Methods for a linear optical quantum Fredkin gate

Yan-Xiao Gong,[1,2,*] Guang-Can Guo,[2] and Timothy C. Ralph[1]
[1]*Centre for Quantum Computer Technology, Department of Physics, University of Queensland, St Lucia 4072, Australia*
[2]*Key Laboratory of Quantum Information, University of Science and Technology of China, CAS, Hefei, 230026, People's Republic of China*

(Received 7 April 2008; published 2 July 2008)

We consider the realization of a quantum Fredkin gate with only linear optics and single photons. First we construct a heralded Fredkin gate using four heralded controlled-NOT (CNOT) gates. Then we simplify this method to a post-selected one utilizing only two CNOT gates. We also give a possible realization of this method which is feasible with current experimental technology. Another post-selected scheme requires time entanglement of the input photons but needs no ancillary photons.

DOI: 10.1103/PhysRevA.78.012305 PACS number(s): 03.67.Lx, 42.50.Dv

I. INTRODUCTION

Quantum computing [1], due to its potential to solve problems far beyond classical computers, has attracted great attention in recent years. Many physical systems have been considered for a quantum computer [2]. One promising system is to use single photons, showing benefits such as low decoherence and easy single-qubit manipulation. However, such systems suffer a major disadvantage—the lack of interaction between individual photon qubits, which is needed for implementing nontrivial multiqubit gates. Surprisingly, Knill, Laflamme, and Milburn demonstrated that scalable quantum computing was possible using linear optical elements, single photons and photon detection [3]. Since then there has been considerable progress in improving the original scheme and demonstrating its basic elements [4].

Here we focus on the implementation of a linear optical Fredkin gate [5]. The Fredkin gate plays an important role in both classical computing and quantum computing [1]. The Fredkin gate is a three-qubit controlled-swap gate, that is, if the control qubit is in state $|1\rangle$, the two target qubits swap their states, otherwise they remain in their initial states. In the context of universal quantum computer, multiqubit gates are usually thought to be built by a combination of single- and two-qubit gates. Smolin and DiVincenzo have shown that five two-qubit gates [two controlled-NOT (CNOT) and three controlled-square-root-NOT (CSRNOT) gates] are sufficient to implement the Fredkin gate [6]. The most efficient known CNOT implementation requires two ancillary photons and has a probability of success of 1/4 [7], while the CSRNOT also requires two ancillary photons but has a probability of success of 1/8 [8]. Hence the Smolin, Divincenzo gate needs ten ancillary photons and the total success probability is $2^{-13}\approx 1.2\times 10^{-4}$. Therefore, their scheme is too difficult to be realized with current experimental technology. Recently, another scheme was proposed in Ref. [9] by simulating the Kerr medium in Milburn's optical Fredkin gate [10] with linear optical elements. It needs only six ancillary photons with the success probability 4.1×10^{-3}.

Recently, the complexity of the Toffoli gate was greatly reduced and the success probability was improved by exploiting additional photonic degrees of freedom [11]. In this paper, we wish to see if a similar effect can be achieved by applying those techniques to the Fredkin gate. We propose some methods for implementing the Fredkin gate with linear optics and single photons. The qubits in our schemes are all encoded in polarization states of single photons, so that $|0\rangle \equiv |H\rangle$ and $|1\rangle \equiv |V\rangle$, where $|H\rangle$ ($|V\rangle$) denotes the horizontal (vertical) polarization state. The rest of the paper is organized as follows. In the next section we propose a heralded Fredkin gate using four heralded CNOT gates. In Sec. III we give a post-selected Fredkin gate, i.e., working in the coincidence basis, and we also present a possible optical realization which is feasible with existing technology. In Sec. IV we replace the four heralded CNOT gates in the heralded scheme with four post-selected CNOT gates assisted by time entanglement but without ancillary photons. We conclude in Sec. V.

II. HERALDED FREDKIN GATE

Our heralded Fredkin gate is built up from four CNOT gates. The schematic structure is shown in Fig. 1. To show how the scheme works, we consider an arbitrary input state written as

$$\begin{aligned} &a_1|H\rangle_{c_{\rm in}}|H\rangle_{t_{1\rm in}}|H\rangle_{t_{2\rm in}} + a_2|H\rangle_{c_{\rm in}}|H\rangle_{t_{1\rm in}}|V\rangle_{t_{2\rm in}} \\ &+ a_3|H\rangle_{c_{\rm in}}|V\rangle_{t_{1\rm in}}|H\rangle_{t_{2\rm in}} + a_4|H\rangle_{c_{\rm in}}|V\rangle_{t_{1\rm in}}|V\rangle_{t_{2\rm in}} \\ &+ a_5|V\rangle_{c_{\rm in}}|H\rangle_{t_{1\rm in}}|H\rangle_{t_{2\rm in}} + a_6|V\rangle_{c_{\rm in}}|H\rangle_{t_{1\rm in}}|V\rangle_{t_{2\rm in}} \\ &+ a_7|V\rangle_{c_{\rm in}}|V\rangle_{t_{1\rm in}}|H\rangle_{t_{2\rm in}} + a_8|V\rangle_{c_{\rm in}}|V\rangle_{t_{1\rm in}}|V\rangle_{t_{2\rm in}}, \end{aligned} \tag{1}$$

where $a_i (i=1,2,\ldots,8)$ is an arbitrary complex number satisfying normalization condition.

First the polarizing beam splitter PBS1 (PBS2) transmits the horizontally polarized photons to beam 1 (4) and vertically polarized photons to beam 2 (3). Then the photons in each of the beams 1, 2, 3, and 4 undergo a CNOT gate controlled by the control state. Therefore, the input state becomes

$$\begin{aligned} &a_1|H\rangle_c|H\rangle_1|H\rangle_4 + a_2|H\rangle_c|H\rangle_1|V\rangle_3 + a_3|H\rangle_c|V\rangle_2|H\rangle_4 \\ &+ a_4|H\rangle_c|V\rangle_2|V\rangle_3 + a_5|V\rangle_c|V\rangle_1|V\rangle_4 + a_6|V\rangle_c|V\rangle_1|H\rangle_3 \\ &+ a_7|V\rangle_c|H\rangle_2|V\rangle_4 + a_8|V\rangle_c|H\rangle_2|H\rangle_3. \end{aligned} \tag{2}$$

*yxgong@mail.ustc.edu.cn

 ©2008 The American Physical Society

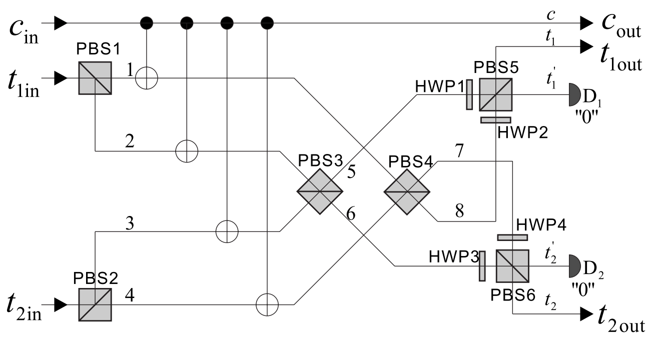

FIG. 1. Schematic of the heralded linear optical Fredkin gate comprising four heralded CNOT gates. Lowercase letters and numbers label the beams. The input control qubit is in beam c_{in} and the two input target qubits are in beams $t_{1\text{in}}$ and $t_{2\text{in}}$. Polarizing beam splitters (PBS) transmit horizontally polarized photons and reflect vertically polarized photons. Half-wave plates HWP1 and HWP3 are oriented at 67.5°. HWP2 and HWP4 are set to 22.5°. The gate succeeds if the two photon number resolving detectors D1 and D2 detect no photons. The output control and target states lie in modes c, t_1, and t_2.

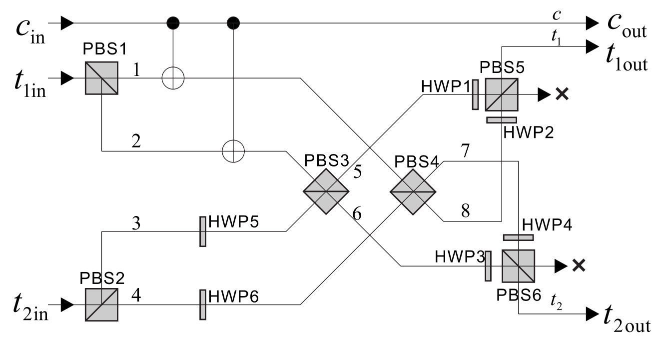

FIG. 2. Schematic of the post-selected linear optical Fredkin gate using two controlled-NOT (CNOT) gates. Lowercase letters and numbers label the beams. The control and two target qubits are input in beams c_{in}, $t_{1\text{in}}$, and $t_{2\text{in}}$. Polarizing beam splitters (PBSs) transmit horizontally polarized photons and reflect vertically polarized photons. Half-wave plates HWP1, HWP3, and HWP5 are oriented at 67.5°. HWP2, HWP4, and HWP6 are set to 22.5°. This gate succeeds conditioned on exactly one photon in each of the output beams c, t_1, and t_2. Symbols "×" denote the discarded outputs.

Next the photons in modes 2 (1) and 3 (4) are mixed at PBS3 (PBS4), followed by half-wave plates (HWPs). Of these, HWP1 and HWP3 oriented at 67.5° induce the transformations

$$|H\rangle \rightarrow \frac{1}{\sqrt{2}}(-|H\rangle + |V\rangle), \quad (3)$$

$$|V\rangle \rightarrow \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), \quad (4)$$

while HWP2 and HWP4 set to 22.5° result in

$$|H\rangle \rightarrow \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle), \quad (5)$$

$$|V\rangle \rightarrow \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle). \quad (6)$$

Finally, PBS5 (PBS6) combines the photons in modes 5 (4) and 8 (7). Thus, conditioned on a simultaneous zero detection in each of the modes t_1' and t_2' we can obtain the successful output state in modes c, t_1, and t_2,

$$\begin{aligned} &a_1|H\rangle_c|H\rangle_{t_1}|H\rangle_{t_2} + a_2|H\rangle_c|H\rangle_{t_1}|V\rangle_{t_2} + a_3|H\rangle_c|V\rangle_{t_1}|H\rangle_{t_2} \\ &+ a_4|H\rangle_c|V\rangle_{t_1}|V\rangle_{t_2} + a_5|V\rangle_c|H\rangle_{t_1}|H\rangle_{t_2} + a_6|V\rangle_c|V\rangle_{t_1}|H\rangle_{t_2} \\ &+ a_7|V\rangle_c|H\rangle_{t_1}|V\rangle_{t_2} + a_8|V\rangle_c|V\rangle_{t_1}|V\rangle_{t_2}. \end{aligned} \quad (7)$$

The success probability for vacuum detections at detectors $D1$ and $D2$ is 1/4. If we use the heralded CNOT gate proposed by Pittman *et al.* [7], we need eight ancillary photons and the success probability is $4^{-5} \approx 1.0 \times 10^{-3}$. Compared with the scheme by Smolin and DiVincenzo [6], our scheme has higher success probability and needs less ancillary photons. However, our scheme is not as efficient as Fiurášek's scheme [9], as we need more ancillary photons and have lower success probability. However, as we shall see, our scheme can be simplified to a post-selected gate using only two ancillary photons with higher success probability, which may be feasible with existing experimental technology.

III. POST-SELECTED FREDKIN GATE USING TWO CNOT GATES

We now consider the construction of a post-selected gate. By this we mean that a gate succeeds conditioned on simultaneous successful detection of exactly one photon for each qubit, so-called coincidence detection. Figure 2 is the schematic of a post-selected Fredkin gate. Comparing this scheme with the heralded one shown in Fig. 1, we can see that the simplification is replacing the two CNOT gates implementing on the photons in beams 3 and 4 controlled by the photon in beam c by HWP5 (67.5°) and HWP6 (22.5°) with the transformations given by Eqs. (4) and (5), respectively.

Therefore, for the input state given by Eq. (1), the state before PBS3 and PBS4 is

$$\begin{aligned} &(a_1|H\rangle_c|H\rangle_1 + a_3|H\rangle_c|V\rangle_2 + a_5|V\rangle_c|V\rangle_1 \\ &+ a_7|V\rangle_c|H\rangle_2) \otimes \frac{1}{\sqrt{2}}(|H\rangle_4 + |V\rangle_4) \\ &+ (a_2|H\rangle_c|H\rangle_1 + a_4|H\rangle_c|V\rangle_2 + a_6|V\rangle_c|V\rangle_1 \\ &+ a_8|V\rangle_c|H\rangle_2) \otimes \frac{1}{\sqrt{2}}(|H\rangle_3 + |V\rangle_3). \end{aligned} \quad (8)$$

Then through the analogous analysis in Sec. II and in the case of coincidence detection of the output modes c, t_1, and t_2, we can obtain the success output state the same as Eq. (7).

Figure 3 shows a possible optical realization of this scheme. We utilize a heralded CNOT gate proposed in Ref. [7], with the success probability 1/4. Another CNOT gate need not be heralded and a post-selected CNOT gate given in Ref. [12] can work with the success probability 1/9. However, as

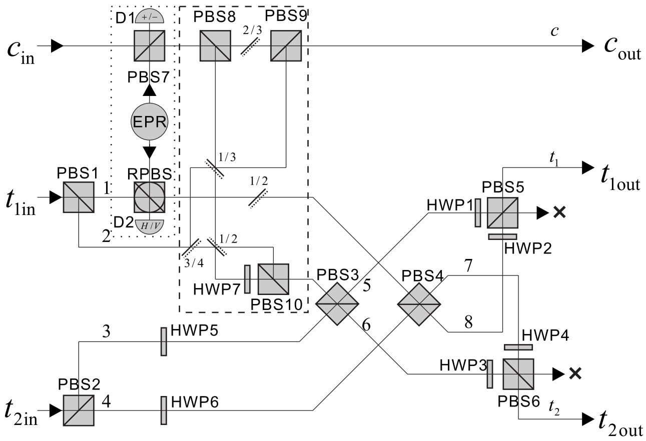

FIG. 3. Optical realization of a post-selected Fredkin gate. Lowercase letters and numbers label the beams. Photons in beams c_{in}, $t_{1\text{in}}$, and $t_{2\text{in}}$ are the control and two target qubits. Polarizing beam splitters (PBSs) transmit horizontally polarized ($|H\rangle$) photons and reflect vertically polarized ($|V\rangle$) photons. A heralded controlled-NOT (CNOT) gate proposed in Ref. [7] lies in the dotted box. EPR is an ancillary Bell state source. RPBS, transmitting 45° polarized ($|+\rangle$) photons and reflecting −45° polarized ($|V\rangle$) photons, can be realized by inserting one half-wave plate (HWP) oriented at 22.5° in each of the two inputs and two outputs of a PBS. D1 and D2 are photon detectors detecting +/− basis and H/V basis, respectively. A post-selected CNOT gate suggested in Ref. [12] is enclosed by the dashed box. Beam splitters are represented as black lines with their reflectivity indicated aside and dotted line indicates the surface from which a sign change occurs upon reflection. HWP1, HWP3, and HWP5 are oriented at 67.5°. HWP2, HWP4, and HWP6 are set to 22.5°. HWP7 is set to 45°. This gate succeeds conditioned on the coincidence of the successful detection at D1 and D2 and exactly one photon in each of the output beams c, t_1, and t_2. Symbols "×" denote the discarded outputs.

in our scheme the target state of the second CNOT gate is known ($|V\rangle$ or vacuum), it turns out that the gate can be optimized for maximum success probability 1/6 [11,13] (see the gate in the dashed box). Therefore in the case of fivefold coincidence, i.e., detection of exactly one photon in each of the output modes c, t_1, and t_2 and successful detection at D1 and D2, the gate succeeds with a total probability of success $1/4\times 1/6\times 1/8=1/192\approx 5.2\times 10^{-3}$. Note that the success probability for coincidence detection in Figs. 2 and 3 is reduced to 1/8 compared with that of 1/4 in Fig. 1 due to HWP5 and HWP6. However, this is more than offset by the halving of the number of CNOTs required. As an ancillary Bell state is needed, to implement this scheme requires at least a five-photon source, which is available at present [14–16], and therefore our scheme is feasible with current technology. However, the low success probability of our scheme would make the experiment more difficult and longer time detection would be needed.

IV. POST-SELECTED FREDKIN GATE ASSISTED BY TIME ENTANGLEMENT

In this section we introduce another post-selected Fredkin gate assisted by time entanglement. Let us first remind the

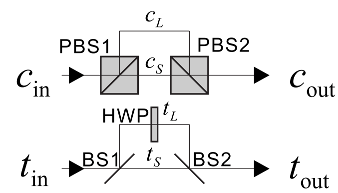

FIG. 4. Schematic of the controlled-NOT (CNOT) gate demonstrated in Ref. [17]. Lowercase letters label the beams. Polarizing beam splitters (PBS) transmit horizontally polarized photons and reflect vertically polarized photons. The control (target) photon is input in mode c_{in} (t_{in}) and output is in mode c_{out} (t_{out}). BS1 and BS2 are both balanced beam splitters. HWP is a half-wave plate set to 45°.

reader of the CNOT gate presented by Sanaka *et al.* [17] (see Fig. 4).

The control and target photons are a photon pair generated by spontaneous parametric down-conversion pumped by a continuous wave laser. Such a source is said to be time-energy entangled [18] as the photon pair is in a superposition of many possible emission times. The control photon is split along the short (c_S) or long (c_L) path at PBS1 and combined again in the same path at PBS2. The target photon is split along the short (t_S) or long (t_L) path at the first beam splitter BS1 and combined again in the same path at BS2. A HWP oriented at 45° rotates the polarization state of the photon taking the long path by 90°. The path-length difference ΔL of c_L and c_S is the same as that of t_L and t_S and satisfies the condition

$$l_{\text{SPDC}} \leq \Delta L \leq l_{\text{pump}}, \tag{9}$$

where l_{SPDC} is the coherence length of the down-converted photon and l_{pump} is the spectral width of the pump laser. Conditioned on coincidence of detection with the time window of the coincidence counter satisfying $\Delta T < \Delta L/c$, we can write the evolution of an arbitrary input state as

$$\begin{aligned} & b_1|H\rangle_{c_{\text{in}}}|H\rangle_{t_{\text{in}}} + b_2|H\rangle_{c_{\text{in}}}|V\rangle_{t_{\text{in}}} + b_3|V\rangle_{c_{\text{in}}}|H\rangle_{t_{\text{in}}} + b_4|V\rangle_{c_{\text{in}}}|V\rangle_{t_{\text{in}}} \\ & \quad \to b_1|H^S\rangle_{c_{\text{out}}}|H^S\rangle_{t_{\text{out}}} + b_2|H^S\rangle_{c_{\text{out}}}|V^S\rangle_{t_{\text{out}}} + b_3|V^L\rangle_{c_{\text{out}}}|V^L\rangle_{t_{\text{out}}} \\ & \qquad + b_4|V^L\rangle_{c_{\text{out}}}|H^L\rangle_{t_{\text{out}}}, \end{aligned} \tag{10}$$

where $b_i (i=1,2,3,4)$ is an arbitrary complex number satisfying normalization condition, and the superscript S (L) denotes the photon passing the short (long) path. Here the coincidence counting has post-selected out unwanted state components in which the control and target photons followed paths of different lengths. Because of the time-energy entanglement, paths of the same length are indistinguishable and so add coherently. The success probability is 1/4.

Figure 5 shows an optical realization of a post-selected Fredkin gate by replacing the four CNOT gates in Fig. 1 with the CNOT gates we have just introduced. Based on the analysis above, in the case of the input state given by Eq. (1), the successful output state can be found to be

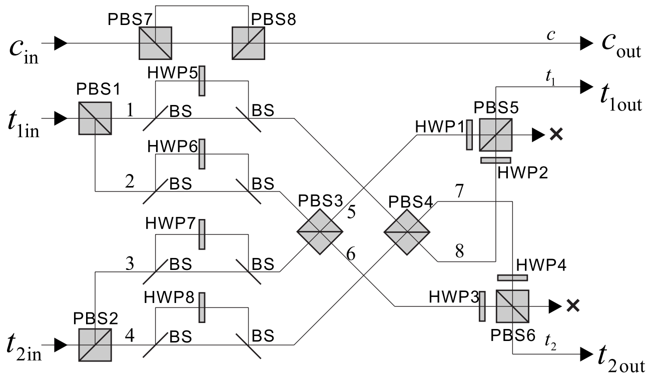

FIG. 5. Optical realization of a post-selected linear optical Fredkin gate assisted by time entanglement. Lowercase letters and numbers label the beams. Photons in beams c_{in}, $t_{1\text{in}}$, and $t_{2\text{in}}$ are the control and two target qubits. Polarizing beam splitters (PBS) transmit horizontally polarized photons and reflect vertically polarized photons. BS is balanced beam splitter. Half-wave plates HWP1 and HWP3 are oriented at 67.5°. HWP2 and HWP4 are set to 22.5°. HWP5, HWP6, HWP7, and HWP8 are set to 45°. The gate succeeds in the case of threefold coincidence detection in the output modes c, t_1, and t_2. Symbols "×" denote the discarded outputs.

$$\begin{aligned}
&a_1|H^S\rangle_c|H^S\rangle_{t_1}|H^S\rangle_{t_2} + a_2|H^S\rangle_c|H^S\rangle_{t_1}|V^S\rangle_{t_2}\\
&+ a_3|H^S\rangle_c|V^S\rangle_{t_1}|H^S\rangle_{t_2} + a_4|H^S\rangle_c|V^S\rangle_{t_1}|V^S\rangle_{t_2}\\
&+ a_5|V^L\rangle_c|H^L\rangle_{t_1}|H^L\rangle_{t_2} + a_6|V^L\rangle_c|V^L\rangle_{t_1}|H^L\rangle_{t_2}\\
&+ a_7|V^L\rangle_c|H^L\rangle_{t_1}|V^L\rangle_{t_2} + a_8|V^L\rangle_c|V^L\rangle_{t_1}|V^L\rangle_{t_2}. \qquad (11)
\end{aligned}$$

From Eqs. (10) and (11), we can see that to make the output state entangled the three input photons need to be time-entangled in the two time bins, "S" and "L." This scheme needs no ancillary photons and the probability of success is 1/64. Three-qubit time entangled states of the type required, i.e., in which a triple coincidence is in a superposition of many times, have been described in Refs. [19–21], however, an experimental demonstration of such states has not yet been made.

V. CONCLUSIONS

We have discussed the implementation of the Fredkin gate with linear optics and single photons. We have presented a heralded method using four heralded CNOT gates. Our method needs eight ancillary photons which is less than that of ten in Ref. [6], but more than that of six in Fiurášek's scheme [9]. The success probability of our scheme is $4^{-5} \approx 1.0\times 10^{-3}$, which is higher than that of 1.2×10^{-4} in Ref. [6], but is less than that of 4.1×10^{-3} in Ref. [9]. We have also simplified the heralded scheme to a post-selected one by replacing two CNOT gates with two HWPs. This scheme needs only two ancillary photons, and therefore is feasible with existing technology. However, the low success probability of $1/192 \approx 5.2\times 10^{-3}$ would make the experiment very difficult. The other post-selected Fredkin gate we have proposed is assisted by time entanglement. Although this scheme needs no ancillary photons and has higher success probability of 1/64, the three-photon time entangled source required is not available at present.

It should be noted that since the post-selected schemes work in the coincidence basis, such schemes could not be scalable unless photon-number quantum nondemolition (QND) detectors were added to each output beam, nevertheless they open the door to experimental tests of an optical Fredkin gate and would make its application possible. We hope our proposals will stimulate such investigations of the Fredkin gate.

ACKNOWLEDGMENTS

Y.X.G. thanks Yun-Feng Huang for useful discussions. Y.X.G. and G.C.G. were funded by National Fundamental Research Program (Grant No. 2006CB921907), National Natural Science Foundation of China (Grants No. 60121503 and No. 60621064), Innovation Funds from Chinese Academy of Sciences, International Cooperate Program from CAS, and Ministry of Science & Technology of China. T.C.R. was supported by the DTO-funded U.S. Army Research Office Contract No. W911NF-05-0397 and the Australian Research Council.

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University, Cambridge, 2000).
[2] T. P. Spiller, W. J. Munro, S. D. Barrett, and P. Kok, Contemp. Phys. **46**, 407 (2005).
[3] E. Knill, R. Laflamme, and G. J. Milburn, Nature (London) **409**, 46 (2001).
[4] P. Kok, W. J. Munro, K. Nemoto, T. C. Ralph, J. P. Dowling, and G. J. Milburn, Rev. Mod. Phys. **79**, 135 (2007).
[5] E. Fredkin and T. Toffoli, Int. J. Theor. Phys. **21**, 219 (1982).
[6] J. A. Smolin and D. P. DiVincenzo, Phys. Rev. A **53**, 2855 (1996).
[7] T. B. Pittman, B. C. Jacobs, and J. D. Franson, Phys. Rev. A **64**, 062311 (2001).
[8] B. P. Lanyon, M. Barbieri, M. P. Almeida, T. Jennewein, T. C. Ralph, K. J. Resch, G. J. Pryde, J. L. O'Brien, A. Gilchrist, and A. G. White, e-print arXiv:0804.0272.
[9] J. Fiurášek, Phys. Rev. A **73**, 062313 (2006).
[10] G. J. Milburn, Phys. Rev. Lett. **62**, 2124 (1989).
[11] T. C. Ralph, K. J. Resch, and A. Gilchrist, Phys. Rev. A **75**, 022313 (2007).
[12] T. C. Ralph, N. K. Langford, T. B. Bell, and A. G. White, Phys. Rev. A **65**, 062324 (2002).
[13] T. C. Ralph, Phys. Rev. A **70**, 012312 (2004).
[14] Z. Zhao, Y.-A. Chen, A.-N. Zhang, T. Yang, H. J. Briegel, and J.-W. Pan, Nature (London) **430**, 54 (2004).
[15] Q. Zhang, A. Goebel, C. Wagenknecht, Y. A. Chen, B. Zhao, T. Yang, A. Mair, J. Schmiedmayer, and J. W. Pan, Nat. Phys. **2**,

678 (2006).
[16] C. Y. Lu, X. Q. Zhou, O. Gühne, W. B. Gao, J. Zhang, Z. S. Yuan, A. Goebel, T. Yang, and J. W. Pan, Nat. Phys. **3**, 91 (2007).
[17] K. Sanaka, K. Kawahara, and T. Kuga, Phys. Rev. A **66**, 040301(R) (2002).
[18] J. D. Franson, Phys. Rev. Lett. **62**, 2205 (1989).
[19] T. E. Keller, M. H. Rubin, Y. Shih, and L.-A. Wu, Phys. Rev. A **57**, 2076 (1998).
[20] A. A. Hnilo, Phys. Rev. A **71**, 033820 (2005).
[21] F. DellAnno, S. D. Siena, and F. Illuminati, Phys. Rep. **428**, 53 (2006).