Solitary-wave solutions for few-cycle optical pulses

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Propagation of short optical pulses in a one-dimensional nonlinear medium is considered without the use of the slow envelope and unidirectional propagation approximations. The existence of uniformly moving solitary solutions is predicted for a Sellmeier-type dispersion function in the anomalous dispersion domain. A four-parametric family of such solutions is found that contains the classical envelope soliton in the limit of large pulse durations. In the opposite limit we get another family member, which, in contrast to the envelope soliton, strongly depends on the nonlinearity model and represents the shortest and the most intense pulse that can propagate in a stationary manner.

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I. INTRODUCTION

Ultrashort optical pulses have numerous applications which include measurements of fast relaxation times, processing of materials, testing of high-speed devices, tracing of chemical reactions, and investigations of light-matter interactions [1,2]. In the present paper we consider the most extreme representatives of ultrashort pulses—few-cycle pulses—containing only several oscillations of the electromagnetic field. We investigate propagation of such pulses in a transparent nonlinear medium assuming that the electric field depends on time and one space variable.

The description of ultrashort processes requires a modification of the standard slowly varying envelope (SVE) models based on the nonlinear Schrödinger equation (NSE) [3,4]. Apart from straightforward numerical solution of the Maxwell equations, two theories of ultrashort pulses are common in the literature. The first one adds several higher-order dispersion terms to the NSE (higher-order NSE [5,6]). These terms are especially important near the so-called zero-dispersion frequency (ZDF), where the second-order dispersion vanishes. Note that higher-order NSEs still assume that the pulse spectral width $\Delta\omega$ is much smaller than the carrier frequency ω_c, so that the SVE approximation can be used. On the contrary, when considering few-cycle pulses with $\Delta\omega \sim \omega_c$ the SVE approximation is not valid. In the latter case an analog of a higher-order NSE can be derived [7]. Several improvements of this model were suggested later [8–13]. These models are also referred to as higher-order NSEs and are now routinely used to model short pulses and continuum generation [14].

The second approach to the description of few-cycle pulses, which does not make use of the pulse envelope concept, is to derive a simplified model directly for the wave fields. The first-order evolution equations are obtained under the so-called unidirectional approximation. In the most simple form it states that

$$\partial_t^2 - \partial_z^2 \approx 2\partial_t(\partial_t + \partial_z) \tag{1}$$

for the normalized space-time variables and pulses propagating along the z axis. Following this approach several nonenvelope models were proposed [15–17]. The relation between NSE and unidirectional models was discussed in [18,19].

Another characteristic feature of ultrashort optical pulses is that the traditional representation of either the response function $\epsilon(\omega)$ or dispersion relation $k(\omega)$ as a Taylor series near ω_c can become invalid. This happens if $\Delta\omega$ becomes comparable with the spectral width of the transparency window. Since, due to resonances, $\epsilon(\omega)$ always has singularity points in the complex plane, the convergence radius for any Taylor expansion is finite and determined by the singularity that is nearest to ω_c. Therefore, an adequate description of material dispersion cannot be given by the higher-order NSE within the whole frequency range covered by the ultrashort pulse spectrum, whatever the number of terms in the dispersion operator [20].

A possible way to overcome this difficulty is to deal with the full medium description in terms of Bloch equations. Thereby a modified Korteweg–de Vries (KdV) equation was derived in the optical transparency limit and a sine-Gordon-type equation in the opposite case (see [21–26] and references cited therein). In addition, several phenomenological nonenvelope models were developed, where the medium polarization was described by introducing artificial equations for nonlinear oscillators [27–30].

Another possibility to avoid Taylor expansions is to construct a suitable fit to $\epsilon(\omega)$ in the desired spectral range. For instance, a simple expression with three fit parameters A, B, and $\bar{\epsilon}$,

$$\epsilon(\omega) \approx \bar{\epsilon}\left(1 + A\omega^2 - \frac{B}{\omega^2}\right), \tag{2}$$

provides an accurate approximation to the dispersion function in the transparency window between two resonances [31]. Using Eq. (2) together with the unidirectional approximation (1), the following reduced model for the normalized electric field $E(z,\tau)$:

$$\partial_z E - a\partial_\tau^3 E + b\int_{-\infty}^{\tau} E\, d\tau \pm E^2\partial_\tau E = 0, \tag{3}$$

was derived for a pulse propagating in a Kerr medium [32]. Here $\tau = t - \beta_1 z$ corresponds to the coordinate frame moving with the group velocity $1/\beta_1$, the parameters a and b are proportional to A and B, respectively, and the sign of the cubic term is determined by the sign of $\chi^{(3)}$.

In the situation when the pulse spectrum is located above the ZDF the last term in Eq. (2) can be neglected. In this case $b=0$ and Eq. (3) reduces to a modified KdV model which is completely integrable by the inverse scattering technique [33]. If, on the other hand, the pulse spectrum is located below the ZDF the A term in Eq. (2) can be dropped and therefore one has $a=0$ in Eq. (3). This particular case of Eq. (3) was introduced recently [34,35] and it is referred to as a short-pulse equation. The latter is integrable in full analogy with the standard NS, KdV, and sine-Gordon equations [36,37]. The model (3) can be considered as a possible replacement of the higher-order NSE for few-cycle pulses [31,32].

The most interesting class of solutions of the above mentioned models corresponds to solitary pulses propagating uniformly without changing their average shape as long as the dissipative effects can be neglected. Several such solutions are known for the higher-order NSE [38–43]. With a special choice of model parameters even a general solution with an arbitrary number of solitons can be constructed. On the other hand, to our knowledge, solitary solutions are not known for the modifications of the NSE accounting for fast envelope evolution.

Solitary solutions of nonenvelope equations were reported for integrable unidirectional models as cited above and were further obtained for more complex unidirectional models [44]. Many of them do not contain any internal field oscillations (video pulses). On the other hand, any localized solution of Eq. (3) with $b\neq 0$ must satisfy the area condition $\int_{-\infty}^{-\infty}E(z,\tau)d\tau=0$, and therefore it contains at least one oscillation. Although oscillating solitary solutions are usually nonstationary even in the comoving frame of reference and therefore are quite difficult to find, they are natural representatives of the extremely short electromagnetic waves we are interested in. The oscillating pulses can coexist with the video pulses. For instance, both breathers and ordinary solitons are solutions of the modified KdV equation [33]. An oscillating localized solution of the short-pulse equation [Eq. (3) with $a=0$] was reported in [45]. A similar solution for a circularly polarized short pulse [a two-component generalization of Eq. (3) with $a=0$] was found in [46]. All reported solutions have an important common property with breathers, namely, as the number of oscillations increases they become similar to the usual envelope solitons of the NSE, thus providing a link between the two theories of short pulses.

The goal of our paper is to investigate the transition between the few-cycle pulses and envelope solitons. The main result is that both the classical envelope and the nonenvelope solitons reported recently are representatives of the same family of localized solutions. To obtain this family, neither the SVE approximation nor the unidirectional approximation is necessary. Therefore, by construction the obtained solitons satisfy the above mentioned modifications of the NSE. Such one-dimensional solitons exist under the following essential assumptions: instantaneous Kerr-like or saturable nonlinearity, negligible dissipation effect for the time scales of interest, and a simple response function [see Eq. (5) below]. Our solution method uses ideas from the recent paper [46] which is devoted to unidirectional equations.

The paper is organized as follows. The assumptions on dispersion and nonlinearity are quantified and the model equations are introduced in the next section. Soliton tails are analyzed in Sec. III, where the solution ansatz is explained. The full nonlinear problem is posed and solved in Sec. IV. Finally, in Sec. V, we summarize the results of our analysis.

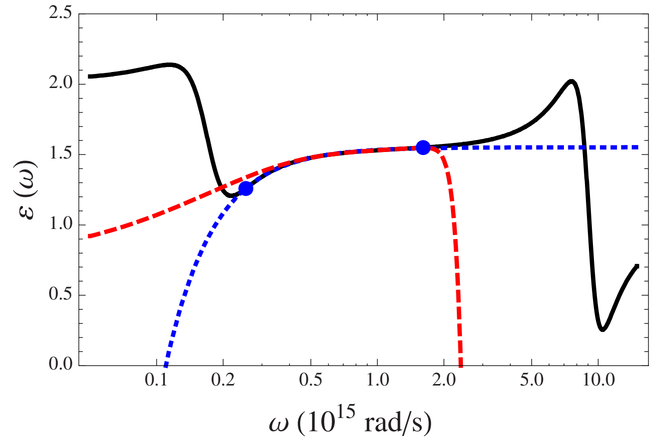

FIG. 1. (Color online) Real part of $\epsilon(\omega)$ for a fluoride glass (solid line) and a fit (5) (dotted line). We have chosen $\omega_0=1$ PHz and obtained $\bar{\epsilon}=1.552$ and $\mu^2=0.0121$ for the best fit between two inflection frequencies (thick points). For comparison, the eighth-order Taylor expansion of $\epsilon(\omega)$ with the central frequency ω_0 (close to the ZDF of 0.98 PHz) is also shown (dashed line).

II. MODEL EQUATIONS

We assume that the radial dependence of the electric field $\mathbf{E}=(E_x,E_y)$ is negligible, so that the field can be described in the $(1+1)$-dimensional approximation $\mathbf{E}=\mathbf{E}(z,t)$, for which we write

$$\partial_t^2\mathbf{D}-c^2\partial_z^2\mathbf{E}=0. \tag{4}$$

The electric displacement vector $\mathbf{D}(\mathbf{E})$ contains both linear and nonlinear parts. The linear part is given in the frequency domain as $\mathbf{D}_\omega^{\mathrm{lin}}=\epsilon(\omega)\mathbf{E}_\omega$. Below we presuppose the following fit for the response function:

$$\epsilon(\omega)\approx\bar{\epsilon}\left(1-\mu^2\frac{\omega_0^2}{\omega^2}\right), \tag{5}$$

where ω_0 is some reference frequency and both $\bar{\epsilon}$ and μ^2 are dimensionless fit parameters.

Relation (5) is motivated by Sellmeier's expression [34]. In the one-dimensional case (5) can provide a good approximation to the dispersion of a bulk medium in the anomalous dispersion range. An exemplary fit (5) for a fluoride glass is shown in Fig. 1 for a frequency range where $\mathrm{Re}[\epsilon(\omega)]$ is concave.

The physical origin of Eq. (5) was recently discussed in [46]. In the coordinate space it corresponds to

$$\partial_t^2\mathbf{D}^{\mathrm{lin}}=\bar{\epsilon}(\partial_t^2\mathbf{E}+\mu^2\omega_0^2\mathbf{E}).$$

Furthermore, it is worth noting that Eq. (5) allows explicit analysis.

We assume that the nonlinear part of the electric displacement vector is described by an instantaneous self-focusing

Kerr expression $\mathbf{D}^{\mathrm{nonl}}=4\pi\chi^{(3)}|\mathbf{E}|^2\mathbf{E}$ with positive frequency-independent $\chi^{(3)}$. Inserting $\mathbf{D}=\mathbf{D}^{\mathrm{lin}}+\mathbf{D}^{\mathrm{nonl}}$ into Eq. (4), we obtain the following basic evolution equation:

$$\bar{\epsilon}(\partial_t^2\mathbf{E}+\mu^2\omega_0^2\mathbf{E})-c^2\partial_z^2\mathbf{E}+4\pi\chi^{(3)}\partial_t^2(|\mathbf{E}|^2\mathbf{E})=0.$$

Introducing a wave vector k_0 such that $\omega_0/k_0=c/\sqrt{\bar{\epsilon}}$ and normalized variables

$$\bar{t}=\omega_0 t,\quad \bar{z}=k_0 z,\quad \mathbf{u}=\sqrt{\frac{4\pi\chi^{(3)}}{\bar{\epsilon}}}\mathbf{E},$$

we transform our basic equation to the dimensionless form

$$\mathbf{u}_{tt}-\mathbf{u}_{zz}+\mu^2\mathbf{u}+(|\mathbf{u}|^2\mathbf{u})_{tt}=0, \tag{6}$$

where derivatives are denoted by indices and overbars are omitted. Note that we do not use the unidirectional approximation in Eq. (6).

To establish a relation to unidirectional models one can introduce $\mathbf{U}(z,\tau)=\mathbf{u}(z,t)$ with $\tau=t-z$ and rewrite Eq. (6) in the equivalent form

$$2\mathbf{U}_{\tau z}+\mu^2\mathbf{U}+(|\mathbf{U}|^2\mathbf{U})_{\tau\tau}=\mathbf{U}_{zz}, \tag{7}$$

where in the spirit of the unidirectional approximation the dependence of $\mathbf{U}$ on z is weak. Ignoring the right-hand side of Eq. (7) we obtain a unidirectional counterpart of Eq. (6),

$$2\mathbf{U}_{\tau z}+\mu^2\mathbf{U}+(|\mathbf{U}|^2\mathbf{U})_{\tau\tau}=0, \tag{8}$$

which is a vector analog to Eq. (3). Clearly Eqs. (7) and (8) are similar when looking for traveling-wave solutions that are stationary in the comoving coordinate frame. However, such solitons do not exist (see the Appendix) and all localized solutions reported for Eq. (8) are nonstationary in the comoving coordinate frame [45,47]. As we will see, the corresponding localized solutions can be found for the full equation (6). They also move in one direction; however, the nontrivial contribution of the right-hand side of Eq. (7) is exactly accounted for.

In what follows we deal with the space-symmetric ($z\to -z$) formulation (6) in which $|\mathbf{u}|^2\mathbf{u}$ is replaced by a general instantaneous nonlinearity $f(|\mathbf{u}|^2)\mathbf{u}$ accounting for, e.g., saturation effects. It is convenient to introduce a complex quantity $\psi=u_x+iu_y$ and rewrite Eq. (6) as

$$\psi_{tt}-\psi_{zz}+\mu^2\psi+[f(|\psi|^2)\psi]_{tt}=0, \tag{9}$$

where

$$f(\zeta)=\zeta+O(\zeta^2)\quad \text{as}\quad \zeta\to 0. \tag{10}$$

Note that ψ as opposed to a complex amplitude in the NSE represents the components of the electric field directly. In the next sections we use also the auxiliary function

$$F(\zeta)=\frac{1}{\zeta}\int_0^{\zeta} f(\zeta)d\zeta,$$

where $F(\zeta)=\zeta/2+O(\zeta^2)$ as $\zeta\to 0$.

III. SOLITON TAILS

Before proceeding with the nonlinear case let us first discuss the small-amplitude limit of Eq. (9),

$$\psi_{tt}-\psi_{zz}+\mu^2\psi=0, \tag{11}$$

which is the Klein-Gordon equation. This equation describes two decaying tails of any localized solution of the full model. Let us show that such solutions can exist only if some necessary conditions are satisfied. First, introducing the field amplitude a and phase φ,

$$\psi=u_x+iu_y=ae^{i\varphi}, \tag{12}$$

we replace Eq. (11) with the following system:

$$a_{tt}-a_{zz}+(\mu^2-\varphi_t^2+\varphi_z^2)a=0, \tag{13}$$

$$(a^2\varphi_t)_t-(a^2\varphi_z)_z=0. \tag{14}$$

Next, substituting the usual traveling-wave ansatz for the field phase,

$$\varphi=\Omega\left(t-\frac{z}{\lambda}\right), \tag{15}$$

with Ω and λ being free parameters, into Eq. (14) we get $\lambda(a^2)_t+(a^2)_z=0$. This equation immediately suggests the following ansatz for the wave amplitude:

$$a=a(\xi),\quad \xi=t-\lambda z. \tag{16}$$

Note that the normalized velocities in Eqs. (15) and (16) are different, indicating that the solution is nonstationary in any moving frame of reference. Finally, due to Eq. (16) the phase Eq. (14) is satisfied automatically and the amplitude Eq. (13) reduces to the form

$$\ddot{a}-\left(\frac{\mu^2}{\lambda^2-1}-\frac{\Omega^2}{\lambda^2}\right)a=0, \tag{17}$$

where the overdot denotes a derivative with respect to ξ. Decaying soliton tails exist only if

$$s^2=\frac{\mu^2}{\lambda^2-1}-\frac{\Omega^2}{\lambda^2}>0, \tag{18}$$

which means that the necessary conditions

$$\lambda^2>1,\quad \mu^2>\frac{\lambda^2-1}{\lambda^2}\Omega^2 \tag{19}$$

must be satisfied.

Equations (15) and (16) correspond to the circular polarization of the electromagnetic field. The scaling parameter λ determines the normalized soliton velocity, while the second parameter Ω is the carrier frequency normalized by ω_0. Since our dispersion approximation (5) is valid for $\Omega\simeq 1$ and μ^2 is small (see Fig. 1), it follows from Eq. (19) that λ^2 is only slightly greater than 1. As we will see, the solitary solution exists in the parameter range $1<\lambda_{\min}^2<\lambda^2<\lambda_{\max}^2$.

IV. NONLINEAR SOLUTION

Now we turn to the solitary solution of the full nonlinear model (9). Using the amplitude-phase representation (12), we obtain the amplitude equation

$$(a_{tt} - a\varphi_t^2) - (a_{zz} - a\varphi_z^2) + \mu^2 a + [f(a^2)a]_{tt} - f(a^2)a\varphi_t^2 = 0 \tag{20}$$

instead of Eq. (13), and the phase equation

$$(2a_t\varphi_t + a\varphi_{tt}) - (2a_z\varphi_z + a\varphi_{zz}) + 2[f(a^2)a]_t\varphi_t + f(a^2)a\varphi_{tt} = 0 \tag{21}$$

instead of Eq. (14). The simplest expression (15) for the phase should be generalized to account for the last two terms in Eq. (21). Following [47] we introduce the ansatz

$$\varphi = \Omega\left(t - \frac{z}{\lambda} + \int_{-\infty}^{\xi} g(a)d\xi\right), \tag{22}$$

where the function $g(a)$ accounting for the nonlinear phase correction will be specified later. We require $g(0)=0$ such that Eq. (15) from the previous section can be recovered for the linear case. Inserting Eqs. (22) and (16) into Eq. (21), we rewrite the phase equation as an ordinary differential equation for $g(a)$,

$$[a - f_\lambda(a^2)a]g' + 2[a - f_\lambda(a^2)a]'g = 2[f_\lambda(a^2)a]'. \tag{23}$$

Here, a derivative with respect to a is denoted by a prime and the notations

$$f_\lambda(a^2) = \frac{f(a^2)}{\lambda^2 - 1}, \quad F_\lambda(a^2) = \frac{F(a^2)}{\lambda^2 - 1} \tag{24}$$

are introduced. Integrating Eq. (23) and using the condition $g(0)=0$, we obtain

$$g(a) = \frac{f_\lambda(a^2) - F_\lambda(a^2)}{[1 - f_\lambda(a^2)]^2} + \frac{f_\lambda(a^2)}{1 - f_\lambda(a^2)}. \tag{25}$$

Next, we derive an equation for $a(\xi)$ by inserting Eqs. (16) and (22) into Eq. (20),

$$[\ddot{a} - \Omega^2(1+g)^2 a] - [\lambda^2\ddot{a} - \Omega^2(\lambda^{-1} + \lambda g)^2 a] + \mu^2 a + \frac{d^2}{d\xi^2}[f(a^2)a] - \Omega^2(1+g)^2 f(a^2)a = 0,$$

and rewriting it in the form

$$\frac{d^2}{d\xi^2}[a - f_\lambda(a^2)a] - s^2 a + \Omega^2[(1+g)^2 f_\lambda(a^2) - g^2]a = 0, \tag{26}$$

where s^2 is defined by Eq. (18).

The amplitude $a(\xi)$ is completely defined by Eqs. (25) and (26). The linear part of Eq. (26) is identical to Eq. (17) and therefore ensures the correct soliton asymptotic at $\xi \to \pm\infty$. Equation (26) can be multiplied by $(d/d\xi)[a - f_\lambda(a^2)a]$ and integrated once. The result can be transformed into the form

$$\frac{\dot{a}^2}{2} + U(a) = \text{const} \tag{27}$$

with an effective potential function $U(a)$, which can be derived by a straightforward integration for any nonlinearity function $f(a^2)$. The solitary solution of Eq. (9) corresponds to a homoclinic orbit of Eq. (27). If any, the orbit must satisfy the equation

$$\dot{a} = \pm\sqrt{2[U(0) - U(a)]}.$$

After $U(a)$ is calculated, it is straightforward to check if the desired homoclinic orbit exists. Moreover, such an orbit always exists for the self-focusing nonlinearity (10) at least for small values of the soliton amplitude. The solution is equivalent to the classical envelope soliton of the NSE. As the amplitude increases the pulse is shortened and at some critical amplitude the shortest soliton is obtained. The latter has a nonenvelope nature. These issues are discussed in the remainder of this section.

A. Envelope solitons

In this subsection Eqs. (25) and (26) are considered in the limiting case

$$f_\lambda(a^2) = \frac{f(a^2)}{\lambda^2 - 1} \ll 1, \tag{28}$$

corresponding to a small but finite solution amplitude. Since λ^2 is usually only slightly greater than 1, inequality (28) implies a strong restriction on $a(\xi)$. Let us show that the soliton always exists in this limiting case and has a universal shape of the NSE soliton. First, omitting the high-order terms in Eq. (25), we get

$$g(a) = 2f_\lambda(a^2) - F_\lambda(a^2).$$

Next, we reduce Eq. (26) to the form

$$\ddot{a} - s^2 a + \Omega^2 f_\lambda(a^2)a = 0.$$

Finally, we simplify $f_\lambda(a^2)$ using Eq. (10) and obtain

$$\ddot{a} - s^2 a + \frac{\Omega^2}{\lambda^2 - 1}a^3 = 0. \tag{29}$$

Equation (29) has a standard solitary solution

$$a(\xi) = \frac{\sqrt{\lambda^2 - 1}}{\Omega}\frac{s\sqrt{2}}{\cosh(s\xi)} \tag{30}$$

similar to that of the NSE. It follows from Eq. (30) that in order to satisfy the condition (28) we need $s \ll 1$. Therefore, in accord with Eq. (17), the soliton tails are slowly decaying and contain many oscillations of the carrier wave. That is, Eq. (30) describes an envelope soliton.

B. Nonenvelope solitons

In this subsection we consider a solitary solution of the full system (25) and (26) with the simplest nonlinear function $f(\zeta)=\zeta$ corresponding to the self-focusing Kerr medium. In accord with Eq. (24), it is helpful to introduce a new amplitude variable

$$b(\xi) = \frac{a(\xi)}{\sqrt{\lambda^2 - 1}}$$

such that $f_\lambda(a^2)=b^2$ and $F_\lambda(a^2)=b^2/2$. Then Eq. (25) reduces to the form

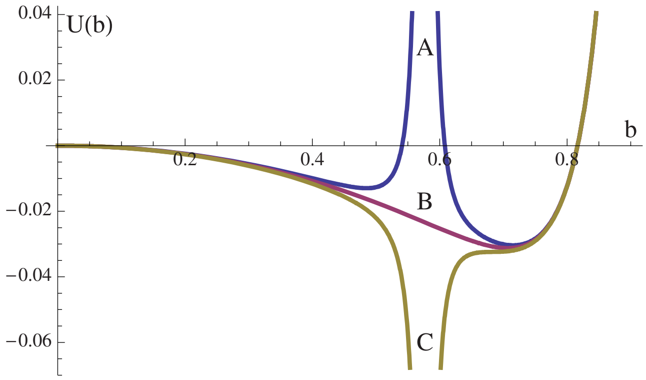

FIG. 2. (Color online) Effective potential $U(b)$ from Eq. (31) for $\Omega=1$ and three values of the parameter s: $8s^2=$ (A) 0.97, (B) 1, and (C) 1.03.

$$g(b) = \frac{b^2(3-2b^2)}{2(1-b^2)^2}.$$

Now, the amplitude Eq. (26) takes the form

$$\frac{d^2}{d\xi^2}(b-b^3) - s^2 b + \Omega^2[(1+g)^2 b^2 - g^2]b = 0$$

or

$$\frac{d^2}{d\xi^2}(b-b^3) - s^2 b + \Omega^2 \frac{4-9b^2+4b^4}{4(1-b^2)^3} b^3 = 0,$$

which is a generalization of Eq. (29). After one integration we obtain

$$\frac{\dot{b}^2}{2} - s^2 \frac{b^2(2-3b^2)}{4(1-3b^2)^2} + \Omega^2 \frac{b^4(2-7b^2+6b^4)}{8(1-b^2)^2(1-3b^2)^2} = 0 \quad (31)$$

with the integration constant being zero for a localized solution.

Equation (31) belongs to the general class (27) with a rather complicated effective potential $U(b)$. A similar equation was investigated in [47]. Solitary solutions exist as long as $s^2 \leq \Omega^2/8$. For $s^2 \to 0$ they reduce to the envelope solitons described in the previous section. In the limit $s^2 \to \Omega^2/8$ we get nonenvelope short-pulse solutions. To understand the behavior of the solution in this limit, we plot $U(b)$ from Eq. (31) for three different situations: (A) s^2 is slightly below $\Omega^2/8$, (B) $s^2=\Omega^2/8$, and (C) s^2 is slightly above $\Omega^2/8$ (see Fig. 2). In the last case a regular solution for $b(\xi)$ does not exist. Exactly for $s^2=\Omega^2/8$ two singular terms in $U(\xi)$ cancel each other. This is also evident from the expansion of the potential

$$U(b) \to \frac{\Omega^2}{96(1-3b^2)^2} - \frac{s^2}{12(1-3b^2)^2} \quad (32)$$

at $b^2 \to 1/3$. The critical value of s^2 was actually found from the expansion (32). At $s^2=\Omega^2/8$ Eq. (31) becomes

$$\dot{b}^2 = \Omega^2 \frac{b^2(2-3b^2)}{16(1-b^2)^2} \quad (33)$$

and can be solved in quadratures,

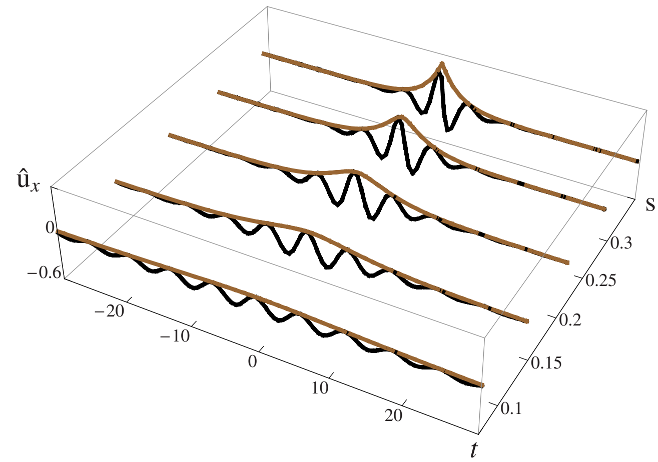

FIG. 3. (Color online) Normalized electric field $\hat{u}_x = u_x(\xi)/\sqrt{\lambda^2-1}|_{z=0}$ and pulse envelope $b(\xi)|_{z=0}$ computed from Eq. (31) for $\Omega=1$ and different values of the parameter s.

$$\frac{1}{3}\sqrt{2-3b^2} - \frac{1}{\sqrt{2}}\operatorname{arccosh}\left(\frac{\sqrt{2}}{\sqrt{3}b}\right) = \pm \frac{\Omega\xi}{4}. \quad (34)$$

Here $b^2(0)=2/3$ is the normalized pulse peak intensity.

For $s^2<\Omega^2/8$ we have a continuous family of pulses. Figure 3 illustrates how the soliton shape evolves as s^2 approaches the critical value. It is interesting to note that the limiting pulse for $s^2 \nearrow \Omega^2/8$ (the last curve in Fig. 3) has a cusplike shape and differs from the exact solution (34) obtained for $s^2=\Omega^2/8$. At the cusp point we have $b^2(0)=1/3$.

Thus, the soliton exists if the condition

$$0 < \frac{(\mu/\Omega)^2}{\lambda^2-1} - \frac{1}{\lambda^2} < \frac{1}{8}$$

is satisfied. This condition defines the range $\lambda^2_{\min}<\lambda^2<\lambda^2_{\max}$ of the soliton velocities. Note that, similarly to the envelope solitons, the soliton family found in this paper depends on four parameters: the velocity $1/\lambda$, the carrier frequency Ω, an arbitrary initial phase, and an arbitrary initial position.

V. CONCLUSIONS

To summarize, generalizing the results of [46], we have found a family of ultrashort optical solitons for Eq. (6) which does not assume the slow-envelope and unidirectional approximations. The following assumptions were made: one-dimensional nondissipative pulse propagation, instantaneous self-focusing nonlinearity, and an anomalous dispersion specified by Eq. (5) in the frequency range of interest (see Fig. 1).

Our mathematical procedure leads to the qualitatively simple equation (27), which allows for a straightforward analysis of the solitary solutions. All solitons have the following universal property: with increase of the pulse duration they reduce to a standard envelope soliton. Such behavior was first predicted in [48]. On the other hand, when the pulse duration decreases, the soliton evolves to a limiting shape representing the shortest pulse for which dispersive spreading is still compensated by nonlinearity (Fig. 3). The half-width of this pulse is of the order of the carrier wave period (single-cycle solution). The appearance of a cusplike singularity which prevents the existence of shorter pulses is discussed.

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APPENDIX

In this appendix, we demonstrate that Eq. (6) does not have traveling-wave solitary solutions that are stationary in a comoving coordinate frame. To this end we use a complex representation $u_x+iu_y=\dot{\Psi}(\xi)$ in Eq. (6). Here, the overdot denotes a derivative with respect to $\xi=t-\lambda z$. By integrating Eq. (6) we obtain an ordinary differential equation for $\Psi(\xi)$,

$$(1-\lambda^2)\ddot{\Psi}+\mu^2\Psi+\partial_\xi(|\dot{\Psi}|^2\dot{\Psi})=0.$$

The latter equation can be presented as an extremum condition $\delta L/\delta\Psi^*=0$ with

$$L[\Psi,\Psi^*]=\int_{-\infty}^{\infty}\left((\lambda^2-1)|\dot{\Psi}|^2+\mu^2|\Psi|^2-\frac{|\dot{\Psi}|^4}{2}\right)d\xi.$$

Assume that $\Psi=h(\xi)$ is a localized solitary solution. Now we insert a test function $\Psi=h(\sigma\xi)/\sqrt{\sigma}$ with a positive scaling parameter σ into $L[\Psi,\Psi^*]$ and rewrite the result as

$$L(\sigma)=\int_{-\infty}^{\infty}\left((\lambda^2-1)|\dot{h}(\xi)|^2+\frac{\mu^2|h(\xi)|^2}{\sigma^2}-\frac{\sigma|\dot{h}(\xi)|^4}{2}\right)d\xi.$$

Since the test function for $\sigma=1$ corresponds to the solitary solution, $L(\sigma)$ should have an extremum at this point. However, this contradicts

$$\left.\frac{dL(\sigma)}{d\sigma}\right|_{\sigma=1}=-\int_{-\infty}^{\infty}\left(2\mu^2|h(\xi)|^2+\frac{|\dot{h}(\xi)|^4}{2}\right)d\xi\neq 0.$$

We conclude that the localized solitary solution $h(\xi)$ cannot exist. In other words, all solitary solutions of Eq. (6) are nonstationary in the comoving frame of reference.

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