Analysis of averaged multichannel delay times

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The physical significances and the pros and cons involved in the usage of different time-delay formalisms are discussed. The delay-time matrix introduced by Eisenbud, where only s waves participate in a reaction, is in general related to the definition of an angular time delay which is shown not to be equivalent to the so-called phase time delay of Eisenbud and Wigner even for single channel scattering. Whereas the expression due to Smith which is derived from a time-delayed radial wave packet is consistent with a lifetime matrix which is Hermitian, this is not true for any Eisenbud-type lifetime matrix which violates time-reversal invariance. Extending the angular time delay of Nussenzveig to multiple channels, we show that if one performs an average over the directions and subtracts the forward angle contribution containing an interference of the incident and scattered waves, the multichannel angle-dependent average time delay reduces to the one given by Smith. The present work also rectifies a recently misinterpreted misnomer of the relation due to Smith.

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I. HISTORICAL DEVELOPMENT OF DIFFERENT DELAY-TIME CONCEPTS

The concept of time delay was introduced around the year 1950 by different authors [1–3]. Eisenbud and Wigner introduced it [1,2] by following the peak of spherical wave packets. This approach is widely used in tunneling and is commonly known as a phase time delay since it involves the energy derivative of the scattering phase shift. Eisenbud [1] also considered multichannel time delay, however, in the context of s-wave scattering only. He obtained an expression which is closely related to the angular time delay as defined by Froissart, Goldberger, and Watson [4]. An entirely different approach based on the concept of "dwell time" was introduced by Smith in 1960 [5]. The average dwell time had a rather straightforward meaning of the average time spent by particles interacting in a certain region (with the average being over the scattering channels). The time delay was then the difference of the time spent with and without interaction. This concept became quite popular in the later years in different branches of physics and is also used currently in the context of chaotic scattering [6]. The average dwell time concept finds application in tunneling problems [7] too, where the average is over the reflection and transmission channels. Smith defined a lifetime matrix, **Q**, which is Hermitian and is nicely related to the scattering **S** matrix, such that one can even define a time operator and find a relation which when inverted allows one to compute **S** from **Q**. Indeed, Lippmann defined a similar time delay operator in 1966 [8]. A mathematically rigorous discussion of the multichannel time delay as derived by Smith can be found in a paper by Martin [9]. The application of delay-time relations to study resonances ranges from atomic physics [10] to particle and nuclear physics, where, the method was applied to study baryon, meson, and pentaquark unstable states [11,12].

II. TIME-DELAYED WAVE PACKETS

The discussions on delay times often begin with a narrow one-dimensional wave packet, where one tries to answer at what value of x would the wave packet $\Psi(x,t)$ be peaked at a given point of time t. It is in this spirit that Wigner started with a time-dependent wave function composed of two frequencies [2], wrote down the spherical wave packet in the asymptotic region and evaluated the delay in the emergence of the wave packet due to interaction. Thus, starting with the asymptotic form of a wave packet, such as

$$\Psi(r,t) \simeq \frac{1}{\sqrt{r}}[e^{-i(kr+\omega t)}\cos(r\Delta k + t\Delta\omega) - e^{i(kr-\omega t+2\delta)}\cos(r\Delta k - t\Delta\omega + 2\Delta\delta)], \quad (1)$$

one finds that the first term has a peak at $r_p=-t(d\omega/dk)$, where $d\omega/dk$ is the group velocity. The second term has a peak at $r_p=(d\omega/dk)t-2d\delta/dk$ and the interaction has delayed the particle by a time $\Delta t=2(d\omega/dk)^{-1}d\delta/dk$ $=2\hbar d\delta/dE$, where δ is the scattering phase shift.

One could start with a wave packet corresponding to the asymptotic form of the full wave function,

$$\Psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + f(\vec{k})\frac{e^{ikr}}{r} \quad (2)$$

and choose to perform a partial wave expansion of $\Psi_{\vec{k}}(\vec{r})$, namely,

$$\Psi_{\vec{k}}(\vec{r}) = 4\pi\sum_{lm} i^l \frac{u(l,k,r)}{kr} Y^*_{lm}(\hat{k})Y_{lm}(\hat{r}), \quad (3)$$

such that,

$$u(l,k,r) = \frac{i}{2}[e^{-i(kr-l\pi/2)} - S^l(E)e^{i(kr-l\pi/2)}], \quad (4)$$

with the l-dependent S-matrix given by $S^l(E)=\exp(2i\delta_l)$. If one now performs the same procedure as that of Wigner, but with 2δ replaced rather by $\{\mathrm{Im}\ln[S^l(E)]\}$, then one obtains an l-dependent time delay, $\Delta t^l(E)$ given by

$$\Delta t^l(E) = \hbar \, \mathrm{Im} \frac{d}{dE}[\ln S^l(E)] = \mathrm{Re}[-i\hbar([S^l]^{-1} dS^l/dE)]. \tag{5}$$

Depending on the wave packet whose peak we choose to follow, which can either contain the full asymptotic wave (2) or the radial one (4), two different delay times emerge. We perform this analysis for the multichannel case.

Equation (4), for scattering with multiple channels takes the form

$$u_i(l,k,r) = \phi_i(l,k,r) - \sum_n S^l_{in}(E)\phi^*_n(l,k,r), \tag{6}$$

where $\phi_i(l,k,r)$ is the incoming plane wave and $\phi^*_n(l,k,r)$ are outgoing ones in all possible channels. The complex multichannel S-matrix elements, $S^l_{in}(E)$, can in general be parametrized in terms of an energy-dependent function $\eta^l_{in}(E)$ and a phase $\xi^l_{in}(E)$ as $S^l_{in}(E) = \eta^l_{in}(E)e^{i\xi^l_{in}(E)}$. To evaluate the time delay of a particle entering in the ith channel and leaving in the jth one, we rewrite $2\xi^l_{ij} = \mathrm{Im}(\ln S^l_{ij})$. Once again, following the peak of the wave packet in the radial direction [similar to that given by Eq. (1) but with the incoming and outgoing plane waves replaced by the $\phi_{i,j}(l,k,r)$'s mentioned above and 2δ replaced by $\mathrm{Im}(\ln S^l_{ij})$], one obtains

$$\Delta t^l_{ij}(E) = \hbar \frac{d}{dE}(\mathrm{Im} \ln S^l_{ij}) = \mathrm{Re}[-i\hbar([S^l_{ij}]^{-1} dS^l_{ij}/dE)], \tag{7}$$

where S^l_{ij} are elements of a multichannel S-matrix, $\mathbf{S}$, however, for a given partial wave l. The superscript l was not given by Smith, but was implicit in the entire derivation. If one does not perform a partial wave expansion, but rather prefers to work with the full wave function as in Eq. (2), then the time delay is that of the full wave packet and is proportional to the argument of the full scattering amplitude $f(\vec{k})$ [13]. For the single channel case, one can simply write

$$\Delta t^{\mathrm{ang}}(E,\theta) = \hbar \frac{\partial}{\partial E} \arg[f(k,\theta)]. \tag{8}$$

This is an angular time delay of the full wave packet [14] and is not trivially connected to the time delay of Wigner or Smith even in the single channel case. If however, one averages over the angles, this delay becomes proportional but not equal to the energy derivative of the phase shift as defined by Wigner and also obtained by Smith for single channel resonances. In the more general case of multichannel resonances, the angular time delay takes the following form [15]:

$$\begin{aligned}\Delta t^{\mathrm{ang}}_{ij}(E,\theta) &= \hbar \frac{d}{dE} \arg(S_{ij} - \delta_{ij}) \\ &= \hbar \frac{d}{dE}[\mathrm{Im} \ln(S_{ij} - \delta_{ij})] \\ &= \mathrm{Re}\{-i\hbar[(S_{ij} - \delta_{ij})^{-1} dS_{ij}/dE]\},\end{aligned} \tag{9}$$

where, δ_{ij} is the Kronecker δ and the elements S_{ij} belong to the full S-matrix which is not decomposed into partial waves. Equations (7) and (9) appear to be of a similar form when $i \neq j$, however, Eq. (7) involves an l-dependent S-matrix and Eq. (9) involves the full one which has not been expanded into partial waves. We shall see below that one cannot directly use the relation (9) to evaluate the time delay in a single partial wave l.

III. OBSERVABLE ANGULAR TIME DELAY

With the time delay of Smith [Eq. (7)] being given for each value of l, it is quite useful in analyzing hadron resonances which are classified according to their l values. The angular time delay in Eq. (9), however, corresponds to the delay of an entire wave packet. We shall now perform a partial wave expansion of the S-matrix in order to investigate if this relation can be brought into a form useful for characterizing hadron resonances. We restrict the discussion to the scattering of spin zero by spin one-half particles (since it corresponds to the realistic case of pion-nucleon scattering which we will discuss later as an explicit example). Alternatively, all of our formulas are valid for the scattering of spinless particles. For a given state of angular momentum J, l may take values, $J+1/2$ or $J-1/2$ and due to conservation of parity, the initial and final state carry the same angular momentum l. Neglecting the possibility of a spin-flip amplitude for the present discussion, one can write the partial wave expansion of the non-spin-flip amplitude in the case of single-channel scattering as

$$f(E,\theta) = \sum_l T_l P_l(\cos\theta), \tag{10}$$

where the T-matrix here is given as $(2l+1)(S_l - 1)/2ik$, with $S_l = \exp(2i\delta_l)$ and δ_l is the scattering phase shift. The argument of the scattering amplitude $f(E,\theta)$ which appears in the definition of angular time delay (8), is then,

$$\arg f(E,\theta) = \arctan\left(\frac{\sum_l |T_l| P_l(\cos(\theta))\sin(\phi_l)}{\sum_l |T_l| P_l(\cos(\theta))\cos(\phi_l)}\right) \tag{11}$$

with $T_l = |T_l|e^{i\phi_l}$. The energy derivative of $\arg f(E,\theta)$ involves an expression entangled in all l's which is not of much use to analyze hadron resonances. However, if one evaluates an angle-averaged time delay with the f written in terms of partial waves, one obtains

$$\hbar \int |f(E,\theta)|^2 \frac{\partial}{\partial E} \arg f(E,\theta) d\Omega = 4\pi\hbar \sum_l |T_l|^2 \frac{d}{dE}(\arg T_l). \tag{12}$$

In the single channel case, $\arg T_l = \delta_l$, where δ_l is the scattering phase shift, and

$$\begin{aligned}&\hbar \int |f(E,\theta)|^2 \frac{\partial}{\partial E} \arg f(E,\theta) d\Omega \\ &\quad = \frac{\pi}{k^2} \sum_l (2l+1) 2 \sin^2 \delta_l \left(2\hbar \frac{d\delta_l}{dE}\right).\end{aligned} \tag{13}$$

One could now use such a relation for characterizing resonances classified by partial waves. This angle-averaged relation is not equal but somewhat similar to that obtained by Wigner, namely, $2\hbar d\delta_l/dE$.

Nussenzveig [14] introduced an additional term to subtract the time delay in the forward direction where the incident beam interferes with the scattered one. Performing an angle average as above but with the new term included, it is easy to show that for single channel scattering,

$$\hbar \int |f(E,\theta)|^2 \frac{\partial}{\partial E} \arg f(E,\theta) d\Omega + \frac{2\pi}{k^2}\frac{d}{dE}[k \operatorname{Re} f(E,0)] = \frac{\pi}{k^2}\sum_l (2l+1)2\hbar \frac{d\delta_l}{dE}. \quad (14)$$

One recovers back the Wigner's phase time delay result now.

Eisenbud's multichannel delay-time matrix. In 1948, Eisenbud [1] obtained the phase time delay in single channel scattering which can also be found in [2] and is given by the energy derivative of the scattering phase shift. He further derived the multichannel generalization by restricting to the case where "all subsystems have spin zero and only waves with $l=0$ participate in the reaction." He wrote down the following relation for a time-delay matrix with elements:

$$\Delta t_{ij}^{\mathrm{Eisen}}(E) = \frac{d}{dE}[S-1]_{ij}. \quad (15)$$

This expression is similar to the angular time delay discussed above. Since Eisenbud did not consider anything but s waves, in connection with the example of a constant phase multichannel Breit-Wigner resonance, he concluded that his relation in the case of single channel scattering, reduces to that of the phase time delay defined by him before. Eisenbud's time delay for s waves was obviously angle independent. However, in the general (and more realistic) case of several partial waves participating in a reaction, we have just seen that (a) it does not reduce to any useful form unless we perform an angle averaging and (b) even after the angle averaging, it reduces only to Eq. (13) which has an extra $\sin^2 \delta_l$ term as compared to the phase time delay.

IV. LIFETIME MATRICES

The concept of a lifetime matrix in quantum collisions was first introduced by Smith [5]. A collision lifetime was first defined as the limit, $R\to\infty$, of the difference between the time that the scattering particles spend within a distance R of each other, with and without interaction. The definition of this time as introduced by Smith, is now popularly known as the dwell time and is used in a variety of quantum mechanical tunneling problems [16].

A. Smith's Hermitian matrix

The collision lifetime was used to construct a lifetime matrix $\mathbf{Q}$, such that $\mathbf{Q}$ was Hermitian and a diagonal element Q_{ii} gave the average lifetime of a collision beginning in the ith channel. This $\mathbf{Q}$ matrix has some elegant properties,

$$\mathbf{Q} = i\hbar \mathbf{S} d\mathbf{S}^\dagger/dE = -i\hbar(d\mathbf{S}/dE)\mathbf{S}^\dagger = \mathbf{Q}^\dagger,$$

$$\mathbf{Q} = -\mathbf{S}\mathbf{t}\mathbf{S}^\dagger = (\mathbf{t}\mathbf{S})\mathbf{S}^\dagger,$$

$$\mathbf{S} = \mathbf{1} - (i/\hbar)\int_E^\infty \mathbf{Q}(E')\mathbf{S}(E')dE'. \quad (16)$$

The first line of these equations shows that $\mathbf{Q}$ is Hermitian. The second line identifies a time operator and the third line shows that one can find $\mathbf{S}$ from $\mathbf{Q}$ as a function of energy, by inverting the relation given above. The diagonal elements Q_{ii} define an average delay time for a particle injected in the ith channel. Since the particle has probability $|S_{ij}|^2$ of emerging in the jth channel, one can write,

$$\langle \Delta t_{ii}^l\rangle(E) = \sum_j S_{ij}^{*l} S_{ij}^l \Delta t_{ij}^l = \operatorname{Re}\left(-i\hbar \sum_j S_{ij}^{*l} dS_{ij}^l/dE\right) = Q_{ii}^l. \quad (17)$$

The angular brackets indicate the average over channels. We remind the reader once again that the above S-matrix elements, S_{ij}^l and hence the Q-matrix elements, Q_{ii}^l (or Q_{ii} as in Smith's paper), are for a given value of l.

B. Ohmura's angle-dependent lifetime matrix

One can try to construct an angle-dependent lifetime matrix from the angular time-delay expression (9) in the same spirit and one obtains [15]

$$\langle \Delta t_{ii}^{\mathrm{ang}}\rangle(E,\theta) = \sum_j (S_{ij}^* - \delta_{ij})(S_{ij}-\delta_{ij})\Delta t_{ij}^{\mathrm{ang}} = \operatorname{Re}\left(-i\hbar\sum_j (S_{ij}^*-\delta_{ij})dS_{ij}/dE\right) = Q_{ii}^{\mathrm{ang}}(E,\theta). \quad (18)$$

Note that the average time delay in the angular case involves weighting the $\Delta t_{ij}^{\mathrm{ang}}$ $(=\operatorname{Re}\{-i\hbar[(S_{ij}-\delta_{ij})^{-1}dS_{ij}/dE]\})$ by the probabilities, $|S_{ij}-\delta_{ij}|^2$, with S_{ij} being elements of the full S-matrix with no partial wave expansion. It can be easily seen that

$$\mathbf{Q}^{\mathrm{ang}}(E,\theta) = i\hbar(\mathbf{S}-\mathbf{1})\frac{d}{dE}(\mathbf{S}-\mathbf{1})^\dagger \neq -i\hbar\left(\frac{d}{dE}(\mathbf{S}-\mathbf{1})\right)(\mathbf{S}-\mathbf{1})^\dagger = \mathbf{Q}^{\mathrm{ang}\,\dagger}(E,\theta). \quad (19)$$

The elements of $\mathbf{Q}^{\mathrm{ang}}$, namely,

$$Q_{ij}^{\mathrm{ang}}(E,\theta) = i\hbar\sum_n (S_{in}-\delta_{in})dS_{jn}^*/dE, \quad (20)$$

are not symmetric $(Q_{ij}^{\mathrm{ang}} \neq Q_{ji}^{\mathrm{ang}})$ and hence the time-delay is not time-reversal invariant. Indeed, it would be odd to expect of a time delay which depends on the angle of the outgoing particles to be time-reversal invariant.

C. Angle- and channel-averaged time delay

In the preceding section, we see that in the single channel scattering, if one averages the angular time delay over all directions and subtracts a forward scattering term, one indeed recovers Wigner's phase time delay which is consistent with Smith's time-delay matrix expression (7). We now demonstrate that using a similar procedure one can reduce the average time delay (average over channels) given by Ohmura to the l-dependent average time delay, Q^l_{ii}, given by Smith. For a particle entering in the ith channel and with the possibility of emerging in some jth channel, we begin by considering the angle- and channel-averaged expression, namely,

$$\langle \Delta t^{\rm ang}_{ii}\rangle_{\rm av}(E) = \hbar \sum_j \int |f_{ij}(E,\theta)|^2 \frac{\partial}{\partial E} \arg f_{ij}(E,\theta) d\Omega + \frac{2\pi\hbar}{k^2}\frac{d}{dE}[k\delta_{ij}\,{\rm Re}\, f_{ij}(E,0)]. \quad (21)$$

The angular brackets indicate the average over channels, and the subscript av indicates the average over directions. Expressing the f_{ij} in terms of partial waves as before, noticing that $T^l_{ij}=(S^l_{ij}-\delta_{ij})/2ik$, and using as before, $\arg(S^l_{ij}-\delta_{ij}) = {\rm Im}\ln(S^l_{ij}-\delta_{ij})$, the first term on the right-hand side of the above equation becomes

$$\begin{aligned}\langle \Delta t^{\rm ang}_{ii}\rangle_{\rm av1} &= 4\pi\hbar \sum_j \sum_l (2l+1)|T^l_{ij}|^2 \frac{d}{dE}(\arg T^l_{ij}) \\ &= \frac{\pi}{k^2}\sum_l (2l+1)\sum_j (S^l_{ij}-\delta_{ij})^*(S^l_{ij}-\delta_{ij}) \\ &\quad \times {\rm Re}\left(-i\hbar(S^l_{ij}-\delta_{ij})^{-1}\frac{d}{dE}S^l_{ij}\right) \\ &= \frac{\pi}{k^2}\sum_l (2l+1)\sum_j (S^l_{ij}-\delta_{ij})^*(S^l_{ij}-\delta_{ij})[\Delta t^l_{ij}]^{\rm new}.\end{aligned} \quad (22)$$

Starting with the angle-dependent multichannel time-delay matrix, we have now obtained a new definition of an angle-averaged and l dependent time-delay matrix with elements $[\Delta t^l_{ij}]^{\rm new}$. Since the factor $(S^l_{ij}-\delta_{ij})^*(S^l_{ij}-\delta_{ij})$ is real, one can rewrite the above equation as

$$\begin{aligned}\langle \Delta t^{\rm ang}_{ii}\rangle_{\rm av1} &= \frac{\pi}{k^2}\sum_l (2l+1){\rm Re}\left(-i\hbar\sum_j (S^l_{ij}-\delta_{ij})^*\frac{d}{dE}S^l_{ij}\right) \\ &= \frac{\pi}{k^2}\sum_l (2l+1){\rm Re}\left[\left(-i\hbar\sum_j S^{*l}_{ij}\frac{d}{dE}S^l_{ij}\right) + i\hbar\frac{d}{dE}S^l_{ii}\right] \\ &= \frac{\pi}{k^2}\sum_l (2l+1)\left(Q^l_{ii} - \hbar\,{\rm Im}\,\frac{d}{dE}S^l_{ii}\right).\end{aligned} \quad (23)$$

The Q^l_{ii} above is indeed a diagonal element of Smith's lifetime matrix and hence also his channel-averaged time delay for a particle being injected in the channel i. If one now evaluates the second term in Eq. (21) in a similar way as the first, it is easily verified to exactly cancel the second term in the last line of Eq. (23), such that

$$\langle \Delta t^{\rm ang}_{ii}\rangle_{\rm av}(E) = \frac{\pi}{k^2}\sum_l (2l+1)Q^l_{ii}(E). \quad (24)$$

V. NEGATIVE TIME DELAY

The relevance of a "delay time" for resonances comes from the fact that the creation and propagation of an unstable intermediate state in a scattering reaction, delays the reaction. Below, we shall discuss different aspects of negative time delay and try to clarify some misconceptions in literature.

A. Misinterpreted misnomer

In a recent work [17], the example with a Breit-Wigner resonance from Eisenbud [1] was reproduced to demonstrate that the time-delay relations due to Smith and Eisenbud differ. The authors, however, wrongly interpreted a misnomer to be an error which propagated in literature due to Smith. It is probably due to the fact that Eisenbud's paper remained as a thesis that the multichannel time-delay relation (7) which is entirely due to Smith was often mentioned to be due to Eisenbud. Repeating the example from Eisenbud's paper of a two-channel Breit-Wigner resonance with one phase in the T-matrix, the authors reached the conclusion that Eisenbud's time delay is positive and the same for both channels in contrast to that due to Smith which can be negative for a channel with a branching fraction less than one-half. There are several points worth noting:

(i) The general form of Eisenbud's time delay which depends on the S-matrix which has not been decomposed into partial waves cannot be directly compared with that of Smith which is written for a particular l [see Eqs. (7)–(9) and the discussion below Eq. (9)].

(ii) If one assumes the existence of only s waves in reactions (as Eisenbud did in his paper), then the two delay times can be compared. However, as we shall see below, even in this case, there is no need for Eisenbud's time delay to be equal in both channels and positive.

(iii) Probably not having noticed that the lifetime matrix definition for an angular time delay differs from that of the radial time delay as given by Smith [see Eqs. (16)–(20)], on the one hand the authors in [17] claimed that Smith's work has propagated an error (which was actually a misnomer) in literature, and on the other hand, used his definition of the lifetime matrix (which is consistent with his time-delay matrix). This definition was indeed used to demonstrate the validity of speed plots for characterizing resonances.

(iv) Referring to Smith's radial time delay in the multichannel case as a misquoted relation, it was inferred that the conclusions regarding negative time delay in [11,12] were wrong. In what follows, we shall see how regions of negative time delay manifest themselves with the opening of inelastic channels in elastic scatterings.

In order to be able to compare the angular and radial time-delay relations, let us restrict ourselves to the situation in Eisenbud's paper where only s waves participate in reactions. We start with a multichannel, unitary, 2×2 matrix [18],

$$\mathbf{S} = \begin{pmatrix} \eta e^{2i\delta_1} & i(1-\eta^2)^{1/2}e^{i(\delta_1+\delta_2)} \\ i(1-\eta^2)^{1/2}e^{i(\delta_1+\delta_2)} & \eta e^{2i\delta_2}\end{pmatrix}, \quad (25)$$

where δ_1 and δ_2 are the real scattering phase shifts for the elastic scattering ($1\to 1$ and $2\to 2$) in channels 1 and 2 and

η is the inelasticity parameter (with $0<\eta\leq 1$). Smith's radial time-delay matrix then takes the form

$$\Delta\mathbf{t}^{\mathrm{Smith}} = 2\hbar \begin{pmatrix} d\delta_1/dE & \frac{1}{2}\frac{d}{dE}(\delta_1+\delta_2) \\ \frac{1}{2}\frac{d}{dE}(\delta_1+\delta_2) & d\delta_2/dE \end{pmatrix}. \quad (26)$$

The time-delay matrix of Eisenbud has elements with terms involving energy derivatives of the phase shift as well as inelasticity. For example,

$$\Delta t_{ii}^{\mathrm{Eisenbud}} = \frac{2[\eta^2 - \eta\cos(2\delta_i)]d\delta_i/dE - \sin(2\delta_i)d\eta/dE}{1+\eta^2-2\eta\cos(2\delta_i)}. \quad (27)$$

It can be seen that $\Delta t_{11}^{\mathrm{Eisenbud}}$ may not necessarily be the same as $\Delta t_{22}^{\mathrm{Eisenbud}}$ for the obvious reason that δ_1 may not be the same as δ_2. Both the delay times, due to Smith and Eisenbud may not necessarily be positive in the two channels. As an example, we consider the phase shifts and the t-matrix phase for pion-nucleon elastic scattering in the vicinity of the nucleon resonances, $N^*(1510)$ and $N^*(1655)$, where the masses in parentheses correspond to the pole positions. These resonances occur in the $l=0$ partial wave, with total spin $J=1/2$ and isospin $I=1/2$ (with the label S_{11} in the notation $l_{2J,2I}$).

One can see that the regions of negative time delay grow with increasing inelasticity. Both the N^* resonances nevertheless show up as positive peaks in both the Eisenbud and Smith delay times.

B. Density of states and the classical example of averages

Beth and Uhlenbeck [19], while calculating virial coefficients B,C in the equation of an ideal gas, $pV=RT(1+\frac{B}{V}+\frac{C}{V^2}+\cdots)$, found that the difference between the density of states (of scattered particles) with interaction $dn_l(E)/dE$ and without $dn_l^{(0)}(E)/dE$ is

$$\frac{dn}{dE} = \frac{dn_l(E)}{dE} - \frac{dn_l^{(0)}(E)}{dE} = \frac{2l+1}{\pi}\frac{d\delta_l(E)}{dE}. \quad (28)$$

In a resonant scattering, this is the density of states of a resonance (in terms of the decay products) [20]. For instance $T=(\Gamma_R/2)/(E_R-E-i\Gamma_R/2)$, gives

$$\frac{d\delta}{dE} = \frac{\Gamma_R/2}{(E_R-E)^2+\Gamma_R^2/4}. \quad (29)$$

This interpretation is also useful in analyzing the survival probabilities of resonances in all partial waves and at all energies [20] except the near-threshold s-wave resonances [21]. The right-hand side of the Beth-Uhlenbeck formula can be seen to be the so-called phase time delay [and so are the diagonal elements of Smith's multichannel time delay matrix (26)]. In the case of s-wave resonances, the phase and dwell time delay become equal at high energies [21]. Coming back to Smith's definition of time delay which is the difference between the time with and without interaction, taken together with the results in [22] which showed that dwell times correspond to densities of states in a region, the connection between time delay and the difference in the density of states becomes more transparent.

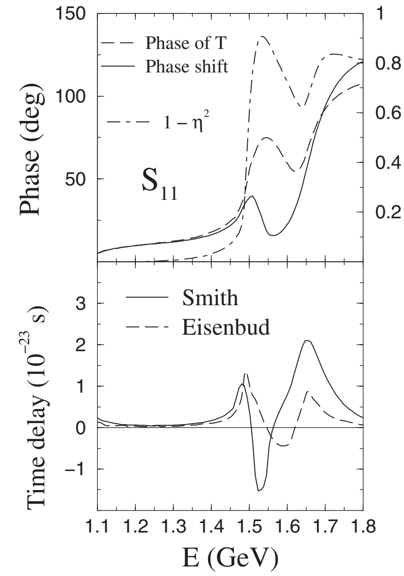

FIG. 1. The T-matrix phase (dashed lines) and scattering phase shifts (solid lines) for the S_{11} resonances occurring in pion-nucleon elastic scattering. The dotted-dashed lines are inelasticities $(1-\eta^2)$ with the scale given on the right-hand side. The lower part of the figure contains the time-delay plots using Smith's and Eisenbud's relations.

In the case of single channel scattering, one expects the time delay of Wigner and Smith to peak in the vicinity of a resonance. This is intuitively based on the fact that an unstable particle (seen as an intermediate state becoming on-shell) has a finite lifetime which delays the reaction. In such a case, the time interval δt, spent by the particle during the scattering process is $\delta t=\delta t_0+\Delta t$ where $\Delta t>0$ is the contribution due to interaction and δt_0 is the time spent without interaction. In the case of multichannel resonances, this picture must be modified. If the branching ratio, Γ_i/Γ (Γ_i and Γ are the partial and full widths, respectively) of the elastic channel i is close to one, the validity of the above interpretation remains. Smaller branching ratios lead, however, to negative delay times ($\Delta t<0$) due to the loss of flux in this channel. We can explain it by using classical averages which mimic the quantum ones in the delay times. In the multichannel case, δt is now the time $\delta t_{\mathrm{interaction}}$ say, weighted with the probability of arrival. If we start with N particles scattering on a target, due to absorption (i.e., the opening of a new channel) only $N_a<N$ will emerge after scattering. Hence, $\delta t=(N_a/N)\delta t_{\mathrm{interaction}}$, but δt_0 remains unchanged. As a result, we obtain $\Delta t=(N_a/N)\delta t_{\mathrm{interaction}}-\delta t_0$. If due to a resonance $\delta t_{\mathrm{interaction}}>\delta t_0$, it can still happen that $(N_a/N)\delta t_{\mathrm{interaction}} < \delta t_0$, and therefore $\Delta t<0$ if N_a/N is small corresponding to a small branching ratio in this channel. This classical picture is not very far from what really happens to the quantum concept of time delay. Indeed, in Fig. 1, Δt has a negative dip exactly at the very same position where the inelasticity is largest.

VI. CONCLUSIONS

According to the wave-packet analysis of the scattered wave, there are only two viable time-delay concepts in quantum scattering. The first one is due to Smith involving the S-matrix elements after a partial wave decomposition has been performed. As a result, this time delay depends only on energy. The other possibility is conceptually similar, but makes use of the full T-matrix ($S=1+2iT$) and is energy and angle dependent. Whereas Smith's approach is more suitable for resonant scattering with resonances classified according to the l quantum number, the angular time delay is a directly measurable concept. Not even in a single channel scattering formalism will these two agree. Only in an artificial world of a single channel with the restriction to s waves only, we obtain $\arg f = \arg T_{l=0} = \arg S_{l=0}$ as considered by Eisenbud who introduced the time delay concept. It is however possible to make a connection between the two quantum time concepts. Averaging the angular time delay over angles and channels gives us back the average time delay of Smith.

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