von Neumann lattices in finite-dimensional Hilbert spaces

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The prime number decomposition of a finite-dimensional Hilbert space reflects itself in the representations that the space accommodates. The representations appear in conjugate pairs for factorization to two relative prime factors which can be viewed as two distinct degrees freedom. These, Schwinger's quantum degrees of freedom, are uniquely related to a von Neumann lattices in the phase space that characterizes the Hilbert space and specifies the simultaneous definitions of both (modular) positions and (modular) momenta. The area in phase space for each quantum state in each of these quantum degrees of freedom, is shown to be exactly h, Planck's constant.

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I. INTRODUCTION

Studies of finite-dimensional quantum mechanics were undertaken in the early days of the development of quantum mechanics by Weyl [1] who showed thereby the connection of the commutation relation to Schrodinger's wave equation. A systematic study of this, finite-dimensional quantum mechanics problem, was initiated 30 years later by Schwinger [2]. His work introduced to the field some of the basic issues that are under intensive investigations in the most recent literature. Among these are the relevance to the physics of finite dimensional quantum mechanics of whether the dimensionality M is a prime number or is made up of product of distinct primes. In the latter case, which is our main concern in the present work, Schwinger [2] noted what he termed "quantum degrees of freedom" (see also in Ref. [3]) which pertain to the relatively prime number factors of M—these are given a concrete representation in this paper. He also considered sets of complete and orthonormal bases that span the M-dimensional space and which are what he termed of "maximum degree of incompatibility." Such vector bases under their modern name of "mutually unbiased bases" (MUB) [4,5] or "conjugate bases" [6,7] have led, with the recent developments in our understanding of the foundation of quantum mechanics, to intensive research in a great variety of problems where implementations seems to involve finite dimensions. These include the pioneering studies of von Neumann lattices and magnetic orbitals on a finite phase space [8], quantum measurements [9], teleportation [10], and others. A cogent review is given, e.g., by Vourdas [11].

The kq representation was introduced by Zak [12,13] in his study of the continuum (infinite-dimensional space) to conveniently handle electrons in a periodic potential subjected to an external magnetic field. Zak's work was based on Weyl's [1] unitary displacement operators in both coordinates and momenta which, for some particular choice of the parameters involve commuting operators whose eigenvalues may be labeled by both the space position and the momentum of the electron. (No violation of the uncertainty principle incurs since the values of both the coordinates and momenta are modular, see Ref. [14].) The Zak transform formalism proves of wide use also in signal processing problems [15] where the corresponding modular variables are frequency and time. The transform is of considerable theoretical interest in conjunction with it being parametrized by both the coordinate and momentum. Only recently [16] the conjugate basis to the kq representation was explicitly given. The finite dimensional kq representation was formulated in Ref. [17]. This can be done only for cases wherein the dimensionality M of the Hilbert space is not prime [17–19,23]. For extended dimensionality (i.e., $M \to \infty$), already at the very early stages of the development of quantum mechanics von Neumann [20] suggested a physical way of accounting for states in phase space by discretizing the phase space with an area of h (in the units $\hbar=1$ this equals 2π) for each state. This problem was studied extensively, e.g., Ref. [21]. Recently it was shown [22] that on a finite phase plane the kq coordinates and the sites on a von Neumann lattice are closely related.

In his studies of finite-dimensional quantum mechanics, Schwinger [2,3] showed that if M, the dimensionality of the Hilbert space under consideration, is not a prime or a power of a prime, then states in the space may be viewed as having "quantum degrees of freedom." Thus, if $M=M_1M_2$ with M_1, M_2 relative primes, the state may be considered as accounting for two distinct degrees of freedom one of dimension M_1 and the other of dimension M_2. This mapping of the one degree of freedom, M-dimensional Hilbert space on a line (i.e., spanned by $|q\rangle$, $q=1,\ldots,M$) onto two (Schwinger's "quantum") degrees of freedom on a torus (spanned by $|q_1q_2\rangle$, $q_1=1,\ldots,M_1$; $q_2=1,\ldots,M_2$) and their conjugate representations were introduced earlier [19]. These representations are reviewed herewith and the equivalence of these partially localized state (PLS) and states over the von Neumann lattice is established. Thus we show that a "quantum degree of freedom," very similar to a proper degree of freedom, occupies a phase space area equals to h, Planck's constant. These states are distinct form the finite dimension

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Zak's kq representation [13] although when M_1, M_2 are co primes both bases are eigenstates of the commuting modular operators, one pertaining to the position (q) and the other to the momentum (k). The latter (i.e., Zak's) involve Bloch-like symmetry in one of the variables. The phase relation between Zak's representations (kq) and the representations used by us when both are applicable (i.e., when the factors M_1, M_2 are coprimes) is given in the Appendix. The present work utilizes Schwinger's [2,3] quantum degrees of freedom for finite-dimensional Hilbert space to consider states with two quantum degrees of freedom as being labeled by the position of one and the momentum of the other.

The paper is organized as follows. Sections II and III outline some previous results and briefly derive the formulae needed in our later discussion. Section IV contains our main result, i.e., relating partially localized states (PLS), the kq representations states and states on von Neumann's lattices. These relations bring to the fore the role of the quantum degrees of freedom that were introduced by Schwinger [2,3] and give meaning to the localization in both the coordinates and momenta in these variables (the price is complete delocalization in their conjugate variables) and relate it to states over the von Neumann lattice. The last section gives the conclusions and some remarks. The Appendix gives the phase relation among different representations.

II. MAPPING OF DISCRETE LINE TO DISCRETE TORUS

Schwinger [2] showed that M-dimensional vector spaces allow the construction of two unitary operators U and V (in his notation), which form a complete operator basis. This means that if an operator commutes with both U and V it is, necessarily, a multiple of the unit operator. These operators have a period M, i.e.,

$$U^M = V^M = 1, \tag{1}$$

where M is the smallest integer for which this equality holds. The eigenvalues of both U and V are distinct: they are the M roots of unity, i.e., with $|x\rangle$ the eigenfunction of U,

$$U|q\rangle = e^{i(2\pi/M)q}|q\rangle, \quad |q+M\rangle = |q\rangle, \quad q = 1, \ldots, M.$$

The operator V is defined over these eigenvectors as

$$V|q\rangle = |q-1\rangle. \tag{2}$$

Schwinger then showed that the absolute value of the overlap between any eigenfunction of U $|q\rangle$ and any one of V $|k\rangle$ is a constant:

$$|\langle k|q\rangle| = \frac{1}{\sqrt{M}}. \tag{3}$$

Vector bases with this attribute are referred [4] to as mutually unbiased (MUB) or conjugate vector bases (we use these terms interchangeably).

A specific example of the M-dimensional space is the following: consider M points on a line, i.e., consider discretized and truncated spatial coordinate x and its conjugate momentum p as our M-dimensional space. This may be realized by imposing boundary conditions on the spatial coordinate x of the wave functions under study $\psi(x)$ and on their Fourier transforms Fp (we take $\hbar=1$) [13]:

$$\psi(x+Mc) = \psi(x), \quad F\left(p + \frac{2\pi}{c}\right) = F(p).$$

Here M is an integer—it is the dimensionality of the Hilbert space, and we term c the "quantization length." As a consequence of the above boundary conditions we have that the value of the spatial coordinate x and the value of the momentum p are discrete and finite:

$$x = sc, \quad s = 1, \ldots, M; \quad p = \frac{2\pi}{Mc}t, \quad t = 1, \ldots, M.$$

In this case we may replace the operators x and p by the unitary operators

$$\tau(M) = e^{i(2\pi/Mc)x}, \quad T(c) = e^{ipc}. \tag{4}$$

These operators satisfy the basic commutator relation

$$\tau(M)T(c) = T(c)\tau(M)e^{-i2\pi/M}. \tag{5}$$

They exhibit the dimensionality (i.e., periodicity) automatically

$$[\tau(M)]^M = [T(c)]^M = 1 \tag{6}$$

and we may associate Schwinger's operator U with $\tau(M)$ and his V with $T(c)$ (henceforth we take $c=1$).

For our analysis where we take M as factorizable: $M = M_1 \cdot M_2$, it is convenient to represent the number M in terms of prime numbers P_j

$$M = \prod_{j=1}^{N} P_j^{n_j}, \quad P_j \neq P_i, \ j \neq i, \tag{7}$$

where the n_j are integers, and more concisely we denote $P_j^{n_j}$ by m_j, i.e.,

$$M = \prod_{j=1}^{N} m_j. \tag{8}$$

We find thus that the greatest common divisor (gcd) among the $m_j s$ is 1:

$$\gcd(m_j, m_i) = 1, \quad \forall\, j \neq i, \tag{9}$$

i.e., distinct m_i's are relatively prime. In our study we consider bipartitioning of the product that represents M [Eq. (8)] into two factors

$$M = M_1 M_2. \tag{10}$$

Here M_1 incorporates one part of the N factors of Eq. (8) and M_2 contains the other part. Our way of bipartitioning implies that the two numbers M_1 and M_2 are relatively prime, viz. $\gcd(M_1, M_2)=1$. In our discussion of the kq representation [12,17,19] the above was used to show that the number of kq representations $\chi(M)$ each with its conjugate, which form a complete basis can be accommodated in the M-dimensional space, is simply related to the number of primes N that appear in M:

$$\chi(M) = 2^{N-1}. \tag{11}$$

(It should be noted that the familiar finite-dimensional Fourier representation is included in this counting.) The mapping of the M-dimensional, one degree of freedom (a line) to (fake) two degrees of freedom (a torus) whose dimensions are M_1 and M_2 can be accomplished in $\chi(M)-1$ ways (the finite-dimensional Fourier transformation should not be included). We now introduce:

$$L_1 = \frac{M}{M_1} = M_2, \quad L_2 = \frac{M}{M_2} = M_1. \tag{12}$$

This implies that the equation,

$$q = q_1 L_1 + q_2 L_2 \,[\text{mod } M], \quad q = 1, \ldots, M,$$
$$q_1 = 1, \ldots, M_1, \quad q_2 = 1, \ldots, M_2, \tag{13}$$

has a unique solution q for every pair $[q_1, q_2]$, with q running over its whole range of M values. We will now [2,9,11] modify Eq. (13) to attain this simpler relation among the solutions. This is obtained by applying the Chinese remainder theorem [9] to the solution of the two congruences

$$q = q_1 \,[\text{mod } M_1], \quad q = q_2 \,[\text{mod } M_2]. \tag{14}$$

The solution of these is

$$q = q_1 N_1 L_1 + q_2 N_2 L_2 \,[\text{mod } M], \tag{15}$$

with

$$N_2 = L_2^{-1} \,[\text{mod } M_2] \text{ and } N_1 = L_1^{-1} \,[\text{mod } M_1]. \tag{16}$$

For example, with $M=15$, we have

$$L_1 = 5, \quad L_2 = 3 \rightarrow N_1 = N_2 = 2. \tag{17}$$

We have then

$$e^{i(2\pi/M)} = e^{i(2\pi/M_1)N_1} e^{i(2\pi/M_2)N_2} \rightarrow e^{i(2\pi/M)x}$$
$$= e^{i(2\pi/M_1)N_1 x} e^{i(2\pi/M_2)N_2 x} \tag{18}$$

and

$$e^{ip} = e^{ipN_1 L_1} e^{ipN_2 L_2}.$$

Further, we may label

$$|q\rangle = |q_1 N_1 L_1 + q_2 N_2 L_2\rangle, \quad |k\rangle = |k_1 N_1 L_1 + k_2 N_2 L_2\rangle.$$

We have thus that the (complete) operator basis pair may be replaced by the two pairs (note the removal of N_i from the V terms) as follows:

$$\tau(M) = e^{i2\pi/Mx} \rightarrow \tau(M_1)\tau(M_2) = e^{i2\pi/M_1 x} e^{i2\pi/M_2 x},$$

$$T(1) = e^{ip}, \quad \rightarrow, \quad T(L_1)T(L_2) = e^{ipL_1} e^{ipL_2}. \tag{19}$$

While the basis $|q\rangle$ may be expressed via $|q_1\rangle|q_2\rangle$ with

$$e^{i(2\pi/M_i)x}|q_i\rangle = e^{i(2\pi/M_i)q_i}|q_i\rangle; \quad e^{ipL_i}|k_i\rangle = e^{i(2\pi/M_i)k_i}|k_i\rangle. \tag{20}$$

Defining

$$\Delta^{M_i}(x) = 1, \;\; x = 0 \text{ mode } [M_i] = 0 \;\text{ otherwise}$$

allows equating

$$\langle q|q_1 N_1 L_1 + q_2 N_2 L_2\rangle = \Delta^M(q - q_1 N_1 L_1 - q_2 N_2 L_2)$$
$$= \Delta^{M_1}(q - q_1)\Delta^{M_2}(q - q_2)$$
$$= \langle q|[|q_1\rangle|q_2\rangle] \tag{21}$$

and

$$\langle k|k_1 N_1 L_1 + k_2 N_2 L_2\rangle = \Delta^M(k - k_1 N_1 L_1 - k_2 N_2 L_2)$$
$$= \Delta^{M_1}(k - k_1)\Delta^{M_2}(k - k_2)$$
$$= \langle k|[|k_1\rangle|k_2\rangle]. \tag{22}$$

The formulas

$$|q\rangle \rightarrow |q_1\rangle|q_2\rangle, \quad |k\rangle \rightarrow |k_1\rangle||k_2\rangle \tag{23}$$

with [19] constitutes the transcription of the M-dimensional, one degree of freedom system to the (Schwinger's quantum) two degrees of freedom one of M_1 and the other of M_2 dimensions ($M=M_1M_2$) which is our mapping of the M-dimensional line to a torus. Our numerical subscripts designates the dimensionality; thus, e.g., $|q_1\rangle$ refers to the spatial coordinate with $q_1=1,2,\ldots,M_1$.

We shall now discuss briefly some attributes of what is termed [19] the $|q_1\rangle|q_2\rangle$ representation (representation henceforth). The conjugate basis viz $|k_1\rangle|k_2\rangle$ may be evaluated directly via the result

$$\langle q|k\rangle = \frac{e^{i(qk2\pi/M)}}{\sqrt{M}}$$

to give

$$\langle q_1|\langle q_2||k_1\rangle|k_2\rangle = \frac{e^{i(q_1k_1L_1N_1+q_2k_2L_2N_2)2\pi/M}}{\sqrt{M}} \tag{24}$$

and, by direct evaluation

$$\langle q_i|k_i\rangle = \frac{e^{i(2\pi/M_i)N_i q_i k_i}}{\sqrt{M_i}} \quad i = 1,2. \tag{25}$$

Within the description in terms of two (quantum) degrees of freedom (QDF). For example, when accounting for $|q\rangle$ via $|q_1\rangle|q_2\rangle$ [see Eq. (23)] we may construct eigenfunctions the above operators by Fourier transformation in either one of the variables (we show in the Appendix that we need not differentiate between the two forms)

$$|q_1,k_2;\rangle = |q_1\rangle|k_2\rangle = \frac{1}{\sqrt{M_2}}\sum_{q_2} e^{i(2\pi/M_2)k_2q_2N_2}|q_1\rangle|q_2\rangle,$$

$$= \frac{1}{\sqrt{M_1}}\sum_{k_1} e^{-i(2\pi/M_1)k_1q_1N_1}|k_1\rangle|k_2\rangle. \tag{26}$$

One can readily check that these are indeed the eigenfunctions of the operators Eq. (A6) with the same eigenvalues. In this form the state $|q_1,k_2;\rangle$ can be described as partially localized (PLS) as we have

$$\langle q_1|\langle q_2|q_{01},k_{02};\rangle = \frac{\Delta^{M_1}(q_1 - q_{01})e^{i2\pi/M_2k_2q_2N_2}}{\sqrt{M_2}}, \tag{27}$$

thus it is localized in the q_1 variable while completely delocalized in the q_2 variable. (In Ref. [17] we used e^{ipM_2} instead of the present $T(N_1L_1)$; these two operators have the same eigenvalues and eigenstates, but enumerated differently.) In the next section we expand our presentation [22], to show that these states are states over the von Neumann lattice.

III. VON NEUMANN LATTICES

We now consider the von Neumann lattice for the M-dimensional Hilbert space. To this end we first describe the phase space of our system. The spatial-like coordinates q are now discrete, $q=0,1,\ldots,M-1$ (this in dimensionless units: $q=cM$ is the "size" of our Hilbert space) and label the eigenfunction $|q\rangle$ of the operator $\exp[i(\frac{2\pi}{M})x]$. These label the abscissa of our phase space. The momentumlike coordinates are $p=\frac{2\pi}{M}k$; $k=0,1,\ldots,M-1$, the label k of $|k\rangle$, the eigenfunctions of $\exp[ip]$ are used for the ordinate of our phase space.

We now consider the case wherein $M=M_1,M_2$, gcd $[M_1,M_2]=1$. Since as k runs from 1 to M the momenta runs from $p=\frac{2\pi}{Mc}\hbar$ *to* $\frac{2\pi}{c}\hbar$, the "distance" $\delta k=1$ represents $\delta p=\frac{2\pi}{Mc}\hbar$. (In dimensionless units, i.e., $c=\hbar=1-p$ runs from $\frac{2\pi}{M}$ *to* 2π and $\delta p=\frac{2\pi}{M}$.) The "distance" $\delta q=1$ along the abscissa is, obviously, c (1 in dimensionless units). In this way we see that each "point" marked on our phase space by (q,k) may be viewed as representing an "area" $\delta q\,\delta k=\frac{2\pi}{M}$.

Our aim in this section is to show that $|q_1,k_2\rangle$ states, i.e., the eigenfunctions of

$$e^{i[2\pi/M_1x]}e^{i[pM_1]}$$

as given above, Eq. (A6) have the attractive physical properties of being states over the von Neumann lattice, a term that will also defined herewith. A von Neumann lattice in this phase space are the M points whose coordinates are

$$q=nM_1,\quad k=mM_2,\quad n=0,1,2,\ldots,M_2-1,$$

$$m=0,1,2,\ldots,M_1-1. \tag{28}$$

A displaced von Neumann lattice are the M points—here the points are defined mode $[M]$.

$$q=q_{01}+nM_1,\quad n,k_2=0,1,2,\ldots,M_2-1,$$

$$k=k_{02}+mM_2,\quad m,q_{01}=0,1,2,\ldots,M_1-1. \tag{29}$$

We now define a "state over a von Neumann lattice" to be the state whose density matrix representative ρ is given by

$$|\langle q|\rho|k\rangle|=\frac{1}{\sqrt{M}}\quad \text{on a lattice point,}$$

$$=0\ \ \text{otherwise}. \tag{30}$$

(This requirement implies that only M out of the M^2 matrix elements are nonvanishing.) It is easily verified that the partially localized state (PLS) Eq. (27) with localization coordinates at the origin ($q_{01}=k_{02}=0$) is such a state (summation over repeated indices is implied):

$$\langle q|\rho|k\rangle=\langle q|q_1q_2\rangle\langle q_1q_2|\psi\rangle\langle\psi|k_1k_2\rangle\langle k_1k_2|p\rangle$$

$$=\Delta(x-q_1N_1L_1-q_2N_2L_2)\frac{\Delta^{M_1}(q_1)}{\sqrt{M_2}}\frac{\Delta^{M_2}(k_2)}{\sqrt{M_1}}$$

$$\times\Delta(p-k_1L_1N_1-k_2L_2N_2). \tag{31}$$

Thus the PLS is a state over a von Neumann lattice. For example, $M=15$, $M_1=3$, $M_2=5$, $L_2=3$, $N_2=2$ leads to nonvanishing matrix elements for

$$q=q_2N_2L_2\to q=0,6,12,3,9,\quad k=k_1L_1\to k=0,5,10.$$

We take it as obvious that with each von Neumann lattice we have its conjugate obtained by interchanging q with k.

To study the case of displaced von Neumann lattices we consider briefly the PLS with (at least) one coordinate not at the origin. For example,

$$\langle q_1q_2|\psi\rangle=\frac{\Delta^{M_1}(q_1-q_{10})e^{i[q_2k_{20}2\pi/M_2]N_2}}{\sqrt{M_2}},\quad q_{01},\ k_{02}\neq 0. \tag{32}$$

The nonvanishing elements of the density matrix are now the M points

$$q=q_{01}N_1L_1+q_2N_2L_2,\quad q_2=0,\ldots,M_2-1,$$

$$k=k_1L_1N_1+k_{02}L_2N_2,\ k_1,\quad =0,\ldots M_1-1. \tag{33}$$

This is a von Neumann lattice with its origin shifted to $q_{01}N_1L_1$, $k_{02}L_2$. Note that corresponding to PLS $|q_{01},k_{02}\rangle$ the nonvanishing terms involve the coordinates of the conjugate state $|k_1,q_2\rangle$. This corresponds to a localized state at x say implies all momentum state equally probable. This state is orthogonal to the von Neumann state considered above. Indeed each of the M pairs of coordinates q_{01},k_{02} represents a state over the von Neumann lattice with its origin shifted to the designated phase space point. These states are easily shown to be orthogonal. Thus, considering the overlap between arbitrary such states (summation convention implied)

$$\langle\psi_1|\psi_2\rangle=\langle\psi_1|q_1q_2\rangle\langle q_1q_2|\psi_2\rangle$$

$$=\frac{\Delta^{M_1}(q_1-q_{01})e^{i[q_2k_{02}2\pi/M_2]}}{\sqrt{M_2}}$$

$$\times\frac{\Delta^{M_1}(q_1-q'_{01})e^{-i[q_2k'_{02}2\pi/M_2]}}{\sqrt{M_2}}$$

$$=\Delta^{M_1}(q_{01}-q'_{01})\Delta^{M_2}(k_{02}-k'_{02}). \tag{34}$$

Thus these M states span the space and form a complete orthonormal basis. We shall show below that these are the PLS considered above. In closing this section we wish to emphasize that by construction each von Neumann lattice point represents an area of 2π in our phase space diagram. The total number of such points within a rectangle labeled by (q_{01},k_{02}) is M. Thus, e.g., for the rectangle labeled by $(q_{01}=0,k_{02}=0)$ includes the phase space coordinates within the area contained in the rectangle (0, 0); $(M_1,0)$, $(0,M_2)$, (M_1,M_2). There are M possible values for $\langle x|\rho|p\rangle$ within this rectangle. Thus the 2π area that contain M points—each labels a state over the von Neumann lattice (there are M such states) each state includes M "points": one in each rectangle—this point is discussed further below.

We now show that PLS are kq states, i.e., are the eigenfunctions of

$$e^{i[2\pi/M_1 x]}e^{i[pM_1]}$$

(repeated indices are summed over):

$$\langle q_1 q_2|e^{i[2\pi/M_1 x]}e^{i[pM_1]}|\psi\rangle = e^{i[2\pi/M_1 q_1]}\langle q_1 q_2|k_1 k_2\rangle e^{i[2\pi/M_2 k_2]} \times\langle k_1 k_2|\psi\rangle = e^{i[2\pi/M_1 q_1]}e^{i[2\pi/M_2 k_2]}\langle q_1 q_2|\psi\rangle. \quad (35)$$

We have then that the $|k_1 q_2\rangle$ representation (as well as their concomitant conjugates, $|k_2 q_1\rangle$), PLS (at the corresponding points), and the states over the von Neumann lattice (shifted to the corresponding spots), are one and the same states. (We note that these states differ from Zak's [12] states. The latter are defined regardless whether M_1, M_2 are co primes or not.) The completeness and orthogonality of the $|k_1 q_2\rangle$ states assures their validity for the other two. The states over the von Neumann lattice were shown to occupy precisely an area of h, Planck's constant in phase space for the finite-dimensional case. While the coordinates of the non vanishing density matrix for the states (partially) localized at $k_{02}=q_{01}=0$ were at the appropriate von Neumann lattice (as well as their conjugates $k_{01}=q_{02}=0$). We thereby accounted for the connection between the q_1, q_2 representations and von Neumann lattices that were obtained recently [22].

We now return to the more general case: we consider an arbitrary pair with partially localized phase points q_{01}, k_{02}. These, in turn, lead to different marked points in our phase space-it specifies a lattice point within the rectangles considered above and may serve as a label for a partially localized state localized at q_{01} and having quasimomenta k_{02}. One can readily verify that this state may be viewed as a state over the von Neumann lattice where the M points for which the matrix elements

$$\langle k|q_{01},\ k_{02}\rangle\langle q_{01},\ k_{02}|q\rangle$$

do not vanish are given by

$$q = q_{01} + nM_1, \quad k = k_{02} + mM_2, \quad n = 0,1,2,\ldots, M_2 - 1, \quad m = 0,1,2,\ldots,M_1 - 1. \quad (36)$$

Thus these states are states over the von Neumann lattice with the whole lattice origin being shifted to the point q_{01}, k_{02}.

In closing this section we would like to point out that the results above are based on the particular choice of labeling, viz. on having achieved a mapping of the M-dimensional line geometry ($q=1,2,\ldots,M$) to a torus like geometry with one torus radius M_1 dimensional ($q_1=1,2,\ldots M_1$) and the other M_2-dimensional ($q_2=1,2,\ldots,M_2$). This was shown to be possible for $M=M_1 M_2$ with $\gcd[M_1,M_2]=1$.

IV. CONCLUDING REMARKS

In a finite, M-dimensional Hilbert space with $M=M_1 M_2$ and M_1, M_2 are co primes, i.e., $\gcd[M_1,M_2]=1$, one may, in the manner of Schwinger, view the states as made of two quantum degrees of freedom (QDF): one of dimensionality M_1 and the other M_2. This was utilized to consider the $|q_1,q_2\rangle$ with q_1 being M_1, while q_2 the M_2-dimensional coordinates. We analyzed the two QDF states as eigenfunctions of the two commuting operators e^{ipM_2}, $e^{i2\pi/M_2 x}$, which, in the present context may be viewed as referring to the two QDF. We showed that the eigenstates, of these operators may be obtained via Fourier transformation in one of the QDF, e.g., $|k_1,q_2;C\rangle$ is the Fourier-like transform in one of these QDF variables (q_1 in this example). (Either of these sets spanning the M-dimensional space under study.)

Thus the basis $|k_1,q_2;C\rangle$ specify k_1 as the (modular) momenta of the M_1-dimensional QDF and q_2 the (modular) position of the second [mode M_2], variable. (Hence the conjugate basis is $|k_2,q_1\rangle$ with analogous meaning for the subscripted labels.) The M orthogonal basis vectors span the space, as does the conjugate basis. These states are termed partially localized states as therein (in the above example) the second QDF is localized while the first has its momenta defined. We termed these states partially localized states (PLS). We discuss briefly the distinction of these from Zak's kq states in the Appendix. In this finite-dimensional quantum mechanics, one may draw the phase space of the system as a whole (i.e., M-dimensional) by marking the abscissa by the eigenvalues of $\exp[i\frac{2\pi}{M}x]$ by the discrete, positionlike, eigenvalue of the state $|q\rangle$, ($q=0,1,2,\ldots,M-1$), and the ordinate by eigenvalues of $\exp[ip]$ by the momentumlike eigenvalue of the state $|p\rangle$ ($p=\frac{2\pi}{M}k$; $k=0,1,2,\ldots,M-1$). (Giving thereby M^2 points forming a square, with each point designating an area of $\frac{2\pi}{M}$.) The von Neumann lattice in this phase space and pertaining to the factorization $M=M_1 M_2$, $\gcd[M_1,M_2]=1$, is given by the points

$$q = nM_2, \quad k = mM_1, \quad n = 0,1,\ldots,M_1 - 1, \quad m = 0,1,\ldots,M_2 - 1.$$

We refer to these points as vn points. Clearly each vn point designates an area containing M phase space points i.e., it designates an area of 2π. (Recall that we work in units of $\hbar=1$, i.e., the area in more physical units is h, Planck's constant.) We then defined "states over the von Neumann lattice" as those density matrices whose only non vanishing matrix elements are

$$\langle x|\rho|k\rangle = \frac{1}{\sqrt{M}}, \quad x,\ p \text{ on von Neumann lattice}.$$

Since the total number of von Neumann lattice points is M and each point occupies and area of $\frac{2\pi}{M}$ each von Neumann state occupies exactly an area of 2π (i.e., an area of h, Planck's constant). We then showed that the state $|k_1=0,q_2=0\rangle$ (as well as its conjugate state) are states over von Neumann lattice and thus occupy an area of h, Planck's constant. This led to the demonstration that concomitant to this state over the von Neumann lattice each of the other vectors in this basis designates a shifted von Neumann lattice—the shift being by the coordinates of the designated vector. Thus the vector $k_1=k_{01}\neq 0$, $q_2=q_{02}\neq 0$ is a state over the von Neumann lattice shifted to $(k_{01}[\text{mode } M_1], q_{02}[\text{mode } M_2])$. Here too the area occupied by each of the M points is $\frac{2\pi}{M}$. Thus the states over von Neumann lattice are labeled by the coordinates of one of the QDF and by the momentum of the other.

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APPENDIX: PHASE RELATION AMONG DIFFERENT REPRESENTATIONS

Utilizing the general definitions Eqs. (10) and (24) one obtains directly

$$\begin{aligned}\langle q|q_1N_1L_1+q_2N_2L_2\rangle &= \Delta^M(q-q_1N_1L_1-q_2N_2L_2)\\ &= \Delta^{M_1}(q-q_1)\Delta^{M_2}(q-q_2)\\ &= \langle q|[|q_1\rangle|q_2\rangle],\end{aligned}$$

$$\begin{aligned}\langle k|k_1N_1L_1+k_2N_2L_2\rangle &= \Delta^M(k-k_1N_1L_1-k_2N_2L_2)\\ &= \Delta^{M_1}(k-k_1)\Delta^{M_2}(k-k_2)=\langle k|[|k_1\rangle|k_2\rangle].\end{aligned} \tag{A1}$$

We also have

$$\begin{aligned}&\langle q_1'+q_2'L_2|q_1N_1L_1+q_2N_2L_2\rangle\\ &= \Delta[q_1'-q_1N_1L_1-(q_2-q_2'N_2^{-1})N_2L_2]\\ &= \Delta^{M_1}(q_1'-q_1)\Delta^{M_2}(q_1'-q_2+q_2'N_2^{-1}).\end{aligned}$$

The above implies also shift operators as follows:

$$e^{i2\pi/Mx}|k\rangle=|k+1\rangle,\quad e^{ip}|q\rangle=|q-1\rangle,$$

$$e^{i2\pi/M_1x}|k_1\rangle=|k_1+M_2\rangle,\quad e^{ipL_1}|q_1\rangle=|q_1-M_2\rangle$$

with similar equations for $|k_2\rangle$, $|q_2\rangle$

We now consider the wave functions explicitly Define

$$\begin{aligned}|q_1k_2,C_1\rangle &\equiv \frac{1}{\sqrt{M_1}}\sum_{k_1} e^{-i2\pi/M_1k_1q_1N_1}|k_1N_1L_1+k_2N_2L_2\rangle\\ &= \frac{1}{\sqrt{M_1}}\sum_{k_1} e^{-i2\pi/M_1k_1q_1N_1}|k_1\rangle|k_2\rangle=|q_1\rangle|k_2\rangle.\end{aligned} \tag{A2}$$

We have by direct calculations that (the eigenvalue indices are deleted for brevity),

$$e^{i2\pi/M_1x}|C_1\rangle=e^{i2\pi/M_1q_1}|C_1\rangle,\quad e^{ipL_2}|C_1\rangle=e^{i2\pi/M_2k_2}|C_1\rangle.$$

In similar fashion, define

$$|q_1k_2,C_2\rangle\equiv\frac{1}{\sqrt{M_2}}\sum_{q_2} e^{i2\pi/M_2k_2q_2N_2}|q_1N_1L_1+q_2N_2L_2\rangle. \tag{A3}$$

With

$$e^{i2\pi/M_1x}|q_1k_2,C_2\rangle=e^{i2\pi/M_1q_1}|q_1k_2,C_2\rangle,$$

$$e^{ipL_2}|C_2\rangle=e^{i2\pi/M_2k_2}|q_1k_2,C_2\rangle.$$

Thus $|q_1k_2;C_1\rangle$ and $|q_1k_2;C_2\rangle$ are eigenfunctions of the complete set of commuting operators for the M-dimensional space under study. To calculate the (possible) phase difference between them we evaluate the overlap

$$\langle q_1k_2,C_1'|q_1k_2,C_2\rangle=\Delta^{M_1}(q_1'-q_1)\Delta^{M_2}(k_2'-k_2). \tag{A4}$$

This evaluation requires the evaluation of the Fourier transform $\langle k_1|q_1\rangle$:

$$\langle k_1|e^{i(2\pi/M_1)x}|q_1\rangle=e^{i(2\pi/M_1)q_1}\langle k_1|q_1\rangle=\langle k_1-M_2|q_1\rangle,$$

$$\langle k_1|e^{ipL_1}|q_1\rangle=e^{i(2\pi/M_1)k_1}\langle k_1|q_1\rangle=\langle k_1|q_1-M_2\rangle.$$

This leads to [with the proper normalization, see Eq. (A2)]:

$$\frac{1}{\sqrt{M_1}}e^{-i(2\pi/M_1)q_1k_1N_1}=\langle k_1|q_1\rangle.$$

We check this by evaluating (we do not include the normalization for simplicity)

$$\begin{aligned}\langle k|q\rangle &= e^{-i(2\pi/M)qk}=e^{-i2\pi/M(q_1N_1L_1+q_2N_2L_2)(k_1N_1L_1+k_2N_2L_2)}\\ &= e^{-i(2\pi/M_1)q_1k_1N_1}e^{-i2\pi/M_2q_2k_2N_2}=\langle k_1|q_1\rangle\langle k_2|q_2\rangle.\end{aligned} \tag{A5}$$

The kq representation for finite dimensional were considered by Zak [12,17] and are defined as the eigenfunctions of the (commuting) operators

$$\tau(M_1)=e^{i(2\pi/M_1)x},\quad T(L_2)=e^{ipL_2}, \tag{A6}$$

which are periodic in one variables and quasiperiodic in the other. These can be written in terms of $|q\rangle$, the eigenfunctions of $\tau(M)=e^{i(2\pi/M)x}$ (we label these by E_1) as

$$|q_1,k_2,E_1\rangle\equiv\frac{1}{\sqrt{M_2}}\sum_{q_2} e^{i(2\pi/M_2)k_2q_2}|q_1+q_2L_2\rangle. \tag{A7}$$

In terms of the eigenfunctions of $T(L_2)$ these can be expressed via (the extra label here is E_2)

$$|q_1,k_2,E_2\rangle\equiv\frac{1}{\sqrt{M_2}}\sum_{k_1} e^{-i(2\pi/M_1)k_1q_1}|k_2+k_1M_2\rangle. \tag{A8}$$

The variables in these states cannot be considered as referring to two degrees of freedom. Indeed they are well defined whether or not M_1 and M_2 are relative primes (in the latter case, of course, N_1, N_2 do not exist)

$$|q_1k_2,E_1\rangle\equiv\frac{1}{\sqrt{M_1}}\sum_{k_1} e^{-i(2\pi/M_1)k_1q_1}|k_2+k_1M_2\rangle. \tag{A9}$$

It is easily shown that

$$e^{i(2\pi/M_1)x}|q_1k_2,E_1\rangle=e^{i(2\pi/M_1)q_1}|q_1k_2,E_1\rangle,$$

$$e^{ipL_2}|q_1k_2,E_1\rangle = e^{i(2\pi/M_2)k_2}|q_1k_2,E_1\rangle.$$

Thus this wave function can differ at most by phase from $|q_1k_2;C_1\rangle$. This phase difference is gotten by evaluation of the overlap. In the following we suppress the (common) eigenvalues indices, e.g., $|q_1k_2,C_1\rangle \to C_1\rangle$

$$\langle C_1'|E_1\rangle = \frac{1}{M_1}\sum_{k_1',k_1}\langle k_1'N_1L_1 + k_2'N_2L_2|k_2 + k_1L_1\rangle \times\exp\left[i\frac{2\pi}{M_1}(k_1'q_1'N_1 - k_1q_1)\right], = \Delta^{M_2}(k_2-k_2')\Delta^{M_1}(q_1-q_1')e^{i(2\pi/M_1)k_2q_1N_1},$$

The phase difference between $|C_1\rangle$ and $|C_2\rangle$ may be obtained by evaluating the overlap between these two wave functions

$$\langle C_1'|C_2\rangle = \frac{1}{\sqrt{M}}\sum_{k_1',q_2} e^{i(2\pi/M_1)k_1'q_1'N_1} \times\langle k_1'N_1L_1\cdots|q_1N_1L_1\cdots\rangle e^{i(2\pi/M_2)k_2q_2N_2}.$$

Inserting

$$\langle k_1'N_1L_1\cdots|q_1N_1L_1\cdots\rangle = \frac{1}{\sqrt{M}}\exp - i\left[\frac{2\pi}{M_1}(k_1'q_1N_1) + \frac{2\pi}{M_2}(k_2'q_2N_2)\right].$$

We get

$$\langle C_1'|C_2\rangle = \Delta^{M_1}(q_1-q_1')\Delta^{M_2}(k_2-k_2')$$

thus confirming Eq. (A4). This can be checked by evaluating the overlap $\langle C_2'|E_1\rangle$: it gives the same result that we got above for $\langle C_1'|E_1\rangle$. We now consider

$$|E_2\rangle \equiv \frac{1}{\sqrt{M_2}}\sum_{q_2} e^{i(2\pi/M_2)k_2q_2}|q_1 + q_2L_2\rangle. \qquad (A10)$$

One can check directly that

$$e^{i(2\pi/M_1)x}|E_2\rangle = e^{i(2\pi/M_1)q_1}|E_2\rangle, \quad e^{ipL_2}|E_2\rangle = e^{i(2\pi/M_2)k_2}|E_2\rangle.$$

Thus this function may differ from those above by phase only. We calculate the overlap to get

$$\langle C_2'|E_2\rangle = \Delta^{M_2}(k_2-k_2')\Delta^{M_1}(q_1-q_1')e^{-i(2\pi/M_2)k_2q_1N_2}.$$

This implies that the phase difference between $|E_1\rangle$ and $|E_2\rangle$ is

$$e^{-i(2\pi/M_2)k_2q_1N_2}e^{-(2\pi/M_1)k_2q_1N_1}.$$

We now evaluate this overlap directly to confirm this result

$$\langle E_1'|E_2\rangle = \frac{1}{\sqrt{M}}\sum_{k_1',q_2} e^{i(2\pi/M_1)k_1'q_1'} \times\langle k_2' + k_1L_1|q_1 + q_2L_2\rangle e^{(2\pi/M_2)k_2q_2}. \qquad (A11)$$

The overlap,

$$\langle k_2' + k_1'L_1|q_1 + q_2L_2\rangle = \frac{1}{\sqrt{M}}e^{-i(2\pi/M)(k_2'q_1 + k_2'q_2L_2 + k_1'q_1L_1)}.$$

Substituting this into the Eq. (A11) we get

$$\langle E_1'|E_2\rangle = \frac{1}{M}e^{i(2\pi/M)k_2'q_1}\sum_{k_1'} e^{i(2\pi/M_1)k_1'(q_1-q_1')}\sum_{q_2} e^{i2\pi/M_1k_1'(q_1-q_1')} = \Delta^{M_1}(q_1-q_1')\Delta^{M_2}(k_2-k_2')e^{i(2\pi/M)k_2'q_1},$$

thereby confirming the previous anticipation.

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