# Entanglement monogamy in a three-qubit state

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(Received 3 May 2008; published 14 July 2008)

We investigate the monogamy nature of entanglement in a three-qubit system. A monogamy inequality is presented to describe the exclusive relation between the *A*-*B* two-qubit concurrence $C_{AB}$ and the *AB*-*C* three-qubit concurrence $C_{(AB)C}$, which represents the entanglement between qubits *A* and *B* as a whole and the third qubit *C*. It is found that the entanglement between any two qubits in a three-qubit system is limited by the entanglement between these two qubits and another qubit. As a consequence, we present the upper bounds for the concurrence $C_{AB}$, when the concurrence between qubits *A* and *C* ($C_{AC}$) and the concurrence between qubits *B* and *C* ($C_{BC}$) are both given or one of the two is provided.

DOI: 10.1103/PhysRevA.78.012325 PACS number(s): 03.67.Mn, 03.65.Ta, 03.65.Ud

## I. INTRODUCTION

Entanglement plays an important role in distinguishing quantum theory from classical physics. Owing to its fascinating properties and many promising applications in information science, entanglement is attracting more and more attention. During the last decade, great progress on entanglement has been made in various aspects, such as its quantification [1–5], generation [6], applications [7], and experimental realization [8], etc. However, as a typical counterintuitive concept in quantum word, entanglement is still far from being completely understood. Recently, a very interesting property of entanglement, usually called "monogamy nature of entanglement" was discovered and frequently discussed, that is, the degree of entanglement between any two objects *A* and *B* will be limited if the object *A* (or *B*) is being entangled with another object *C*. In the extreme case with maximally entangled *A* (or *B*) and *C*, the composite system *A*-*C* (or *B*-*C*) must stay in a pure state [9] and thus be separable to the rest object *B* (or *A*).

The first inequality for the quantification of the entanglement monogamy was achieved by Coffman, Kundu, and Wootters in 2000. In a tripartite system composed of qubits *A*, *B*, and *C*, by regarding any two of the three qubits, e.g., *A* and *B*, as a single object and two independent objects, respectively, they find that the squared sum of the concurrence between qubit *A* and *C* and the concurrence between qubit *B* and *C* cannot exceed the squared concurrence between qubit *C* and the composite object (*AB*) [10], i.e.,

$$C_{AC}^2 + C_{BC}^2 \leq C_{(AB)C}^2. \tag{1}$$

The generalization of the above monogamy relation to *n*-qubit states, which is conjectured by Coffman, Kundu, and Wootters in Ref. [10], has been proven recently by Osborne and Verstraete in Ref. [11]. In addition, if the concurrence for two-qubit entanglement is replaced by other entanglement measures, such as distillable entanglement [1], negativity [4], and squashed entanglement [5], the monogamy relation (1) is also valid [12,13]. In continuous variable systems, a similar constraint relation [14], or even a stronger one [15], can be established in *N*-mode Gaussian states.

Another type of monogamy inequality for the entanglement is related to the entanglement of assistance (EOA) [16]. Given a tripartite quantum state $|\Psi_{ABC}\rangle$ with the three parties *A*, *B*, and *C* belonging to three players Alice, Bob, and Charlie, respectively, Charlie can in principle find a suitable set of local measurements to keep the state of the subsystem *A*-*B*, i.e., $\rho_{AB}=\mathrm{Tr}_C(|\Psi_{ABC}\rangle\langle\Psi_{ABC}|)$, invariant and maximize the average entanglement between the other two parties *A* and *B*. This maximum value of the average entanglement between *A* and *B*, with the aid of the local operations of Charlie on party *C*, is called "entanglement of assistance" $E_a(\rho_{AB})$ [16]. As is well-known, provided an entanglement measure $E(|\Phi_{AB}\rangle)$ for an arbitrary bipartite pure state $|\Phi_{AB}\rangle$, the corresponding entanglement for a mixed state $\rho_{AB}$ is then defined as

$$E(\rho_{AB}) \equiv \min_{\{p_i,|\Phi_i\rangle\}} \sum_i p_i E(|\Phi_i\rangle), \tag{2}$$

where $\{p_i,|\Phi_i\rangle\}$ stands for pure state decompositions of the mixed state $\rho_{AB}$, satisfying $\rho_{AB}=\Sigma_i p_i|\Phi_i\rangle\langle\Phi_i|$. Following this line, the EOA mentioned above can be mathematically described by

$$E_a(\rho_{AB}) \equiv \max_{\{p_i,|\Phi_i\rangle\}} \sum_i p_i E(|\Phi_i\rangle). \tag{3}$$

Obviously, for any bipartite state $\rho_{AB}$, the EOA $E_a(\rho_{AB})$ is always larger than or equal to the entanglement $E(\rho_{AB})$ [17]. In terms of concurrence of assistance (COA) [18], which is a special type of EOA and also an entanglement monotone for $2\otimes 2\otimes n$ tripartite pure states [19], the above conclusion can be generalized into the case of *n*-qubit states [20],

$$C_{AB^{(1)}}^2 + C_{AB^{(2)}}^2 + \cdots + C_{AB^{(n-1)}}^2 \leq S_L(\rho_A) \leq C_a^2 + C_a^2(\rho_{AB^{(2)}}) + \cdots + C_a^2(\rho_{AB^{(n-1)}}), \tag{4}$$

where $C_{AB^{(i)}}$ is the concurrence of the system composed of two particles *A* and $B^{(i)}$, and $S_L(\rho_A)$ being linear entropy [21]

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with respect to the reduced density matrix $\rho_A$.

Since a quantum system being entangled with another one would limit its possible entanglement with a third system, we ask ourselves the following two questions. First, given the entanglement between any two subsystems of an $A$-$B$-$C$ tripartite system, e.g., $E(\rho_{AC})$ or $E(\rho_{BC})$, what are the upper and lower bounds for the entanglement $E(\rho_{AB})$? Second, in the same tripartite system, if the entanglements $E(\rho_{AC})$ and $E(\rho_{BC})$ are both provided, what is the limitation on the entanglement $E(\rho_{AB})$? In this paper, we focus on the simplest tripartite system composed of three qubits $A$, $B$, and $C$. After providing a monogamy relation between the entanglement $C_{AB}$ and the entanglement $C_{(AB)C}$, where $C_{AB}$ represents the concurrence between qubits $A$ and $B$, and $C_{(AB)C}$ represents the concurrence between the qubits $A$ and $B$ as a single object and qubit $C$, we will present some typical examples and then give the answers to the above questions.

## II. MONOGAMY RELATION IN A THREE-QUBIT STATE

### A. Entanglement of a bipartite 4⊗2 quantum state

In a tripartite state composed of three qubits $A$, $B$, and $C$, if the first two qubits $A$ and $B$ are considered as a single object, denoted by $(AB)$ in this paper, such a three-qubit state is equivalent to a bipartite $4\otimes 2$ quantum state. To quantify the entanglement of a pure $4\otimes 2$ quantum state, many computable measures could be chosen, such as $\Theta$ concurrence [22] and $I$ concurrence [23]. Just as done in Ref. [10], here we quantify the entanglement of a pure $4\otimes 2$ quantum state, $|\psi_{(AB)C}\rangle$, in terms of linear entropy, which is defined by [21]

$$C^2(|\psi_{(AB)C}\rangle) = S_L(\rho_{(AB)}) = 2(1 - \mathrm{Tr}\ \rho^2_{(AB)}) = S_L(\rho_C) = 2(1 - \mathrm{Tr}\ \rho^2_C), \tag{5}$$

where $\rho_{(AB)}=\mathrm{Tr}_C(|\psi_{(AB)C}\rangle\langle\psi_{(AB)C}|)$ and $\rho_C=\mathrm{Tr}_{(AB)}(|\psi_{(AB)C}\rangle\langle\psi_{(AB)C}|)$ are the reduced density matrices with respect to the first object composed of qubits $A$ and $B$, and the second object with single qubit $C$, respectively. Similar to concurrence for two-qubit states, the linear entropy of entanglement of a bipartite pure $4\otimes 2$ state is also directly determined by the reduced density matrix, and they even present the same result for a pure two-qubit state and a pure $4\otimes 2$ state if the reduced density matrices of the two quantum states have the same eigenvalue spectrum. In this paper, we use the symbol $C_{(AB)C}$ to represent the entanglement between the first object $(AB)$ and the second object (qubit $C$) in a three-qubit state and call it concurrence too. Given two eigenvalues, $\lambda_1$ and $\lambda_2=1-\lambda_1$, for the reduced density matrix ($\rho_{(AB)}$ or $\rho_C$) of a three-qubit pure state $|\psi_{(AB)C}\rangle$ (as mentioned above, considering the two qubits $A$ and $B$ as a whole), the concurrence $C_{(AB)C}$ can be calculated by [10]

$$C_{(AB)C} = 2\sqrt{\lambda_1\lambda_2} = 2\sqrt{\lambda_1(1-\lambda_1)}. \tag{6}$$

### B. Maximal concurrence of two-qubit states having two fixed nonzero eigenvalues

The reduced state of a three-qubit pure state has at most two nonzero eigenvalues, no matter what subsystem, a single-qubit state or a two-qubit state, is under consideration. Here we assume the reduced density matrix of a three-qbuit state, $\rho_{AB}=\mathrm{Tr}_C|\phi_{ABC}\rangle\langle\phi_{ABC}|$, has two eigenvalues $\lambda_1$ and $\lambda_2$, with $\lambda_1\geq\lambda_2$. That is to say, the reduced two-qubit state $\rho_{AB}$ can be generally described by a diagonal matrix $\Lambda$ and a $4\times 4$ unitary transformation $U$,

$$\rho_{AB} = U\Lambda U^\dagger, \tag{7a}$$

with the diagonal matrix $\Lambda$ being determined by two eigenvalues $\lambda_1$ and $\lambda_2$ through

$$\Lambda = \mathrm{diag}(\lambda_1,\lambda_2,0,0). \tag{7b}$$

Now we want to know if the two nonzero eigenvalues $\lambda_1$ and $\lambda_2=1-\lambda_1$ ($\lambda_1\geq\lambda_2$) of a two-qubit state are provided, but the detailed information about the unitary transformation $U$ in Eq. (7a) is not clear, what is the maximal and minimal concurrence we can estimate for such a two-qubit state? As is well-known, the concurrence of a two-qubit state $\rho_{AB}$ is calculated through

$$C_{AB} = \max\{0, \sqrt{\eta_1} - \sqrt{\eta_2} - \sqrt{\eta_3} - \sqrt{\eta_4}\}, \tag{8}$$

with $\eta_1$, $\eta_2$, $\eta_3$, and $\eta_4$ being eigenvalues of the matrix $\rho_{AB}\tilde{\rho}_{AB}=\rho_{AB}(\sigma_y\otimes\sigma_y)\rho^*_{AB}(\sigma_y\otimes\sigma_y)$ in decreasing order. Based on Eqs. (7), the $\rho_{AB}\tilde{\rho}_{AB}$ in our present case could be written as

$$\rho_{AB}\tilde{\rho}_{AB} = U\Lambda U^\dagger(\sigma_y\otimes\sigma_y)(U\Lambda U^\dagger)^*(\sigma_y\otimes\sigma_y) = U\Lambda U^\dagger(\sigma_y\otimes\sigma_y)U^*\Lambda U^T(\sigma_y\otimes\sigma_y). \tag{9}$$

Since the concurrence of $\rho_{AB}$ is strictly determined by the eigenvalues of $\rho_{AB}\tilde{\rho}_{AB}$, we can make a similar transformation on the above matrix,

$$U^\dagger\rho_{AB}\tilde{\rho}_{AB}U = \Lambda S^\dagger\Lambda S, \tag{10}$$

where $S=U^T(\sigma_y\otimes\sigma_y)U$ is a symmetric unitary matrix, and solely determined by the matrix $U$ in Eq. (7a) ($U^T$ represents the transpose of the matrix $U$). Thus the concurrence of the state $(B)$ is now determined by the eigenvalues of $\Lambda S^\dagger\Lambda S$. As a similar matrix of $\Lambda$, the matrix $S^\dagger\Lambda S$ has the same eigenvalue spectrum with $\Lambda$, i.e., $\{\lambda_1,\lambda_2,0,0\}$.

The following question is, what are the minimal and maximal concurrences of the state with the form (7) by scanning the matrix $U$ in Eq. (7a) over all $4\times 4$ unitary matrices, or equivalently scanning the matrix $S$ in Eq. (10) over all $4\times 4$ symmetric unitary matrices? The minimal one is easy to achieve. In Eqs. (7), if we chose the matrix $U$ as the four-dimensional unity matrix, all four eigenvalues of $\rho_{AB}\tilde{\rho}_{AB}$ (and also $\Lambda S^\dagger\Lambda S$) are zero, which corresponds to zero concurrence.

The derivation of the maximal concurrence for the state (7) is a little more complicated. Based on matrix theory, given two semipositive $n\times n$ matrices $M_1$ and $M_2$ with eigenvalues $\{\nu_1,\ldots,\nu_n\}$ and $\{\mu_1,\ldots,\mu_n\}$ in decreasing order, respectively, the eigenvalues of the product of $M_1$ and $M_2$ are bounded by $\nu_1\mu_1$ from above and by $\nu_n\mu_n$ from below [24]. Since the two matrices $\Lambda$ and $S^\dagger\Lambda S$ in Eq. (10) have the same eigenvalue spectrum with $\lambda_1$ being the maximal one, we know that the maximal eigenvalue of the matrix $\Lambda S^\dagger\Lambda S$ is $\eta_1=\lambda_1^2$. This maximal eigenvalue is always achievable, e.g., we can choose such a matrix $S$ with the following form:

$$S=\mathrm{diag}\{1,S_3\}, \tag{11}$$

where the block $S_3$ is an arbitrary three-dimensional symmetric unitary matrix.

Our following task is to find the minimal value of $\sqrt{\eta_2}+\sqrt{\eta_3}+\sqrt{\eta_4}$ to maximize the concurrence in Eq. (8). In the ideal case, the three eigenvalues, $\eta_2$, $\eta_3$, and $\eta_4$, should be set to zero. Fortunately, this ideal case is also achievable. For example, we can choose

$$U=\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ -1 & 0 & 0 & i \end{pmatrix} \tag{12a}$$

in Eq. (7a), corresponding to

$$S=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12b}$$

and

$$\rho_{AB}=\frac{1}{2}\begin{pmatrix} \lambda_1 & 0 & 0 & -\lambda_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\lambda_2 & 0 \\ -\lambda_1 & 0 & 0 & \lambda_1 \end{pmatrix}. \tag{12c}$$

The above two-qubit state (12c) has two nonzero eigenvalues $\lambda_1$ and $\lambda_2$, and the four eigenvalues of $\rho_{AB}\tilde{\rho_{AB}}$ are $\{\lambda_1^2,0,0,0\}$, which implies the maximal concurrence $C_{AB}^{\max}=\lambda_1$. Thus we conclude that for a two-qubit state $\rho_{AB}$ with two fixed nonzero eigenvalues $\lambda_1$ and $\lambda_2=1-\lambda_1$ ($\lambda_1\geq\lambda_2$), the possible concurrence $C_{AB}$ is confined in the range $[0,\lambda_1]$.

### C. Monogamy inequality in a three-qubit state

Now let us consider the above two conclusions together. According to the discussion in Sec. II A, a two-qubit state with two fixed nonzero eigenvalues $\lambda_1$ and $\lambda_2=1-\lambda_1$ can represent the reduced density matrix of an $(AB)$-$C$ three-qubit pure state, and the concurrence between the composite object $(AB)$ and qubit $C$ is evaluated by $C_{(AB)C}=2\sqrt{\lambda_1(1-\lambda_1)}$. At the same time, the possible concurrence of such two-qubit states has a maximal value $C_{AB}^{\max}=\lambda_1$ (we have assumed that $\lambda_1\geq\lambda_2$). That is to say, in a three-qubit pure state $|\psi_{(AB)C}\rangle$, the concurrence between qubits $A$ and $B$, $C_{AB}$, is limited by the concurrence between the composite object $(AB)$ and qubit $C$, $C_{(AB)C}$, through the following relation:

$$C_{AB}\leq\frac{1}{2}\left(1+\sqrt{1-C_{(AB)C}^2}\right). \tag{13}$$

Obviously, the entanglement between qubits $A$ and $B$ does not favor the participation of the third party $C$ because the maximal concurrence $C_{AB}$ decreases with increasing concurrence $C_{(AB)C}$. This phenomenon is in accord with the monogamy nature of entanglement. From Eq. (13), we also see that the disturbance of the third qubit $C$ on the previous two qubits $A$ and $B$ as a whole cannot completely destroy the quantum correlation between qubits $A$ and $B$. That is, given an arbitrary value for the concurrence $C_{(AB)C}$, we can always find a suitable three-qubit state, in which the qubits $A$ and $B$ are entangled with concurrence not less than 0.5. For example, in the extreme case with $C_{(AB)C}=1$, we can choose such a three-qubit state $|\psi\rangle=\frac{1}{2}|100\rangle+\frac{1}{2}|010\rangle+\frac{1}{\sqrt{2}}|001\rangle$, in which $C_{AB}=0.5$.

In the following, we present the proof that the above monogamy inequality (13) also holds for mixed three-qubit states. By tracing a mixed three-qubit state $\rho_{ABC}$ over the degrees of one subsystem, e.g., qubit $C$, the reduced density matrix $\rho_{AB}=\mathrm{Tr}_C(\rho_{ABC})$ is a general two-qubit state, usually having four nonzero eigenvalues, i.e.,

$$\rho_{AB}=U\Lambda U^{\dagger}, \tag{14a}$$

with $\Lambda$ being a diagonal matrix

$$\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_4). \tag{14b}$$

Without loss of generality, we assume the four eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ in Eq. (14) satisfy $\lambda_1\geq\lambda_2\geq\lambda_3\geq\lambda_4$. In the case of $\lambda_1\leq\frac{1}{2}$, we can rewrite the diagonal matrix $\Lambda$ as

$$\begin{aligned}\Lambda=&\left[\lambda_1-\frac{\lambda_1\lambda_2}{1-\lambda_1}+\frac{\lambda_1^2\lambda_3}{(1-\lambda_1)^2}-\frac{\lambda_1^3\lambda_4}{(1-\lambda_1)^3}\right]\mathrm{diag}(1,0,0,0)\\ &+\left[\frac{\lambda_2}{1-\lambda_1}-\frac{\lambda_1\lambda_3}{(1-\lambda_1)^2}+\frac{\lambda_1^2\lambda_4}{(1-\lambda_1)^3}\right]\mathrm{diag}(\lambda_1,1-\lambda_1,0,0)\\ &+\left[\frac{\lambda_3}{1-\lambda_1}-\frac{\lambda_1\lambda_4}{(1-\lambda_1)^2}\right]\mathrm{diag}(0,\lambda_1,1-\lambda_1,0)\\ &+\frac{\lambda_4}{1-\lambda_1}\mathrm{diag}(0,0,\lambda_1,1-\lambda_1)\\ =&\,p_0\Lambda_0+p_1\Lambda_1+p_2\Lambda_2+p_3\Lambda_3,\end{aligned} \tag{15}$$

where the diagonal matrices $\Lambda_j$ ($j=1,2,3$) have two nonzero eigenvalues $\lambda_1$ and $1-\lambda_1$. According to the result in Sec. II A, each one of them corresponds to a pure $4\otimes 2$ state with concurrence $2\sqrt{\lambda_1(1-\lambda_1)}$. The matrix $\Lambda_0$ in Eq. (15) represents a pure state component in a decomposition of the reduced state $\rho_{AB}$, which is related to a product state component in the state $\rho_{(AB)C}$. The coefficients $p_j$ ($j=0,1,2,3$) are the probabilities of components $\Lambda_j(j=0,1,2,3)$, satisfying $\sum_{j=0}^3 p_j=1$. In the case of $\lambda_1\geq\frac{1}{2}$, we can also find a decomposition of $\Lambda$ by exchanging $\lambda_1$ and $1-\lambda_1$ to each other in Eqs. (15). That is

$$\Lambda=p_0'\Lambda_0'+p_1'\Lambda_1'+p_2'\Lambda_2'+p_3'\Lambda_3', \tag{16}$$

where the diagonal matrices $\Lambda_j'$ ($j=1,2,3$) also have two nonzero eigenvalues $\lambda_1$ and $1-\lambda_1$, and each one of them corresponds to a pure $4\otimes 2$ state with concurrence $2\sqrt{\lambda_1(1-\lambda_1)}$ in a decomposition of the mixed state $\rho_{(AB)C}$. Based on the convexity of the entanglement of a mixed state, which is usually defined as the minimum average entanglement among all pure state decompositions [see expression

Eqs. (2)], the concurrence of $\rho_{(AB)C}$ in both cases (15) and (16) should satisfy

$$C_{(AB)C} \leq \sum_{i=0}^{3} p_i C_{\Lambda_i} = 2\sqrt{\lambda_1(1-\lambda_1)}(p_1+p_2+p_3)$$
$$\leq 2\sqrt{\lambda_1(1-\lambda_1)},$$

or

$$C_{(AB)C} \leq \sum_{i=0}^{3} p_i' C_{\Lambda_i'} = 2\sqrt{\lambda_1(1-\lambda_1)}(p_1'+p_2'+p_3')$$
$$\leq 2\sqrt{\lambda_1(1-\lambda_1)}, \quad (17)$$

where we have used the symbols $C_{\Lambda_i}$ (or $C_{\Lambda_i'}$) to represent the concurrence of a pure $4\otimes 2$ state, whose reduced state is similar to the diagonal matrix $\Lambda_i$ (or $\Lambda_i'$).

The upper bound for the concurrence of Eq. (14) can be achieved in a similar way as in Sec. II B. Assuming the four eigenvalues of $\rho_{AB}\tilde{\rho_{AB}}$, or equivalently $\Lambda S^{\dagger}\Lambda S$, in decreasing order are $\eta_1$, $\eta_2$, $\eta_3$, and $\eta_4$, the concurrence $C_{AB}$ is calculated by $\sqrt{\eta_1}-\sqrt{\eta_2}-\sqrt{\eta_3}-\sqrt{\eta_4}$. At the same time, the maximal eigenvalue of $\Lambda S^{\dagger}\Lambda S$ is bounded by the product of the maximal eigenvalues of $\Lambda$ and $S^{\dagger}\Lambda S$, i.e., $\eta_1 \leq \lambda_1^2$. Thus we have

$$C_{AB} \leq \sqrt{\eta_1} \leq \lambda_1. \quad (18)$$

This result provides a mathematical evidence that all maximally entangled two-qubit states with unity concurrence are pure states [9] because the maximal eigenvalue $\lambda_1$ for all maximally entangled states has to be set to one according to the above Eq. (18), which leaves the remaining three eigenvalues vanishing. The combination of Eqs. (17) and (18) leads to the same monogamy inequality described in Eq. (13).

## III. RESULTS AND DISCUSSIONS

In this section, we will illustrate some examples for the monogamy relation in three-qubit states. By combining a previous result, we present a useful application for estimating the concurrence between two qubits $A$ and $B$ on the condition that these two qubits are being entangled to another qubit $C$.

### A. Monogamy relation in *W*-type and GHZ-type three-qubit states

The monogamy inequality (13) in the three-qubit states could be shown in the shadow area of Fig. 1. Let us now examine two famous three-qubit states, which are the $W$ state [25], defined as $|W\rangle=\frac{1}{\sqrt{3}}(|100\rangle+|010\rangle+|001\rangle)$, and the Greenberger-Horne-Zeilinger (GHZ) state [26], defined as $|\mathrm{GHZ}\rangle=\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$. By considering the two qubits $A$ and $B$ as a whole, it is not hard to get the concurrence $C_{(AB)C}^{(W)}=2\sqrt{2}/3$ and $C_{(AB)C}^{(\mathrm{GHZ})}=1$, based on Eq. (5). In the basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$, the reduced states of $W$ state and GHZ state after tracing out the degrees of qubit $C$ are

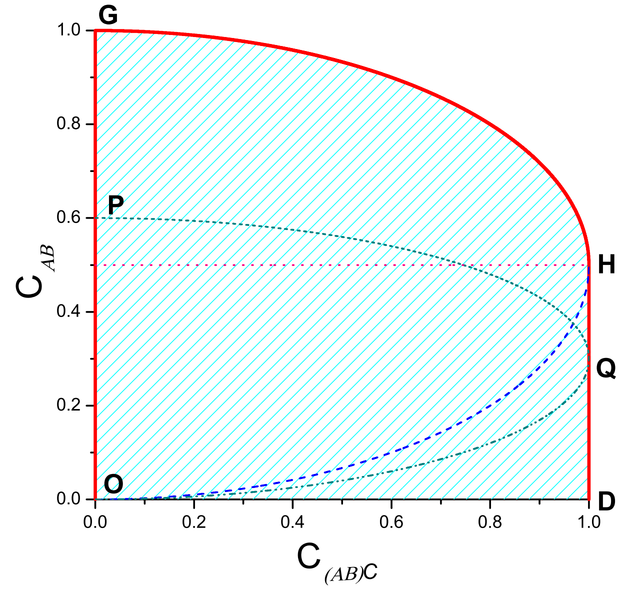


FIG. 1. (Color online) Limitation of the concurrence $C_{AB}$ in a three-qubit state due to the concurrence $C_{(AB)C}$. The arc $\widehat{OHG}$ corresponds to the state $|W'\rangle=\sqrt{1-|\gamma|^2}(\frac{1}{\sqrt{2}}e^{i\phi_1}|100\rangle+\frac{1}{\sqrt{2}}e^{i\phi_2}|010\rangle)+|\gamma|e^{i\phi_3}|001\rangle$, and the arc $\widehat{OQP}$ corresponds to the state $|W''\rangle=\sqrt{1-|\gamma|^2}(\frac{3}{\sqrt{10}}e^{i\psi_1}|100\rangle+\frac{1}{\sqrt{10}}e^{i\psi_2}|010\rangle)+|\gamma|e^{i\psi_3}|001\rangle$. Relation between concurrence $C_{AB}$ and concurrence $C_{(AB)C}$ in an extended GHZ state (20) is shown as the bottom line $OD$ of the shadow area.

$$\rho_{AB}^{(W)} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (19a)$$

and

$$\rho_{AB}^{(\mathrm{GHZ})} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (19b)$$

It is easy to get the concurrence $C_{AB}^{(W)}=2/3$ and $C_{AB}^{(\mathrm{GHZ})}=0$. Here we see the monogamy inequality (13) is saturated for the $W$ state, and the GHZ state only reaches the lower bound of the above monogamy inequality.

We can make a deeper discussion by extending the $W$ state and GHZ state to a more general case. Considering the following $W$-type and GHZ-type states,

$$|W'\rangle = \alpha|100\rangle + \beta|010\rangle + \gamma|001\rangle \quad (|\alpha|^2+|\beta|^2+|\gamma|^2=1) \quad (20a)$$

and

$$|\mathrm{GHZ}'\rangle = a|000\rangle + b|111\rangle \quad (|a|^2+|b|^2=1), \quad (20b)$$

the reduced density matrices by tracing the above two states over the degrees of qubit $C$ are

$$\rho_{AB}^{(W')} = \begin{pmatrix} |\gamma|^2 & 0 & 0 & 0 \\ 0 & |\beta|^2 & \alpha^*\beta & 0 \\ 0 & \alpha\beta^* & |\alpha|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{21a}$$

and

$$\rho_{AB}^{(\mathrm{GHZ}')} = \begin{pmatrix} |a|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |b|^2 \end{pmatrix}. \tag{21b}$$

Based on Eq. (5) for the concurrence of high dimensional bipartite pure states, and standard two-qubit concurrence by Wootters, the above two density matrices lead to

$$C_{(AB)C}^{(W')} = 2|\gamma|\sqrt{|\alpha|^2+|\beta|^2},$$

$$C_{(AB)C}^{(\mathrm{GHZ}')} = 2|a||b|, \tag{22}$$

for the concurrence between qubits $A$ and $B$ as a whole and qubit $C$ in the $W$-type state and the GHZ-type state, and

$$C_{AB}^{(W')} = 2|\alpha||\beta|,$$

$$C_{AB}^{(\mathrm{GHZ}')} = 0, \tag{23}$$

for the concurrence between qubits $A$ and $B$. It is easy to verify that the monogamy inequality (13) is saturated for the above extended $W$ state when $|\alpha|=|\beta|$ and $|\gamma|^2 \le 0.5$ in Eq. (20a) (see the arc $\widehat{GH}$ in Fig. 1). The curve $\widehat{OH}$ in Fig. 1 describes the relation between the concurrence $C_{AB}^{(W')}$ and the concurrence $C_{(AB)C}^{(W')}$ for $|\alpha|=|\beta|$ and $|\gamma|^2 \ge 0.5$. In Fig. 1, we also plot the dependence of $C_{AB}^{(W')}$ on $C_{(AB)C}^{(W')}$ when $|\alpha|=3|\beta|$. The curves $OQ$ and $PQ$ in Fig. 1 correspond to $|\gamma|^2 \ge 0.5$ and $|\gamma|^2 \le 0.5$ in this case, respectively. Since the concurrence between qubits $A$ and $B$ is always zero [see the second equation in (23)] for the extended GHZ state (20b), it can be represented by the line $OD$ in Fig. 1.

The entanglement of a three-qubit GHZ-type state (20b) is very fragile in real noisy environments, because the loss of any qubit, $A$, $B$, or $C$, will lead to complete destruction of entanglement in this system. Comparatively speaking, the entanglement of the $W$-type state (20a) is more robust because the noises on qubit $C$, which induce decoherence between qubit $C$ and object $(AB)$, might not destroy the concurrence $C_{AB}$. To this point of view, the bottom line $OD$ in Fig. 1 represents the most fragile three-qubit entangled states, the entanglement in which cannot resist the noises imposed on qubit $C$, whereas the arc $\widehat{GH}$ in Fig. 1 means the maximal concurrence left in the system, after qubit $C$ is missing, or say, disturbed by some quantum noises, and this maximal value depends on the original concurrence $C_{(AB)C}$ before the noises imposed on qubit $C$ is considered.

### B. Application of the monogamy relation in three-qubit states

The monogamy relation (13) presents an upper bound for a two-qubit state when these two qubits, considered as a single object, are being entangled to another qubit. However, a more practical question is, in an $A$-$B$-$C$ three-qubit system, how the entanglement between qubits $A$ and $B$ is affected by the entanglement between qubits $A$ and $C$? Furthermore, if the entanglement between qubits $A$ and $C$, and the entanglement between qubits $B$ and $C$ are given simultaneously in a three-qubit system, what estimation can we make for the entanglement between qubit $A$ and $B$? To answer these two questions, we need to replace the three-qubit concurrence $C_{(AB)C}$ (actually a bipartite $4\otimes 2$ concurrence) in the monogamy relation (13) by two-qubit concurrences. This can be satisfied by substituting the previous entanglement monogamy relation (1), derived by Coffman, Kundu, and Wootters, into the present result (13), i.e.,

$$C_{AB} \le \frac{1}{2}(1+\sqrt{1-C_{AC}^2-C_{BC}^2}). \tag{24a}$$

The above equation presents a direct relation between three two-qubit concurrences in a three-qubit state. In a direct way, we see that the concurrence between qubits $A$ and $B$ is limited by the concurrence between qubits $A$ and $C$, and the concurrence between qubits $B$ and $C$. Another two similar monogamy inequalities can be achieved under the permutation of the three qubits $A$, $B$, and $C$, that is,

$$C_{AC} \le \frac{1}{2}(1+\sqrt{1-C_{AB}^2-C_{BC}^2}), \tag{24b}$$

$$C_{BC} \le \frac{1}{2}(1+\sqrt{1-C_{AB}^2-C_{AC}^2}). \tag{24c}$$

The combination of the above three equations forms the constraint relation among the three two-qubit concurrences $C_{AB}$, $C_{AC}$, and $C_{BC}$. With these three new monogamy inequalities, we can discuss the above questions now. If only one of the two concurrences, $C_{AC}$ and $C_{BC}$, is provided, and the other one is completely unknown, the above three inequalities confine the concurrence $C_{AB}$ in the range

$$0 \le C_{AB} \le \sqrt{1-C_{AC}^2} \tag{25a}$$

or

$$0 \le C_{AB} \le \sqrt{1-C_{BC}^2}. \tag{25b}$$

In the other situation, if the two concurrences, $C_{AC}$ and $C_{BC}$, are both provided, the inequality (24) gives $C_{AB} \le \sqrt{1-C_{BC}^2-[(C_{AC}-\frac{1}{2})+|C_{AC}-\frac{1}{2}|]^2}$, and the inequality (24) gives $C_{AB} \le \sqrt{1-C_{AC}^2-[(C_{BC}-\frac{1}{2})+|C_{BC}-\frac{1}{2}|]^2}$. In a word, the concurrence between qubits $A$ and $B$ in a three-qubit state, $C_{AB}$, is confined in the range

$$0 \le C_{AB} \le \min\{a_1,a_2,a_3\}, \tag{26a}$$

with

$$a_1 = \frac{1}{2}(1+\sqrt{1-C_{AC}^2-C_{BC}^2});$$

$$a_2 = \sqrt{1-C_{BC}^2-\left[\left(C_{AC}-\frac{1}{2}\right)+\left|C_{AC}-\frac{1}{2}\right|\right]^2};$$

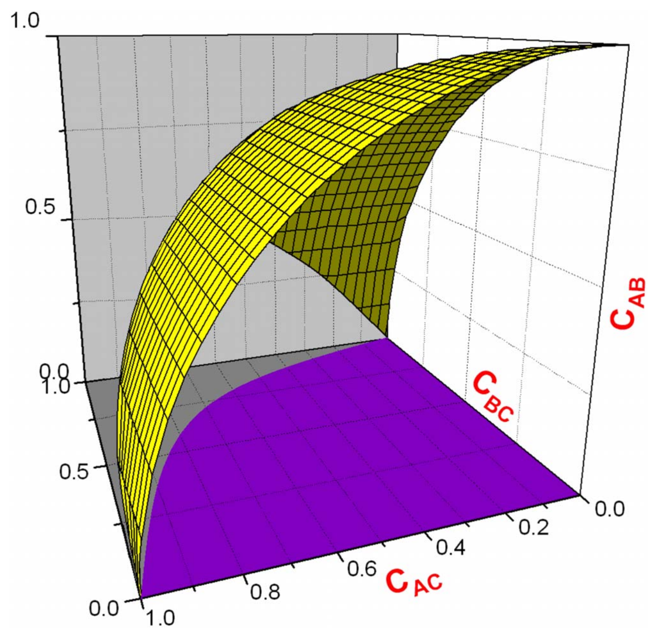


FIG. 2. (Color online) Dependence of the upper bound of concurrence $C_{AB}$ on the other two two-qubit concurrences $C_{AC}$ and $C_{BC}$ in a three-qubit state.

$$a_3 = \sqrt{1 - C_{AC}^2 - \left[\left(C_{BC} - \frac{1}{2}\right) + \left|C_{BC} - \frac{1}{2}\right|\right]^2}. \quad (26b)$$

Obviously, this is a stronger upper bound compared with the result (25), in accord with the more stringent condition that both concurrences $C_{AC}$ and $C_{BC}$ are provided. In Fig. 2, we plot the dependence of the upper bound of concurrence $C_{AB}$ on the other two two-qubit concurrences $C_{AC}$ and $C_{BC}$ in this case. It is easy to verify that the standard three-qubit GHZ state, with $C_{AB}^{(\mathrm{GHZ})} = C_{AC}^{(\mathrm{GHZ})} = C_{BC}^{(\mathrm{GHZ})} = 0$, corresponds to the origin point in Fig. 2. The standard three-qubit $W$ state, with $C_{AB}^{(W)} = C_{AC}^{(W)} = C_{BC}^{(W)} = \frac{2}{3}$, saturates the monogamy inequality (26). The graph in Fig. 2 also tells us that the concurrence $C_{AB}$ is constrained by, but not strictly determined by, the other two concurrences $C_{AC}$ and $C_{BC}$, unless the qubit $C$ is maximally entangled to either one of the two qubits $A$ and $B$.

## IV. CONCLUSIONS

To summarize, we investigate the monogamy nature of entanglement in a three-qubit system, and the inequality (13) is derived to quantify this monogamy relation. It is found that the entanglement between any two qubits in a three-qubit system is limited by the entanglement between these two qubits and another qubit. This monogamy inequality is saturated by a group of $W$-type three-qubit states, while the GHZ-type three-qubit states only reach the bottom line of the inequality. Combining our current result with a previous entanglement monogamy inequality, we present an upper bound for the two-qubit concurrence $C_{AB}$ in an $A$-$B$-$C$ three-qubit state, on condition that the other two two-qubit concurrences $C_{AC}$ and $C_{BC}$ are both given or solely provided, which is practically valuable in a quantum-information science. As is well-known, many mathematical relations derived in qubit systems might not be valid in high dimensional systems. A typical example is the monogamy relation (1) may be violated for qutrits or higher dimensional objects [27]. Then whether and how our current discussions and results could be generalized to high dimensional systems or multipartite systems is worth a further investigation.

## ACKNOWLEDGMENTS

J.H.H. thanks the financial support from Scientific Research Foundation of Nanchang University and the work was partly supported by National Nature Science Foundation of China (Grant No. NSFC/05-06/01).

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