PHYSICAL REVIEW A **78**, 011807(R) (2008)

Quasilinear evolution and saturation of the modulational instability of partially coherent optical waves

T. Hansson,[1] D. Anderson,[1] M. Lisak,[1] V. E. Semenov,[2] and U. Österberg[3]
[1]*Chalmers University of Technology, Göteborg, Sweden*
[2]*Institute of Applied Physics, Nizhny Novgorod, Russia*
[3]*Dartmouth College, Hanover, New Hampshire 03755–8000, USA*
(Received 2 October 2007; published 29 July 2008)

The quasilinear development of the modulational instability of a partially coherent wave in a dispersive and nonlinear medium is analyzed. It is found that the quasilinear diffusion process tends to further stabilize the instability growth in addition to the stabilizing effect which is due to the partial coherence. In an unstable situation, the growth of the perturbation causes a slow change of the coherence spectrum such as to increase the degree of incoherence and eventually saturate the instability. On the other hand, in a stable situation the decay of the perturbation may cause a reversed quasilinear diffusion of the background distribution, which leads to an increasing degree of coherence of the background.

DOI: 10.1103/PhysRevA.78.011807 PACS number(s): 42.65.Tg, 42.65.Sf

The modulational instability (MI) is a generic feature in nonlinear wave propagation governed by the nonlinear Schrödinger equation and is caused by an interplay between nonlinearity and dispersion or diffraction. Over the years MI has been investigated in connection with many nonlinear systems describing the propagation of coherent optical waves [1]. Only recently, theoretical [2–5] as well as experimental [6,7] investigations have shown that MI may also take place for partially coherent optical waves. It was found that the partially coherent character of the light provides a stabilizing effect that tends to suppress the MI.

However, as the modulational instability develops it will ultimately affect the background wave, a problem which has been investigated in detail for the coherent case, e.g., Ref. [8]. The self-consistent interaction between the unstable perturbation and the background wave in the partially coherent wave was first approached in [9] and shown to be described by a quasilinear diffusion equation for the Wigner distribution function, which characterizes the coherence properties of the background wave. In fact, this formalism is mathematically similar to the phenomena of Landau damping and growth of electron plasma waves and the concomitant quasilinear diffusion of the electron velocity distribution in plasma physics. In the plasma case, the interaction between a small (unstable) wave perturbation and the electron velocity distribution leads to a slow change of the distribution in such a way as to quench the instability by forming a plateau in the distribution function in the finite velocity range where the interaction between the particles and the wave takes place, i.e., in the range of resonant velocities; see, e.g., Refs. [10,11].

In the present work, the quasilinear diffusion equation for the Wigner distribution function is analyzed in more detail. The result is found to be a diffusive redistribution of the *total* Wigner distribution that tends to counteract the MI by further degrading the coherence of the partially coherent background until the growth rate of the MI becomes zero and no further interaction occurs. In contrast to the resonant interaction in the plasma case, as described above, the quasilinear dynamics in the Wigner case is *nonresonant* and involves interaction between the wave and all parts of the Wigner distribution. This implies that the stabilization of the instability proceeds by a broadening of the total distribution rather than the plateau formation in a finite range of the distribution, which is characteristic of Landau damping of plasma waves. In the case when the MI is originally stabilized by damping due to the partial coherence, the redistribution tends to increase the coherence of the background until either the perturbation is completely quenched or the damping rate of the MI goes to zero due to the decreasing width of the incoherent background spectrum. The present work is an effort to analyze the dynamics beyond the linear stability analysis of the modulational instability of partially coherent waves.

We consider one-dimensional (1D) propagation of a partially coherent wave in a medium with an intensity dependent refractive index. For simplicity we concentrate on a Kerr-type nonlinearity. Assuming that the relaxation time of the medium response is much longer than the characteristic time of the statistical wave intensity fluctuations, the evolution of the wave field, $\psi(x,t)$, is determined by the nonlinear Schrödinger equation (NLS), cf. [12],

$$i\frac{\partial\psi}{\partial t}+\frac{\beta}{2}\frac{\partial^2\psi}{\partial x^2}+\kappa\langle|\psi|^2\rangle\psi=0, \qquad (1)$$

where β is the diffraction or dispersion coefficient, κ is the nonlinear coefficient, and the bracket $\langle\,\rangle$ denotes statistical ensemble average.

A number of different approaches have been developed for analyzing partial coherence in terms of Eq. (1); for a summary, see [12]. In the present analysis we will use the approach based on the Wigner distribution function, $\rho(p,x,t)$, being the Fourier transform of the correlation function of $\psi(x,t)$, i.e.,

$$\rho(p,x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\langle\psi^*(x+y/2,t)\psi(x-y/2,t)\rangle e^{ipy}dy, \qquad (2)$$

which implies that

$$\langle|\psi|^2\rangle = \int_{-\infty}^{+\infty} \rho(p,x,t)dp. \quad (3)$$

The evolution of the Wigner function is determined by the Wigner-Moyal equation [3,4], which is obtained by applying the Wigner transform to Eq. (1). This yields

$$\frac{\partial\rho}{\partial t} + \beta p\frac{\partial\rho}{\partial x} + 2\kappa\langle|\psi|^2\rangle\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}\right)\rho = 0, \quad (4)$$

where the sine operator is formally defined by its series expansion and the arrows over the derivatives indicate that the derivatives act to the left and right, respectively. This equation is significantly simplified in the geometrical optics limit (also known as the Vlasov limit in plasma physics), which corresponds to keeping only the first term in the expansion of the sine operator, cf. [3]

$$\frac{\partial\rho}{\partial t} + \beta p\frac{\partial\rho}{\partial x} + \kappa\frac{\partial\langle|\psi|^2\rangle}{\partial x}\frac{\partial\rho}{\partial p} = 0. \quad (5)$$

Analogous equations have been obtained in plasma physics for describing the self-consistent interaction between plasma particles and Langmuir waves, cf. [13]. The Vlasov limit generally provides a good approximation to the Wigner-Moyal equation when the coherence length of the field is small compared to the width of the coherent field-intensity profile, a situation that corresponds to a large degree of incoherence.

For the subsequent analysis it is convenient to summarize the main features of the modulational instability analysis as given, e.g., in [3]. The background wave is assumed to have a constant (averaged) intensity, $\langle|\psi_0|^2\rangle$, and its coherence properties are characterized by the background distribution function $\rho_0(p)$, where $\int_{-\infty}^{+\infty}\rho_0(p)dp = \langle|\psi_0|^2\rangle$. The stability of this distribution to plane wave perturbations is analyzed by writing $\rho(p,x,t) = \rho_0(p) + \rho_1(p)\exp[i(kx-\omega t)]$, where $\rho_1 \ll \rho_0$. Inserting this ansatz into Eq. (4) and using Eq. (3), the following implicit dispersion relation is obtained for the perturbation:

$$1 + \frac{\kappa}{\beta}\int_{-\infty}^{+\infty} \frac{\rho_0(p+k/2) - \rho_0(p-k/2)}{k(p-\omega/\beta k)}dp = 0, \quad (6)$$

which in the Vlasov limit corresponding to small k can be written as

$$1 + \frac{\kappa}{\beta}\int_{-\infty}^{+\infty} \frac{d\rho_0/dp}{p-\omega/\beta k}dp = 0. \quad (7)$$

In the case of a Lorentzian distribution function of the form

$$\rho_0(p) = \frac{\psi_0^2}{\pi}\frac{p_0}{p^2+p_0^2}, \quad (8)$$

where p_0^{-1} is the correlation length, the dispersion relation can be evaluated explicitly using either Eq. (6) or (7) with the result

$$\omega = i|\beta k|\left(\frac{1}{2}\sqrt{k_c^2 - k^2} - p_0\right) \quad \text{from Eq. (6)},$$

$$\omega = i|\beta k|\left(\frac{k_c}{2} - p_0\right) \quad \text{from Eq. (7) (small } k \text{ limit)}, \quad (9)$$

where $k_c^2 = 4\kappa\psi_0^2/\beta$ is assumed positive, i.e., $\kappa\beta > 0$. In the coherent case when $p_0 = 0$, an instability occurs if $|k| < k_c$. This result clearly shows how the partial coherence tends to suppress the growth rate of the modulational instability.

However, this analysis only describes the initial linear stage of the instability. As the perturbation grows, it begins to affect the background distribution, thus changing also its own growth rate. This phenomenon is well known in plasma physics where the (linear) theory of the Landau damping and growth of plasma waves is extended to account for the phenomenon of quasilinear diffusion, cf. [10,11]. Starting from Eq. (4), and following the conventional steps in the quasilinear procedure, see, e.g., Refs. [10–13], an evolution equation for the Wigner distribution can be derived by expanding the distribution function and the intensity according to

$$\rho = \rho_0(p,t) + \sum_{n=1}^{\infty}\rho_n(p,x,t) \quad (10)$$

and

$$\langle|\psi|^2\rangle = \langle|\psi|^2\rangle_0(t) + \sum_{n=1}^{\infty}\langle|\psi|^2\rangle_n(x,t) \quad (11)$$

where each term with index n is assumed to be of order n in the small amplitude of the perturbation and the (slow) time variation indicated in the zero order terms [$\rho_0(p,t)$ and $\langle|\psi|^2\rangle_0(t)$] is due to the nonlinear reaction of the instability on the background. Inserting these expansions into Eq. (4), the evolution of the first order perturbation of the Wigner distribution ρ_1 can be written as

$$\frac{\partial\rho_1}{\partial t} + \beta p\frac{\partial\rho_1}{\partial x} + 2\kappa\langle|\psi|^2\rangle_1\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}\right)\rho_0 = 0, \quad (12)$$

where use has been made of the fact that $\langle|\psi|^2\rangle_0$ does not depend on x. An additional equation coupling ρ_0 and ρ_1 is obtained by taking the spatial average of Eq. (4), using the fact that the spatial averages of $\rho_n(p,x,t)$ and $\langle|\psi|^2\rangle_n(x,t)$ vanish since they can be expressed as Fourier integrals with wave number spectra denoted as ρ_{nk} and $\langle|\psi|^2\rangle_{nk}$, respectively. The slow time evolution of $\rho_0(p,t)$ can then be shown to be driven by the perturbations according to the equation

$$\frac{\partial\rho_0}{\partial t} \simeq -2\kappa\int_{-\infty}^{\infty}\langle|\psi|^2\rangle_1\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p}\right)\rho_1 dx, \quad (13)$$

where the right-hand side has been approximated to lowest nonvanishing order in the perturbations. An explicit expression for the perturbation ρ_1 can be obtained from the spatial Fourier transform of Eq. (12), after noting that the perturbations are linear and have an explicit time dependence of the form $\rho_{1k}(p,k,t) = \rho_{1k}(p,k)e^{-i\omega(k)t}$ and $\langle|\psi|^2\rangle_{1k}(k,t) = \langle|\psi|^2\rangle_{1k}(k)e^{-i\omega(k)t}$, which yields

$$\rho_{1k} = - i2\kappa \frac{1}{\beta pk - \omega} \langle |\psi|^2 \rangle_{1k} \sin\left(\frac{ik}{2} \frac{\vec{\partial}}{\partial p} \right) \rho_0, \qquad (14)$$

where $\omega=\omega(k)$ denotes the frequency of the perturbation and is determined by the dispersion relation equation (6). When this result is inserted into Eq. (13), using the orthogonality of the Fourier basis functions and expanding the sine operators in the Vlasov limit, Eq. (13) is reduced to the diffusion equation, cf. [9],

$$\frac{\partial \rho_0}{\partial t} = \frac{\partial}{\partial p}\left(D(p,t) \frac{\partial \rho_0}{\partial p} \right). \qquad (15)$$

The diffusion constant $D(p,t)$ is given by

$$D(p,t) = - \frac{i\kappa^2}{2\pi} \int_{-\infty}^{+\infty} \frac{k^2 |\langle |\psi|^2 \rangle_{1k}|^2}{\beta pk - \omega} dk, \qquad (16)$$

but is more conveniently written as [using the parity condition $\omega(k)=-\omega^*(-k)$]

$$D(p,t) = \frac{\kappa^2}{2\pi} \int_{-\infty}^{+\infty} \frac{\gamma k^2 |\langle |\psi|^2 \rangle_{1k}|^2}{(\beta pk - \omega_r)^2 + \gamma^2} dk, \qquad (17)$$

where $\omega \equiv \omega_r + i\gamma$. Since the diffusion equation is derived in the small k limit, and the analysis will be restricted to situations where the wave would be modulationally unstable in the coherent case, it is clear that $\omega(k)$ is purely imaginary [cf. Eq. (9)] and the diffusion is of nonresonant character. This feature was not analyzed in [9] where the diffusion constant was evaluated in the classical way assuming a dominating resonant contribution. Thus, in the present investigation the quasilinear change in the Wigner distribution function is not restricted to a flattening of the resonant region, but stabilization of the instability will rather occur as a result of a change involving all parts of the distribution.

The time evolution of the perturbation $\rho_1(p,t)$ is determined by $\partial\rho_{1k}/\partial t = \gamma\rho_{1k}$, which implies that the intensity of the perturbation will, depending on the sign of γ, grow or be damped according to

$$|\langle |\psi|^2 \rangle_{1k}(t)|^2 = |\langle |\psi|^2 \rangle_{1k}(0)|^2 \exp\left(2\int_0^t \gamma(k,t')dt' \right). \qquad (18)$$

Finally, to simplify the analysis we assume the spectrum of the perturbation to be concentrated around a single wave number, $\pm k$, which implies that the diffusion coefficient, Eq. (17), has a Lorentzian dependence of p and can be written as

$$D(p,t) = K \frac{\gamma \exp\left(2\int_0^t \gamma dt' \right)}{p^2 + p_1^2}; \quad K = \frac{\kappa^2}{\pi\beta^2} |\langle |\psi|^2 \rangle_{1k}(0)|^2, \qquad (19)$$

where $\gamma=\gamma(k,t)$ and $p_1=\gamma/(\beta k)$. The diffusion equation (15) together with Eqs. (19) and (9) self-consistently determine the evolution of the quasilinear diffusion process.

In order to proceed with an approximate analysis that brings out the main physical features of the interaction dynamics, we make the analytically simplifying assumption that the quasilinear diffusion proceeds in such a way as to approximately preserve the Lorentzian shape of the background distribution while allowing the parameters of the Lorentzian to vary in time, i.e.,

$$\rho_0(p,t) = \frac{\psi_0^2(t)}{\pi} \frac{p_0(t)}{p^2 + p_0^2(t)}. \qquad (20)$$

The diffusing Lorentzian now involves two parameter functions, $\psi_0^2(t)$ and $p_0(t)$, which can be determined from two moments obtained by integrating Eq. (15) as it stands and after multiplication by ρ_0, respectively. The reason for choosing this latter somewhat unusual moment is the fact that the Lorentzian does not have a finite root mean square (RMS) width, which rules out the possibility of using the more conventional moment with respect to p^2. This approximate approach, based on a similarity ansatz for the evolution of the distribution, is inspired by the Goodman moment method, which has been used for analyzing many types of diffusion problems, see [14]. We thus have two independent relations involving $\psi_0^2(t)$ and $p_0(t)$, viz.,

$$\frac{d}{dt} \int_{-\infty}^{\infty} \rho_0 dp = 0,$$

$$\frac{d}{dt} \int_{-\infty}^{\infty} \rho_0^2 dp = -2 \int_{-\infty}^{\infty} D(p,t) \left(\frac{\partial \rho_0}{\partial p} \right)^2 dp. \qquad (21)$$

The first relation in Eq. (21) implies that $\psi_0^2(t) = \psi_0^2(0) = \text{const}$. The second relation simplifies considerably by assuming that $p_1^2 \ll p^2$. The variation of $p_0(t)$ is then found to be determined by

$$p_0^4(t) = p_0^4(0) + 10K\left[\exp\left(2\int_0^t \gamma(t')dt' \right) - 1 \right]. \qquad (22)$$

A modulationally unstable perturbation thus tends to cause a diffusion and a concomitant broadening of the background distribution, which increases the width $p_0(t)$ and the associated degree of incoherence of the background, cf. Figs. 1 and 2. According to Eq. (9) this implies that the growth rate of the instability will decrease until it ultimately becomes zero and the instability is quenched by the quasilinear diffusion process. On the other hand, in the stable case when $\gamma < 0$, the diffusion coefficient becomes negative. This should cause a narrowing of the coherence spectrum (implying that the background becomes more coherent), either until the spectral energy in the perturbation becomes zero and the diffusion process stops, or until the damping ceases due to narrowing of the gain spectrum [decreasing $p_0(t)$]. The situation is analogous to that of quasilinear diffusion of nonresonant electrons interacting with electrostatic plasma waves [10,11].

Self-consistent numerical simulations based on Eq. (3) and the Vlasov equation (5) would involve solving a 3D nonlinear integrodifferential system of equations and is beyond the scope of this paper. However, simulations of the system (15), (19), and (9) were carried out for the case of $\gamma > 0$ and these verify the qualitative behavior of the quasilinear diffusion process obtained from the approximate approach. Equation (22) does provide a good approximation for the evolution of the width of the distribution, although it

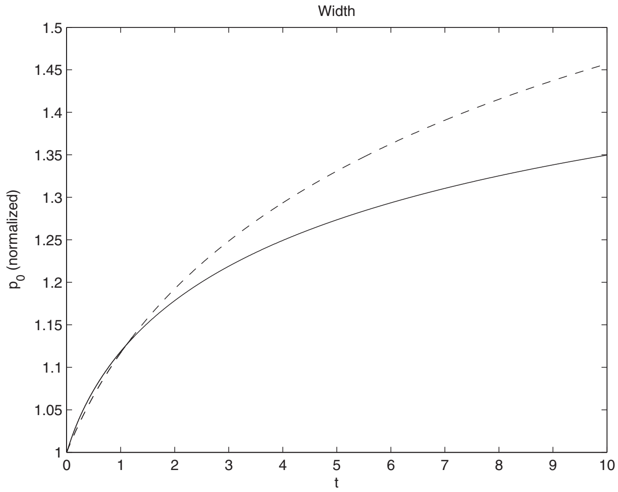

FIG. 1. The variation of the normalized width of the background distribution $(p_0(t)/p_0(0))$ as a function of time—numerically obtained result (fully drawn line) and analytical prediction according to Eq. (22) (dashed line). The parameters are chosen as $\psi_0^2=100$, $p_0(0)=3$, $\beta k=10^{-2}$, $k_c^2=100$, $k^2=20$, and $K=150$.

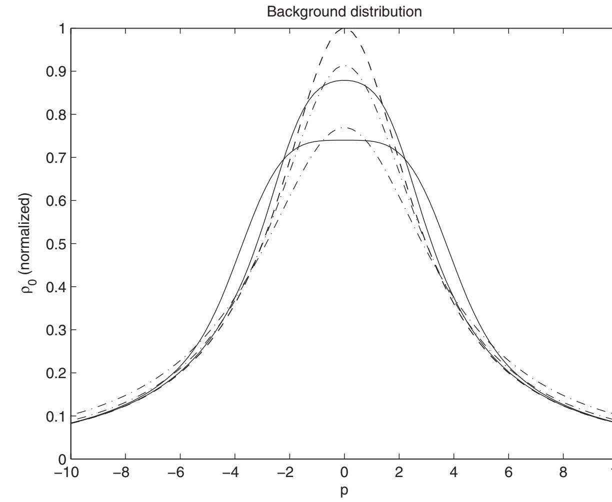

FIG. 2. Comparison of the evolution of the normalized Wigner distribution function $(\rho_0(p,t)/\rho_0(0,0))$, as obtained by numerical simulations (fully drawn line) and by the approximate moment method (dashed line), at the normalized times $t=0$, $t=1.25$, and $t=10$, respectively. Same parameters as in Fig. 1.

tends to overestimate the change in $p_0(t)$ for larger times, cf. Fig. 1. A further comparison was also made for the time evolution of the total distribution as predicted by the Lorentzian similarity ansatz and as obtained numerically, cf. Fig. 2. The qualitative agreement is seen to be good, although the evolving non-Lorentzian form of the numerical solution starts to become significant for larger times.

We emphasize that a behavior, similar to the one predicted and observed above, was found in [5], where an experimental study was made of a continuous wave (CW) light beam with stochastic phase noise, propagating in a nonlinear Kerr medium. The change of the coherence properties of the beam during propagation depended strongly on the sign of dispersion. In the modulationally stable case, the optical spectrum and the coherence properties were essentially unchanged, whereas in the unstable case of anomalous dispersion, strong spectral distortion and coherence degradation were observed.

In conclusion, the present analysis has considered the quasilinear evolution of the modulational instability of a partially coherent CW beam in a dispersive medium with a Kerr-type nonlinearity. The evolution of the background distribution is shown to be determined by a quasilinear *nonresonant* diffusion equation for the corresponding Wigner distribution function, Eq. (15). The nonresonant character of the diffusion implies that the interaction occurs between the wave and the whole distribution—not just a restricted resonant part. An approximate analysis of the solution of the quasilinear diffusion equation shows that the coherence properties of the background distribution change in such a way as to lead to a globally adjusted steady-state distribution. In the modulationally unstable case, the perturbation grows and the background distribution broadens until the increased incoherence stops further growth of the perturbation. In the marginally stable case, the perturbation is damped while making the background distribution more coherent. Two steady-state solutions are possible in this latter case: either the perturbation is damped out completely or the attenuation is halted by the decreasing damping rate as the coherence of the background increases.

[1] G. P. Agrawal, *Nonlinear Fiber Optics*, 3rd ed. (Academic, London, 2001).
[2] M. Soljačić, M. Segev, T. Coskun, D. N. Christodoulides, and A. Vishwanath, Phys. Rev. Lett. **84**, 467 (2000).
[3] B. Hall, M. Lisak, D. Anderson, R. Fedele, and V. E. Semenov, Phys. Rev. E **65**, 035602(R) (2002).
[4] L. Helczynski, D. Anderson, R. Fedele, B. Hall, and M. Lisak, IEEE J. Sel. Top. Quantum Electron. **8**, 408 (2002).
[5] S. B. Cavalcanti, G. P. Agrawal, and M. Yu, Phys. Rev. A **51**, 4086 (1995).
[6] D. Kip, M. Soljačić, M. Segev, E. Eugenieva, and D. N. Christodoulides, Science **290**, 495 (2000).
[7] J. Klinger, H. Martin, and Z. Chen, Opt. Lett. **26**, 271 (2000).
[8] S. Trillo and S. Wabnitz, Opt. Lett. **16**, 986 (1991).
[9] M. Lisak, D. Anderson, L. Helczynski-Wolf, R. Fedele, and V. Semenov, Phys. Scr., T **T113**, 56 (2004).
[10] T. H. Stix, *Waves in Plasmas* (AIP, New York, 1992).
[11] P. M. Bellan, *Fundamentals of Plasma Physics* (Cambridge University Press, Cambridge, 2006).
[12] Yu. S. Kivshar and G. P. Agrawal, *Optical Solitons; From Fibers to Photonic Crystals* (Academic, San Diego, 2003).
[13] S. L. Musher, A. M. Rubenchik, and V. E. Zhakarov, Phys. Rep. **252**, 177 (1995).
[14] T. R. Goodman, *Advances in Heat Transfer*, edited by T. F. Irvine and J. P. Harnett (Academic, New York, 1964), Vol. 1.