Evolution of entanglement entropy following a quantum quench: Analytic results for the *XY* chain in a transverse magnetic field

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The nonequilibrium evolution of the block entanglement entropy is investigated in the *XY* chain in a transverse magnetic field after the Hamiltonian parameters are suddenly changed from and to arbitrary values. Using Toeplitz matrix representation and multidimensional phase methods, we provide analytic results for large blocks and for all times, showing explicitly the linear growth in time followed by saturation. The consequences of these analytic results are discussed and the effects of a finite block length is taken into account numerically.

The nonequilibrium evolution of extended quantum systems is one of the most challenging problems of contemporary research in theoretical physics. The subject is in a renaissance era after the experimental realization [1] of cold atomic systems that can evolve out of equilibrium in the absence of any dissipation and with high degree of tunability of Hamiltonian parameters. A strongly limiting factor for a better understanding of these phenomena is the absence of effective numerical methods to simulate the dynamics of quantum systems. For methods such as time-dependent density matrix renormalization group (TDMRG) [2] this lack of efficiency has been traced back [3] to a too fast increasing of the entanglement entropy between parts of the whole system and the impossibility for a classical computer to store and manipulate such large amounts of quantum information.

This observation partially moved the interest from the study of local observables to the understanding of the evolution of the entanglement entropy and in particular to its growth with time [5,6]. Based on early results from conformal field theory [6,7] and on exact and/or numerical ones for simple solvable models [6,8] it is widely accepted [3] that the entanglement entropy grows linearly with time for a so-called global quench (i.e., when the initial state differs globally from the ground state and the excess of energy is extensive), while at most logarithmically for a local one (i.e., when the initial state has only a local difference with the ground state and so a little excess of energy). As a consequence a local quench is simulable by means of TDMRG, while a global one is not.

However, despite this fundamental interest and a large effort of the community, still analytic results are lacking. In this Rapid Communication we fill this gap providing the analytic expression of the entanglement entropy at any time in the limit of a large block for the *XY* chain in a transverse magnetic field described by the Hamiltonian

$$H(h,\gamma) = -\sum_{j=1}^{N}\left[\frac{1+\gamma}{4}\sigma_j^x\sigma_{j+1}^x + \frac{1-\gamma}{4}\sigma_j^y\sigma_{j+1}^y + \frac{h}{2}\sigma_j^z\right], \quad (1)$$

where σ_j^α are the Pauli matrices at the site *j*. Periodic boundary conditions are always imposed. In spite of its simplicity, the model shows a rich phase diagram being critical for *h* =1 and any γ and for $\gamma=0$ and $|h|\leq 1$, with the two critical lines belonging to different universality classes. The *block*

entanglement entropy is defined as the Von Neumann entropy $S_\ell = -\text{Tr}\,\rho_\ell \ln \rho_\ell$, where $\rho_\ell = \text{Tr}_{n\geq\ell}\,\rho$ is the reduced density matrix of the block formed by ℓ contiguous spins. In the following we will consider the quench with parameters suddenly changed at time $t=0$ from h_0, γ_0 to h, γ.

Our main result is that, in the thermodynamic limit $N \to \infty$ and subsequently in the limit of a large block $\ell \gg 1$, the time dependence of $S_\ell(t)$ can be written as an integral over the momentum variable φ,

$$S_\ell(t) = t\int_{2|\epsilon'|t<\ell}\frac{d\varphi}{2\pi}2|\epsilon'|H(\cos\Delta_\varphi)$$

$$+ \ell\int_{2|\epsilon'|t>\ell}\frac{d\varphi}{2\pi}H(\cos\Delta_\varphi), \quad (2)$$

where $\epsilon' = d\epsilon/d\varphi$ is the derivative of the dispersion relation $\epsilon^2 = (h-\cos\varphi)^2 + \gamma^2\sin^2\varphi$ and represents the momentum dependent sound velocity (that because of locality has a maximum we indicate as $v_M \equiv \max_\varphi|\epsilon'|$), $\cos\Delta_\varphi = [hh_0 - \cos\varphi(h+h_0) + \cos^2\varphi + \gamma\gamma_0\sin^2\varphi]/\epsilon\epsilon_0$ contains all the quench information [9] and $H(x) = -[(1+x)/2\ln(1+x)/2 + (1-x)/2 \times\ln(1-x)/2]$.

We first prove (2) and then discuss its interpretation and physical consequences. The readers not interested in the derivation can jump directly to the latter part.

The method. The entanglement entropy can be written in terms of a block Toeplitz matrix [6,10]. One first introduces Majorana operators $\check{a}_{2l-1} \equiv (\Pi_{m<l}\sigma_m^z)\sigma_l^x$ and $\check{a}_{2l} \equiv (\Pi_{m<l}\sigma_m^z)\sigma_l^y$ and the correlation matrix Γ_ℓ^A through the relation $\langle\check{a}_m\check{a}_n\rangle = \delta_{mn} + i\Gamma_{\ell\,mn}^A$ with $1\leq m, n\leq\ell$, that is a block Toeplitz matrix,

$$\Gamma_\ell = \begin{bmatrix} \Pi_0 & \Pi_1 & \cdots & \Pi_{\ell-1} \\ \Pi_{-1} & \Pi_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \Pi_{1-\ell} & \cdots & \cdots & \Pi_0 \end{bmatrix}, \quad \Pi_l = \begin{bmatrix} -f_l & g_l \\ -g_{-l} & f_l \end{bmatrix},$$

with (as a straightforward generalization of [6])

$$g_l = \int_{-\pi}^{\pi}\frac{d\varphi}{2\pi}e^{-i\varphi l}e^{-i\theta_\varphi}(\cos\Delta_\varphi + i\sin\Delta_\varphi\cos 2\epsilon_\varphi t),$$

$$f_l = i \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} e^{-i\varphi l} \sin \Delta_\varphi \sin 2\epsilon_\varphi t, \quad (3)$$

$\cos \theta_\varphi = -(h - \cos \varphi)/\epsilon$, and $\sin \Delta_\varphi = -\sin \varphi [\gamma h_0 - \gamma_0 h - \cos \varphi(\gamma - \gamma_0)]/\epsilon\epsilon_0$. The entanglement entropy is given by

$$S_\ell = -\mathrm{Tr}\left[\frac{1 + i\Gamma_\ell}{2} \ln \frac{1 + i\Gamma_\ell}{2} \right]. \quad (4)$$

This trace can be numerically evaluated for finite ℓ as done in Ref. [6] in the Ising case ($\gamma = \gamma_0 = 1$) for $h, h_0 \geqslant 1$. Strong numerical evidence supports the fact that for $2v_M t < \ell$, $S_\ell(t)$ increases linearly with time for large enough ℓ, but this remained without any proof until now. It has also been argued that the limit $t \to \infty$ exists and S_ℓ saturates to a value proportional to ℓ (opposite the ground-state expectation where there is at most a logarithmic ℓ dependence [10–12]) that has been calculated with a generalization of the Szego lemma [6]. Equation (2) not only provides the proof for a strictly linear increasing of $S_\ell(t)$ for $t < \ell/2v_M$, but gives also the complete time dependence.

Proof of Eq. (2). Let us first sketch the strategy to prove Eq. (2) and give only after the technical details. The matrix $i\Gamma_\ell$ has the same eigenvalues of the $\ell \times \ell$ Hermitian Hankel+Toeplitz matrices $W_\pm = H \pm iT$, with $H_{kj} = g_{\ell+1-k-j}$ and $T_{kj} = f_{k-j}$. In fact, if $\vec{w}$ is an eigenvector of $W_\pm$ with eigenvalue ω, then the vector $\vec{u}$ with elements $u_{2j-1} = w_j$ and $u_{2j} = \pm w_{\ell+1-j}$ is an eigenvector of $i\Gamma_\ell$ with eigenvalue $\mp \omega$. We will show that $\mathrm{Tr}\, W^{2n}$ with n integer satisfies a relation similar to Eq. (2) with $H(x)$ replaced by the appropriate power. Since Eq. (4) can be written as an expansion in $\mathrm{Tr}\, W^n$ with only even powers, this implies Eq. (2). Another ingredient is that in Eq. (3) for $\ell \gg 1$ the term $e^{-i\theta_\varphi}$ is stationary being ℓ independent. At this point a multidimensional integral for $\mathrm{Tr}\, W^{2n}$ (see below) can be calculated with the stationary phase approximation, which is exact for $\ell \gg 1$.

In order to obtain $\mathrm{Tr}[W^n]/\ell, n \in \mathbb{N}$ as $\ell \to \infty$ with a multidimensional stationary phase approximation (see, e.g., [14]), we consider the Toeplitz and the Hankel symbols

$$T_{kj} = \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} e^{-i(k-j)\varphi} t(\varphi),$$

$$H_{kj} = \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} e^{-i(k+j-\ell-1)\varphi} h(\varphi). \quad (5)$$

Each multiplication between two $H+T$ matrices involves sums such as

$$\sum_{j=1}^{\ell} \exp\{i[j - (\ell+1)/2](\varphi_1 \mp \varphi_2)\}$$

$$= \ell \int_{-1}^{1} d\xi_{12} \frac{\varphi_1 \mp \varphi_2}{4 \sin\left(\frac{\varphi_1 \mp \varphi_2}{2}\right)} \cos\left(\ell \xi_{12} \frac{\varphi_1 \mp \varphi_2}{2}\right), \quad (6)$$

where the sign is plus when a Hankel matrix is on the left and minus otherwise. We can fix the sign to be minus, multiplying any Hankel symbol to the right by the parity operator acting as $Pf(x)P = f(-x)$. When we close the chain with

the trace operator, we cannot change the sign of the first symbol and if there is an odd number of P, the term does not contribute to the leading order, because the phase in the integrand is not stationary anymore. The ξ dependence on the phase implies that all φ_j variables are equal along the stationary curve. Note that all the significant terms have the same number of Hankel symbols with direct and reversed signs. In fact, considering a tensor product of n symbols inside a symmetric integration, it follows (with $\simeq$ we always mean equal in the limit of large ℓ)

$$(t + hP)^{\otimes n} \simeq \sum_k a_k^{(n,t)} h^{\otimes k} \otimes h_-^{\otimes k} + b_k^{(n,t)} h^{\otimes(k+1)} \otimes h_-^{\otimes k} P, \quad (7)$$

with $h_- = PhP$, as can be straightforwardly proved by induction. Using the parity $h^*(\varphi) = h(-\varphi)$, we have that the Hankel symbol phase $e^{-i\theta_\varphi}$, which is not proportional to ℓ, can be dropped. The Toeplitz symbol is odd $t(\varphi) = -t(-\varphi)$ so the symmetrized product $[t(\varphi_1) + h(\varphi_1)P] \otimes [t(\varphi_2) + h(\varphi_2)P] \simeq t(\varphi_1)t(\varphi_2) + h(\varphi_1)h(-\varphi_2)$ leaves no parity terms, and if n is odd the whole integral asymptotically vanishes. Thus, from now on, we use the redefined symbols

$$h(\varphi) = \cos \Delta_\varphi - i \sin \Delta_\varphi \cos 2\epsilon t,$$

$$t(\varphi) = \sin \Delta_\varphi \sin 2\epsilon t, \quad (8)$$

which depend on the initial parameter only through Δ_φ. Repeated application of the multiplication rule leads to

$$\frac{1}{\ell} \mathrm{Tr}\, W^{2n} \simeq \frac{\ell^{2n-1}}{(4\pi)^{2n}} \int_{C_{[-\pi,\pi]}^{(2n)}} d^{2n}\varphi \int_{C_{[-1,1]}^{(2n)}} d^{2n}\xi A \times B, \quad (9)$$

with

$$A = \prod_{j=1}^{2n} \cos\left[\ell \xi_j \frac{\varphi_j - \varphi_{j-1}}{2} \right],$$

$$B = \prod_{j=1}^{n} \{\cos^2 \Delta_\varphi + \sin^2 \Delta_\varphi \cos[2\epsilon(\varphi_{2j-1})t - 2\epsilon(\varphi_{2j})t]\},$$

$$(10)$$

where C is the hypercubic domain. The product A can be moved inside the cosine function turning it in a sum, because of the symmetry of the ξ domain of integration with respect to 0. There is a trivial integration along a direction in the ξ domain since the integrand depends only on the difference between the ξ variables, thus

$$\int_{C_{[-1,1]}^{(2n)}} d^{2n}\xi A \simeq \int d^{2n-1}\zeta \Omega(\zeta) \cos\left[\ell \sum_{j=1}^{2n-1} \zeta_j \frac{\varphi_j - \varphi_{2n}}{2} \right],$$

with

$$\Omega(\zeta) = \max\left[0, \min_j\left(1, 1 - \sum_{i=1}^{j} \zeta_i\right) + \min_j\left(1, 1 + \sum_{i=1}^{j} \zeta_i\right) \right].$$

The permutation symmetry of φ variables allows one to order the cosine products in B, so that we can introduce a set of spin variable $\sigma_j \in \{-1, 1\}$ and bring B to the form

$$B \simeq \sum_{k=0}^{n} \binom{n}{k} \frac{(\cos \Delta)^{2n-2k} (\sin \Delta)^{2k}}{2^k}$$

$$\times \sum_{\{\sigma_i\}_{i=1\ldots k}} \cos \left[\sum_{i=1}^{k} (2\sigma_i \varepsilon_{2i-1} t - 2\sigma_i \varepsilon_{2i} t) \right]. \quad (11)$$

Interchanging the limit $\ell \to \infty$ with the integration with respect to the variable $\varphi \equiv \varphi_{2n}$, the remaining $(2n-2)$-dimensional integral is easily solved by stationary phase methods. For each configuration of $\{\sigma\}$ the Hessian determinant is 2^{2-4n} and the Hessian signature vanishes [this is the reason why in Eq. (2) oscillations, usually present in stationary phase calculations, are not present]. On stationary points we finally have $\Omega = 2$ on time-independent terms and $\Omega = 2(1 - 2|\epsilon'| t/\ell)$ otherwise. Of course the limit exists and the direct computation gives

$$\lim_{\ell \to \infty} \frac{\mathrm{Tr}\, W^{2n}}{\ell} = \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} (\cos \Delta_\varphi)^{2n}$$

$$+ \int_{2|\epsilon'| t < \ell} \frac{d\varphi}{2\pi} [1 - (\cos \Delta_\varphi)^{2n}] \left(1 - 2|\epsilon'| \frac{t}{\ell} \right),$$

$$(12)$$

that is, a "Taylor" expansion of Eq. (2). From Eq. (12) we also have the time dependence of all Rényi entropies $S_R = \ln \mathrm{Tr}[\rho_l^\alpha]/(1-\alpha)$: it is enough to replace $H(\cos \Delta)$ with $\ln(|\cos^2 \frac{\Delta}{2}|^\alpha + |\sin^2 \frac{\Delta}{2}|^\alpha)/(1-\alpha)$ in Eq. (2). In passing, we mention that our exact result obviously satisfies the bound for $S_\ell(t)$ given in Ref. [4].

Description of the result. In Ref. [6] an interpretation of the time dependence of S_ℓ has been provided in terms of causality (later generalized to the correlation functions in [13]). The idea is simple: the initial state has a very high energy relative to the ground state of the Hamiltonian which governs the time evolution, and therefore acts as a source of quasiparticle excitations. Particles emitted from different points (further apart than the correlation length in the initial state) are incoherent, but pairs of particles moving to the left or right from a given point are entangled. Thus $S_\ell(t)$ should just be proportional to the number of coherent particles that emitted from any point reach one a point in $[0,\ell]$ and the other the remainder of the system. Since there is a maximum speed for these excitations v_M, this implies the linear growth for $2v_M t < \ell$ and saturation for very large times.

However, only in the conformal case when ϵ' does not depend on the momentum because of the linear dispersion relation, this scenario makes quantitative predictions on the time evolution, else the rate of production of particles $f(p', p'')$ is an unknown function of the Hamiltonian parameters both before and after the quench. The comparison of Eq. (2) with the general one [Eq. (4.2) in [6]] allows one to identify $f(p', p'')$ with $\delta(p' - p'') H(\cos \Delta_{p'})$. We can also easily read from our result the value of the ratio
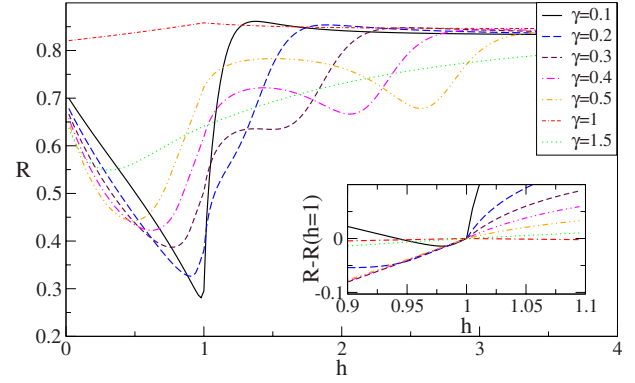

FIG. 1. (Color online) R defined in Eq. (13) for quenches from $(h_0 = 10, \gamma_0 = 2)$. The inset shows the singular behavior in the neighborhood of $h = 1$.

$$R \equiv \frac{\left(\dfrac{\partial S_A}{\partial t} \right)_{t < t^*}}{2v_M \left(\dfrac{\partial S_A}{\partial \ell} \right)_{t \gg t^*}} = \frac{\int_{-\pi}^{\pi} d\varphi |\epsilon'| H(\cos \Delta_\varphi)}{v_M \int_{-\pi}^{\pi} d\varphi H(\cos \Delta_\varphi)} \quad (13)$$

that results in the average of the absolute value of the speed of sound on the $H(\cos \Delta_\varphi)$ distribution. R as a function of the quench parameters is shown in Fig. 1. It is not analytic at the quantum critical point $h = 1$, as a trivial consequence of the nonanalyticity of its building blocks (i.e., ϵ, Δ_φ). However it is clear from the inset that such nonanalyticity is so weak that is unrealistic to say that the out-of-equilibrium behavior of entanglement entropy is sensitive to the phase transition.

From Eq. (2) we also have the large time corrections to the asymptotic result. Since $H(\pm 1) = 0$ with a logarithmic singularity, when the zero-velocity mode giving the large t behavior is at $\varphi = \pm \pi$ (as, e.g., for $h > 1$), one has that the first correction is $\propto \ell^4 \ln t/t^3$, whereas when there are zero velocities not at the border of the Brillouin zone, where $H(x)$ is finite, the leading correction is $\propto \ell^2/t$.

For $t = \infty$ only the second term in Eq. (2) contributes to the entropy that thus is extensive. As already noticed in Ref. [6] this is the same result at finite *large* temperature β_{eff}. An interesting question is whether this effective temperature is observable independent as found in the conformal case [13] or instead depends on the operator and so would not be a well-defined concept. Further checks of this point are mandatory to avoid speculations.

As a consequence of $H(x) \leq H(0) = \ln 2$, we have $S_\ell(t) \leq \ell \ln 2$ for any time, a bound that is just the maximum entanglement allowed by the dimension of the Hilbert space. In Ref. [6] it was noticed that the various curves for the quench from $h_0 = \infty$ to any h apparently collapse on a single curve when rescaling $S_\ell(t)$ to $S_\ell(\infty)$. From Eq. (2), this is exactly true for $|h| \leq 1$, but only approximately otherwise. Finally, the $t = \infty$ result is symmetric under the exchange $(h, \gamma) \leftrightarrow (h_0, \gamma_0)$, because the asymptotic result only depends on $\cos \Delta_\varphi$ that does not distinguish between initial and final values.
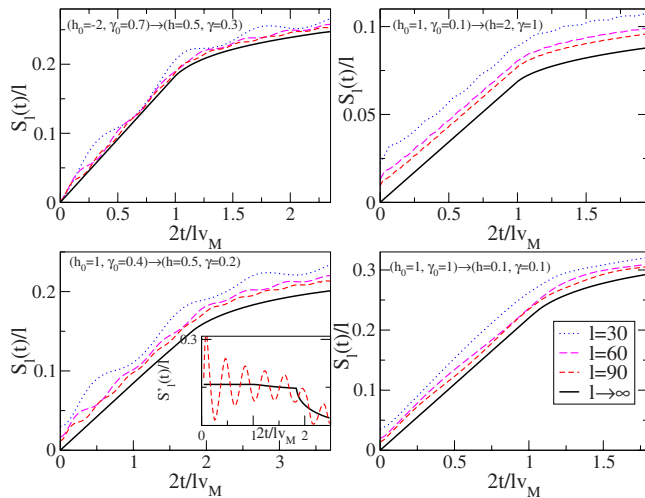
FIG. 2. (Color online) Time evolution of the entanglement entropy $S_\ell(t)/\ell$ for several quenches and ℓ. The straight line is the leading asymptotic result for large ℓ. The inset in the bottom-left graph shows the derivative with respect to time of $S_\ell(t)$ for $\ell \to \infty$ and the numerical derivative for $\ell = 90$.

Finite ℓ. The matrix representation (4) allows for the numerical calculation of S_ℓ for finite and relatively large ℓ. Some results are reported in Fig. 2, where we choose those quench parameters that make the finite ℓ effects more relevant. Increasing ℓ the results always approach Eq. (2), showing unambiguously its correctness, but there are peculiar and interesting finite ℓ effects. The most evident effect is the oscillation of $S_\ell(t)$. These oscillations have been generically seen in numerical studies also on more complicated models [8], but they are absent in the Ising model for h, h_0

$\geqslant 1$ [6]: oscillations can only be present when there is a second *local* maximum of $|\epsilon'|$. The data (also for cases not shown in the figure) provide strong evidence that the first nonoscillating correction at order $O(\ell^0)$ is positive and time independent.

In the bottom-left plot in Fig. 2 the most unexpected effect is shown. For the quench $(h_0 = 1, \gamma_0 = 0.4) \to (h = 0.5, \gamma = 0.2)$, it seems that the linear regime of $S_\ell(t)$ continues after $t^* = \ell/2v_M$. However, looking at the derivative (inset) one realizes that it is not exactly constant, since it slightly bends at t^*. This happens because for this peculiar quench the maximum velocity mode carries very little information, and so a stronger nonanalyticity is present at a local maximum of the velocity smaller than v_M. This effect is pronounced every time that $h_0 \gamma \sim h \gamma_0$, with $|h|, |\gamma|, |h_0|, |\gamma_0| < 1$, because of the functional form of Δ_φ. This anomalous behavior is important because it is nowadays common to extract the speed of propagation of information from t^*. Every time this effect is present, this procedure gives the wrong answer. For example, we plotted in the inset of Fig. 2 the numerical derivative of $S_\ell(t)$ for $\ell = 90$ (a value hardly reached in nonequilibrium simulation). It is evident that at t^* there is no trace of the nonanalyticity. Relying on these results one would have obtained a value of v_M that is almost half of the real one.

Conclusions. The nonequilibrium time evolution of $S_\ell(t)$ for the *XY* chain seems to encode most of the features that have been observed numerically in other contexts [8]. It is then natural to wonder whether slight modifications of Eq. (2) can be true in more complicated situations and not only for models mapped to free fermionic theories as the present one.

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