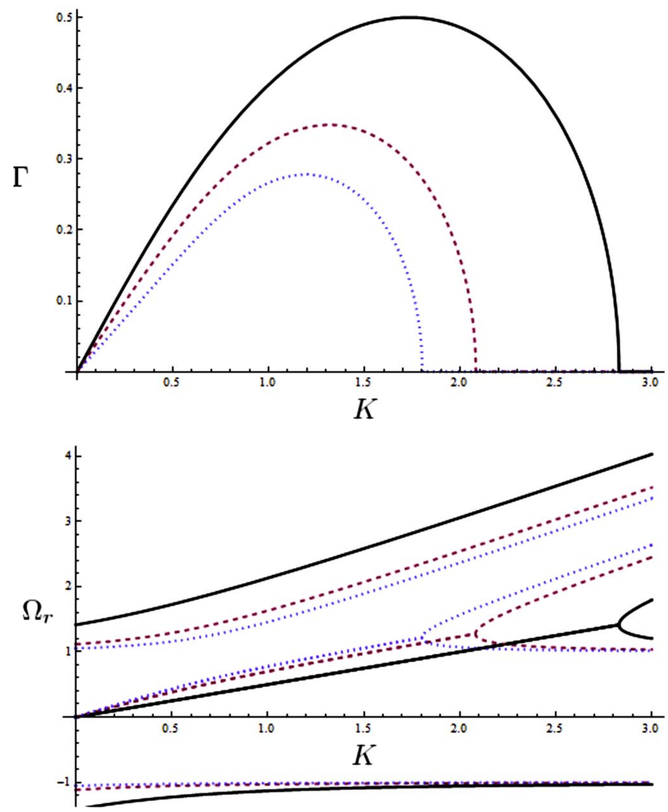

FIG. 3. (Color online) Normalized roots of Eq. (39) in terms of the dimensionless quantities $\Omega=\omega/\omega_p$, $K=\vec{k}\cdot\vec{v}_1/\omega_p$, and $N=n_1/n_0$. (a) Dimensionless growth rate $\Gamma(K)=+\Omega_i(K)$ (unstable solutions) and (b) dimensionless modes $\Omega_r(K)$ (stable solutions). In both plots, $N=1$ (full black line), 1/2 (dashed violet line), and 1/3 (dotted blue line).

VII. CONCLUSIONS

In this work we have used both fluid and kinetic equations to describe the collective oscillations in a cloud of neutral atoms confined in a magneto-optical trap. Our approach is based on a simple but physically relevant model for the forces acting on the cold atoms, which has been well verified by experiments, and can be described by a Poisson equation similar to that describing electrostatic interactions [4,8,13]. Once the physical picture was established, we started by setting the basic equations and parameters. We have also shown that the presence of internal forces leads to the existence of collective waves which have a hybrid character, with properties that are common to both electron plasma waves and acoustic waves. The similarities and differences of the cold atom gas oscillations and those of a plasma were discussed.

Taking into account the finite size effect in a cloud of cold atoms, we have shown that internal resonances of these hybrid oscillations inside the cloud can be excited. These modes should be called Tonks-Dattner resonances, in analogy with similar plasma physics effects, since they are shown here also to exist in a neutral magneto-optical trapped gas as well. Previous models developed for plasmas were limited to planar and cylindrical geometries, and they were extended to the spherical geometry which is more relevant to cold atom clouds. We have extended our discussion of these oscillations to the nonlinear regime, where coupling between dipole oscillations and hybrid waves can take place. Our approximate description suggests that such a nonlinear coupling can lead to the destabilization of hybrid waves or Tonks-Dattner resonances, driven by dipolar oscillations of the cloud center of mass.

Our analysis proceeded with a kinetic approach. Using this more refined description of oscillations, we were able to derive more general dispersion relations where nondissipative Landau damping was included. We have limited our discussion to hybrid wave modes with wavelengths much smaller than the typical dimensions of the atomic cloud. But the results can easily be extended to confined Tonks-Dattner resonances. Finally, we have established a quasilinear kinetic equation, showing the occurrence of diffusion in atomic velocity space. This diffusion effect is a direct consequence of the collective fluctuation spectrum, and implies the existence of additional collective processes preventing the occurrence of laser cooling.

In the present work we have explored the similarities of the cold atom cloud with a plasma, which can be associated with the existence of an effective electric charge for the neutral atoms. The resulting wave modes, however, are not identical to plasma wave modes, but show a hybrid character. We hope that this work will motivate future experimental and theoretical work on the collective oscillations in cold atom traps, and will contribute to launching cold atom research in new directions.

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