Experimentally accessible geometric measure for entanglement in *N*-qubit pure states

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We present a multipartite entanglement measure for *N*-qubit pure states, using the norm of the correlation tensor which occurs in the Bloch representation of the state. We compute this measure for several important classes of *N*-qubit pure states such as Greenberger-Horne-Zeilinger and *W* states and their superpositions. We compute this measure for interesting applications like the one-dimensional Heisenberg antiferromagnet. We use this measure to follow the entanglement dynamics of Grover's algorithm. We prove that this measure possesses almost all the properties expected of a good entanglement measure, including monotonicity. Finally, we extend this measure to *N*-qubit mixed states via convex roof construction and establish its various properties, including its monotonicity. We also introduce a related measure which has all properties of the above measure and is also additive.

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I. INTRODUCTION

Entanglement has proved to be a vital physical resource for various kinds of quantum-information processing, including quantum state teleportation [1,2], cryptographic key distribution [3], classical communication over quantum channels [4–6], quantum error correction [7], quantum computational speedups [8], and distributed computation [9,10]. Further, entanglement is expected to play a crucial role in many-particle phenomena such as quantum phase transitions, transfer of information across a spin chain [11,12], etc. Therefore, quantification of entanglement of multipartite quantum states is fundamental to the whole field of quantum information and, in general, to the physics of multicomponent quantum systems. Whereas the entanglement in pure bipartite states is well understood, classification of multipartite pure states and mixed states, according to the degree and character of their entanglement is still a matter of intense research [13–15]. The principal achievements are in the setting of bipartite systems. Among these, one highlights Wootter's formula for the entanglement of formation of two-qubit mixed states [16], which still awaits a viable generalization to multiqubit case. Others include corresponding results for highly symmetric states [17–19]. The issue of entanglement in multipartite states is far more complex. Notable achievements in this area include applications of the relative entropy [20], negativity [21] Schmidt measure [22], and the global entanglement measure proposed by Meyer and Wallach [23].

A measure of entanglement is a function on the space of states of a multipartite system, which is invariant on individual parts. Thus a complete characterization of entanglement is the characterization of all such functions. Under the most general local operations assisted by classical communication (LOCC), entanglement is expected to decrease. A measure of entanglement that decreases under LOCC is called an entanglement monotone. On bipartite pure states the sums of the k smallest eigenvalues of the reduced density matrix are entanglement monotones. However, the number of independent invariants (i.e., the entanglement measures) increases exponentially as the number of particles N increases and complete characterization rapidly becomes impractical. A pragmatic approach would be to seek a measure which is defined for any number of particles (scalable), which is easily calculated, and which provides physically relevant information or equivalently which passes the tests expected of a *good* entanglement measure [13,14].

In this paper, we present a global entanglement measure for N-qubit pure states which is scalable, which passes most of the tests expected of a good measure and whose value for a given system can be determined experimentally, without having a detailed *prior* knowledge of the state of the system. The measure is based on the Bloch representation of multipartite quantum states [24].

The paper is organized as follows. In Sec. II we give the Bloch representation of an N-qubit quantum state and define our measure $E_{\mathcal{T}}$. In Sec. III we compute $E_{\mathcal{T}}$ for different classes of N-qubit states, namely, the Greenberger-Horne-Zeilinger (GHZ) and W states and their superpositions. In Sec. IV we prove various properties of $E_{\mathcal{T}}$, including its monotonicity, expected of a good entanglement measure. In Sec. V we extend $E_{\mathcal{T}}$ to N-qubit mixed states via the convex roof and establish its monotonicity. In Sec. VI we introduce a related measure which is additive and shares all other properties with $E_{\mathcal{T}}$. Finally, we conclude in Sec. VII.

II. BLOCH REPRESENTATION OF AN *N*-QUBIT STATE AND THE DEFINITION OF THE MEASURE

Consider the generators $\{I,\sigma_x,\sigma_y,\sigma_z\}\equiv\{\sigma_0,\sigma_1,\sigma_2,\sigma_3\}$ of the SU(2) group (Pauli matrices). These Hermitian operators form an orthogonal basis (under the Hilbert-Schmidt scalar product) of the Hilbert space of operators acting on a single-qubit state space. The N times tensor product of this basis with itself generates a product basis of the Hilbert space of operators acting on the N-qubit state space. Any N-qubit density operator ρ can be expanded in this basis. The corre-

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sponding expansion is called the Bloch representation of ρ [24].

In order to give the Bloch representation of a density operator acting on the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ of an N-qubit quantum system, we introduce the following notation. We use k,k_i $(i=1,2,\ldots)$ to denote a qubit chosen from N qubits, so that k,k_i $(i=1,2,\ldots)$ take values in the set $\mathcal{N}=\{1,2,\ldots,N\}$. The variables α_k or α_{k_i} for a given k or k_i span the set of generators of the SU(2) group for the kth or k_ith qubit, namely, the set $\{I_{k_i},\sigma_{1_{k_i}},\sigma_{2_{k_i}},\sigma_{3_{k_i}}\}$ for the k_ith qubit. For two qubits k_1 and k_2 we define

$$\sigma_{\alpha_{k_1}}^{(k_1)} = (I_2 \otimes I_2 \otimes \cdots \otimes \sigma_{\alpha_{k_1}} \otimes I_2 \otimes \cdots \otimes I_2),$$

$$\sigma_{\alpha_{k_2}}^{(k_2)} = (I_2 \otimes I_2 \otimes \cdots \otimes \sigma_{\alpha_{k_2}} \otimes I_2 \otimes \cdots \otimes I_2),$$

$$\sigma_{\alpha_{k_1}}^{(k_1)}\sigma_{\alpha_{k_2}}^{(k_2)} = (I_2 \otimes I_2 \otimes \cdots \otimes \sigma_{\alpha_{k_1}} \otimes I_2 \otimes \cdots \otimes \sigma_{\alpha_{k_2}} \otimes I_2 \otimes I_2), \tag{1}$$

where $\sigma_{\alpha_{k_1}}$ and $\sigma_{\alpha_{k_2}}$ occur at the k_1th and k_2th places (corresponding to the k_1th and k_2th qubits, respectively) in the tensor product and are the α_{k_1}th and α_{k_2}th generators of SU(2), $(\alpha_{k_1}=1,2,3$ and $\alpha_{k_2}=1,2,3)$, respectively. Then we can write

$$\rho = \frac{1}{2^N}\Bigg\{ \otimes^N I_2 + \sum_{\{k\}\subset\mathcal{N}} \sum_{\alpha_k} s_{\alpha_k}\sigma_{\alpha_k}^{(k)} + \sum_{\{k_1,k_2\}} \sum_{\alpha_{k_1}\alpha_{k_2}} t_{\alpha_{k_1}\alpha_{k_2}}\sigma_{\alpha_{k_1}}^{(k_1)}\sigma_{\alpha_{k_2}}^{(k_2)} + \cdots + \sum_{\{k_1,k_2,\ldots,k_M\}} \sum_{\alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_M}} t_{\alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_M}}\sigma_{\alpha_{k_1}}^{(k_1)}\sigma_{\alpha_{k_2}}^{(k_2)}\cdots\sigma_{\alpha_{k_M}}^{(k_M)} + \cdots + \sum_{\alpha_1\alpha_2\ldots\alpha_N} t_{\alpha_1\alpha_2\ldots\alpha_N}\sigma_{\alpha_1}^{(1)}\sigma_{\alpha_2}^{(2)}\cdots\sigma_{\alpha_N}^{(N)}\Bigg\}. \tag{2}$$

where $\mathbf{s}^{(k)}$ is a Bloch vector (see below) corresponding to the kth subsystem, $\mathbf{s}^{(k)}=[s_{\alpha_k}]_{\alpha_k=1}^{3}$, which is a tensor of order 1 defined by

$$s_{\alpha_k} = \mathrm{Tr}(\rho\sigma_{\alpha_k}^{(k)}) = \mathrm{Tr}(\rho_k\sigma_{\alpha_k}), \tag{3}$$

where ρ_k is the reduced density matrix for the kth qubit. Here $\{k_1,k_2,\ldots,k_M\}$, $1\le M\le N$, is a subset of $\mathcal{N}$ and can be chosen in $\binom{N}{M}$ ways, contributing $\binom{N}{M}$ terms in the sum $\Sigma_{\{k_1,k_2,\ldots,k_M\}}$ in Eq. (2), each containing a tensor of order M. The total number of terms in the Bloch representation of ρ is 2^N. We denote the tensors occurring in the sum $\Sigma_{\{k_1,k_2,\ldots,k_M\}}$ $(1\le M\le N)$ by $\mathcal{T}^{\{k_1,k_2,\ldots,k_M\}}=[t_{\alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_M}}]$, which are defined by

$$t_{\alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_M}} = \mathrm{Tr}(\rho\sigma_{\alpha_{k_1}}^{(k_1)}\sigma_{\alpha_{k_2}}^{(k_2)}\cdots\sigma_{\alpha_{k_M}}^{(k_M)}) = \mathrm{Tr}[\rho_{k_1,k_2,\ldots,k_M}(\sigma_{\alpha_{k_1}}\otimes\sigma_{\alpha_{k_2}}\otimes\cdots\otimes\sigma_{\alpha_{k_M}})] \tag{4}$$

where $\rho_{k_1,k_2,\ldots,k_M}$ is the reduced density matrix for the subsystem $\{k_1,k_2,\ldots,k_M\}$. We call the tensor in the last term in Eq. (2) $\mathcal{T}^{(N)}$.

From Eq. (4) we see that all the correlations between M out of N qubits are contained in $\mathcal{T}^{\{k_1,k_2,\ldots,k_M\}}$ and all the N-qubit correlations are contained in $\mathcal{T}^{(N)}$. If ρ is an N-qubit pure state we have

$$\mathrm{Tr}(\rho^2) = \frac{1}{2^N}\left(1 + \sum_{k=1}^{N}\|\mathbf{s}^{(k)}\|^2 + \sum_{\{k_1,k_2\}}\|\mathcal{T}^{\{k_1,k_2\}}\|^2 + \cdots + \sum_{\{k_1,k_2,\ldots,k_M\}}\|\mathcal{T}^{\{k_1,k_2,\ldots,k_M\}}\|^2 + \cdots + \|\mathcal{T}^{(N)}\|^2\right) = 1. \tag{5}$$

Any state $\rho=|\psi\rangle\langle\psi|$ existing in a d^2-dimensional Hilbert space of operators acting on a d-dimensional Hilbert space of kets can be expanded in the basis comprising d^2-1 generators of SU(d) and the identity operator. The set of coefficients in this expansion, namely, $\{\mathrm{Tr}(\rho\lambda_i)\}$, $i=1,2,\ldots,d^2-1$, is a vector in $\mathbb{R}^{d^2-1}$ and is the Bloch vector of ρ. The set of Bloch vectors and the set of density operators are in one-to-one correspondence with each other. The set of Bloch vectors for a given system forms a subspace of $\mathbb{R}^{d^2-1}$ denoted $B(\mathbb{R}^{d^2-1})$. The specification of this subspace for $d\ge 3$ is an open problem [25,26]. However, for pure states, the following results are known [27]:

$$\|s\|_2 = \sqrt{\frac{d(d-1)}{2}},$$

$$D_r(\mathbb{R}^{d^2-1}) \subseteq B(\mathbb{R}^{d^2-1}) \subseteq D_R(\mathbb{R}^{d^2-1}), \tag{6}$$

where D_r and D_R are balls of radii $r=\sqrt{\frac{d}{2(d-1)}}$ and $R=\sqrt{\frac{d(d-1)}{2}}$, respectively, in $\mathbb{R}^{d^2-1}$.

We propose the following measure for an N-qubit pure state entanglement:

$$E_{\mathcal{T}}(|\psi\rangle) = (\|\mathcal{T}^{(N)}\| - 1), \tag{7}$$

where $\mathcal{T}^{(N)}$ is given by Eq. (4) for $M=N$ in the Bloch representation of $\rho=|\psi\rangle\langle\psi|$. The norm of the tensor $\mathcal{T}^{(N)}$ appearing in definition (7) is the Hilbert-Schmidt (Euclidean) norm $\|\mathcal{T}^{(N)}\|^2=(\mathcal{T}^{(N)},\mathcal{T}^{(N)})=\Sigma_{\alpha_1\alpha_2\ldots\alpha_N}t^2_{\alpha_1\alpha_2\ldots\alpha_N}$. Throughout this paper, by norm, we mean the Hilbert-Schmidt (Euclidean) norm. We comment on the normalization of $E_{\mathcal{T}}(|\psi\rangle)$ below.

III. GHZ AND *W* STATES

Before proving various properties of $E_{\mathcal{T}}(|\psi\rangle)$, we evaluate it for states in the N-qubit GHZ or W class. A general N-qubit GHZ state is given by

$$|\psi\rangle = \sqrt{p}|000\ldots0\rangle + \sqrt{1-p}|111\ldots1\rangle, \quad N \ge 2. \tag{8}$$

A general element of $\mathcal{T}^{(N)}$ is given by $t_{i_1i_2\ldots i_N}=\langle\psi|\sigma_{i_1}\otimes\sigma_{i_2}\otimes\cdots\otimes\sigma_{i_N}|\psi\rangle$, $i_k=1,2,3$, $k=1,2,\ldots,N$. The nonzero elements of $\mathcal{T}^{(N)}$ are $t_{11\ldots1}=2\sqrt{p(1-p)}$, $t_{33\ldots3}=p+(-1)^N(1-p)$. Other nonzero elements of $\mathcal{T}^{(N)}$ are those with $2k\sigma_2$'s and $(N-2k)\sigma_1$'s, $k=0,1,\ldots,\lfloor\frac{N}{2}\rfloor$, where $\lfloor x\rfloor$ is the greatest integer less than or equal to x (e.g., for $N=3$, t_{122}, etc.). These are equal to $(-1)^k2\sqrt{p(1-p)}$. This gives

$$\|\mathcal{T}^{(N)}\|^2 = 4p(1-p) + [p + (-1)^N(1-p)]^2 + 4p(1-p)\sum_{k=1}^{[N/2]}\binom{N}{2k}. \tag{9}$$

Thus we get, for $E_{\mathcal{T}}(|\psi\rangle)$,

$$E_{\mathcal{T}}(|\psi\rangle) = \|\mathcal{T}^{(N)}\| - 1 = \sqrt{4p(1-p) + [p + (-1)^N(1-p)]^2 + 4p(1-p)\sum_{k=1}^{[N/2]}\binom{N}{2k}} - 1. \tag{10}$$

Equation (8), with $N=2$, represents a general two-qubit entangled state in its Schmidt decomposition,

$$|\psi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle.$$

Thus Eq. (10) gives the entanglement in a two-qubit pure state. Using Eq. (10) it is straightforward to see that $E_{\mathcal{T}}(|\psi\rangle)$ for an arbitrary two-qubit pure state is related to the concurrence by

$$E_{\mathcal{T}}(|\psi\rangle) = \sqrt{1+2C^2} - 1,$$

where the concurrence C for such a state is $2\sqrt{p(1-p)}$.

Figure 1 plots $E_{\mathcal{T}}(|\psi\rangle)$ in Eq. (10) as a function of p for $N=3$. For the N-qubit GHZ (maximally entangled) state $p=1/2$, so that

$$R_N = E_{\mathcal{T}}(|\text{GHZ}\rangle) = \sqrt{1 + \frac{1}{4}[1+(-1)^N]^2 + \sum_{k=1}^{[N/2]}\binom{N}{2k}} - 1. \tag{11}$$

We see that, as a function of N, $E_{\mathcal{T}}(|\text{GHZ}\rangle)$ increases as a polynomial of degree $\lfloor \frac{N}{2} \rfloor$. Figure 2 plots $E_{\mathcal{T}}(|\text{GHZ}\rangle)$ as a function of N. $E_{\mathcal{T}}(|\text{GHZ}\rangle)$ increases sharply with N as expected. Note that $E_{\mathcal{T}}(|\text{GHZ}\rangle) \geq 0$ for the GHZ class of states. Whenever appropriate, we normalize the entanglement of an N-qubit state $|\psi\rangle$, $E_{\mathcal{T}}(|\psi\rangle)$, by dividing by $R_N = E_{\mathcal{T}}(|\text{GHZ}\rangle)$.

The N-qubit W state is given by

$$|W\rangle = \frac{1}{\sqrt{N}}\sum_j |00\ldots 1_j 0\ldots 00\rangle, \quad N \geq 3,$$

where the jth term has a single 1 at the jth bit. The state $|\tilde{W}\rangle = \otimes_{k=1}^N \sigma_1^{(k)}|W\rangle$ is given by $|\tilde{W}\rangle = \frac{1}{\sqrt{N}}\sum_j |11\ldots 0_j 1\ldots 11\rangle$, $N \geq 3$, and has a single 0 at the jth bit. We note that $|\tilde{W}\rangle$ is locally unitarily connected to $|W\rangle$ so that their entanglements must have the same value. The general element of $\mathcal{T}^{(N)}$ for the state $\rho = |W\rangle\langle W|$ is

$$t_{i_1 i_2 \ldots i_N} = \frac{1}{N}\sum_{j=1}^N \langle 00\ldots 1_j \ldots 00|\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_N}|00\ldots 1_j \ldots 00\rangle + \frac{1}{N}\sum_{j,l=1; j\neq l}^N \langle 00\ldots 1_j \ldots 00|\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_N}|00\ldots 1_l \ldots 00\rangle.$$

Only the first term contributes to $t_{33\ldots 33} = -1$. Other nonzero elements have the form $t_{3\ldots 31_j 3\ldots 31_l 3\ldots 3} = \frac{2}{N} = t_{3\ldots 32_j 3\ldots 32_l 3\ldots 3}$.

There are $\binom{N}{2}$ elements of each of these two types, so that

$$\|\mathcal{T}^{(N)}\|^2 = 1 + 2\left(\frac{2}{N}\right)^2\binom{N}{2} = 1 + 4\frac{N-1}{N}, \tag{12}$$

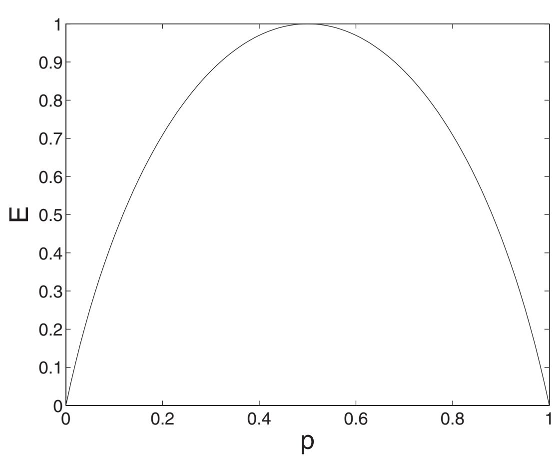

FIG. 1. Variation of $E_{\mathcal{T}}(|\psi\rangle)$ [Eq. (10)] for $N=3$, expressed in units of R_3, with parameter p.

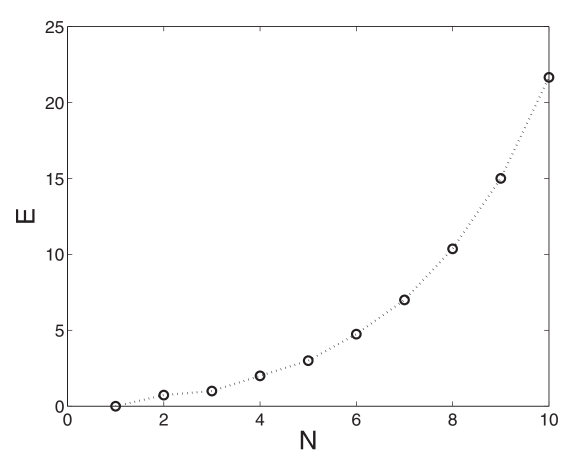

FIG. 2. Variation of $R_N = E_{\mathcal{T}}(|\text{GHZ}\rangle)$ [Eq. (11)] with number of qubits N.

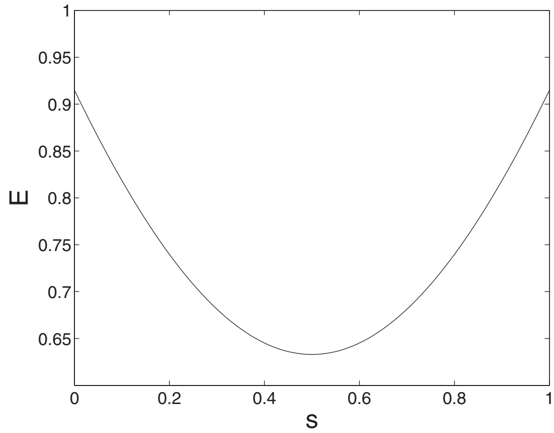

FIG. 3. Variation of $E_{\mathcal{T}}(|\psi_{s,\phi}\rangle)$, expressed in units of R_N, with the superposition parameter s, for $N=3$.

$$E_{\mathcal{T}}(|W\rangle) = \|\mathcal{T}^{(N)}\| - 1 = \sqrt{1 + 4\frac{N-1}{N}} - 1. \qquad (13)$$

It is straightforward to check that $E_{\mathcal{T}}(|W\rangle)=E_{\mathcal{T}}(|\widetilde{W}\rangle)$ as expected. Note that $E_{\mathcal{T}}(|W\rangle)\geq 0$.

Next we consider a superposition of $|W\rangle$ and $|\widetilde{W}\rangle$ states, $|\psi_{s,\phi}\rangle=\sqrt{s}|W\rangle+\sqrt{1-s}e^{i\phi}|\widetilde{W}\rangle$. It is clear that the entanglement of $|\psi_{s,\phi}\rangle$ cannot depend on the relative phase ϕ, as $|\psi_{s,\phi}\rangle$ is invariant under the local unitary transformation $\{|0\rangle,|1\rangle\}\to\{|0\rangle,e^{i\phi}|1\rangle\}$ up to an overall phase factor. As we shall prove below, $E_{\mathcal{T}}$ is invariant under local unitary transformations. Figure 3 shows the entanglement of $|\psi_{s,\phi}\rangle$ as a function of s, calculated using our measure.

An important example of a W state and its generalizations is the one-dimensional spin-$\frac{1}{2}$ Heisenberg antiferromagnet, on a lattice of size N, with periodic boundary conditions, given by the Hamiltonian

$$H_N = \sum_{j=1}^{N} X_jX_{j+1} + Y_jY_{j+1} + Z_jZ_{j+1}, \qquad (14)$$

where the subscripts are mod N and X,Y,Z denote Pauli operators $\sigma_x,\sigma_y,\sigma_z$, respectively. H_N commutes with $S_z=\Sigma Z_j$, so the eigenstate of H_N is a superposition of basis vectors $|b_1\cdots b_n\rangle$ where s of $b_1\cdots b_N$ are ones and $N-s$ are zeros for some fixed $0\leq s\leq N$. When $s=1$, the translational invariance of H_N implies that the eigenstates are

$$|\psi_N^{(k)}\rangle = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} e^{ikj}|00\ldots 1_j0\ldots 0\rangle \qquad (15)$$

where the jth summand has a single 1 at the jth bit just like the W state and the wave number $k=\frac{2\pi m}{N}$ for some integer $0\leq m\leq N-1$. The state $|\psi_N^{(k)}\rangle$ is locally unitarily transformed to the W state so that it has the same value of $E_{\mathcal{T}}(|W\rangle)$ or $E_{\mathcal{T}}(\widetilde{W}\rangle)$.

For $s\geq 2$ the eigenstates of H_N have the form

$$|\psi_N(s)\rangle = \frac{1}{\sqrt{\binom{N}{s}}}\sum_{\{j_1\ldots j_s\}}|00\ldots 1_j0\ldots 1_{j_s}0\rangle, \qquad (16)$$

where 1 occurs at $j_1\ldots j_s$, $\{j_1\ldots j_s\}\subseteq\mathcal{N}=\{1,2,\ldots,N\}$, and can be chosen in $\binom{N}{s}$ ways. We see that for $|\psi_N(s)\rangle$, $t_{33\ldots 3}=1$. For even N, $t_{1\ldots 12\ldots 23\ldots 3}$ with x 1's and y 2's, corresponding to the average of $x\sigma_x$'s, y σ_y's, and $N-x-y$ σ_z's, we get, for even x and even y,

$$t_{1\ldots 12\ldots 23\ldots 3} = \left[2\binom{x}{\frac{x}{2}}\binom{y}{\frac{y}{2}} - \binom{x+y}{\frac{(x+y)}{2}}\right]\binom{N-x-y}{s-\frac{(x+y)}{2}}.$$

Since $|\psi_N(s)\rangle$ is a symmetric state, any permutation of its indices does not change the value of an element of $\mathcal{T}^{(N)}$ [24], so that

$$\|\mathcal{T}^{(N)}_{|\psi_N(s)\rangle}\|^2 = 1 + \frac{1}{\binom{N}{s}^2}\left[\sum_{\substack{x+y=2\\ x,y\ \text{even}}}^{2s}\left[2\binom{x}{\frac{x}{2}}\binom{y}{\frac{y}{2}} - \binom{x+y}{\frac{(x+y)}{2}}\right]^2\binom{N-x-y}{s-\frac{(x+y)}{2}}^2\binom{N}{x}\binom{N-x}{y}\right].$$

Figure 4 shows the variation of $E_{\mathcal{T}}(|\psi_N(s)\rangle)=\|\mathcal{T}^{(N)}_{|\psi_N(s)\rangle}\|-1$ with s. We see that it is maximum at $s=\frac{N}{2}$, which is a characteristic of the ground state of H_N, as expected. Note that $E_{\mathcal{T}}(|\psi_N(\frac{N}{2})\rangle)$ for the state $(s=\frac{N}{2})$ rises far more rapidly than the entanglement of the N-qubit GHZ state $R_N=E_{\mathcal{T}}(|\text{GHZ}\rangle)$ [Eq. (11)] with the number of spins (qubits) N. This can be understood by noting that $|\psi_N(\frac{N}{2})\rangle$ for $s=\frac{N}{2}$ can be written as a superposition of $\frac{1}{2}\binom{N}{N/2}$ N-qubit GHZ states. For example, $|\psi_4(2)\rangle$ can be written as the superposition of three four-qubit GHZ states,

$$|\psi_4(2)\rangle = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle) + \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle) + \frac{1}{\sqrt{2}}(|1001\rangle + |0110\rangle)\right).$$

As N increases, initially $E_{\mathcal{T}}(|\psi_N(\frac{N}{2})\rangle)$ is comparable to R_N, but after $N=16$ the ratio $\frac{E_{\mathcal{T}}(|\psi_N(\frac{N}{2})\rangle)}{R_N}$ increases very rapidly, reaching 10^7 for 100 qubits. Also, as N increases, $E_{\mathcal{T}}(|\psi_N(s)\rangle)$ falls off more rapidly as s deviates from $\frac{N}{2}$. We are presently trying to understand this behavior.

Finally, in this section, we consider the superpositions of W and GHZ states,

$$|\psi_{W+\text{GHZ}}(s,\phi)\rangle = \sqrt{s}|\text{GHZ}\rangle + \sqrt{1-s}e^{i\phi}|W\rangle, \qquad (17)$$

also considered in [28]. For three qubits, $N=3$, a direct calculation gives, for this state,

$$\|\mathcal{T}^{(N)}\|^2 = 4s^2 + 6s(1-s) + \frac{11}{3}(s-1)^2 \quad (0\leq s\leq 1),$$

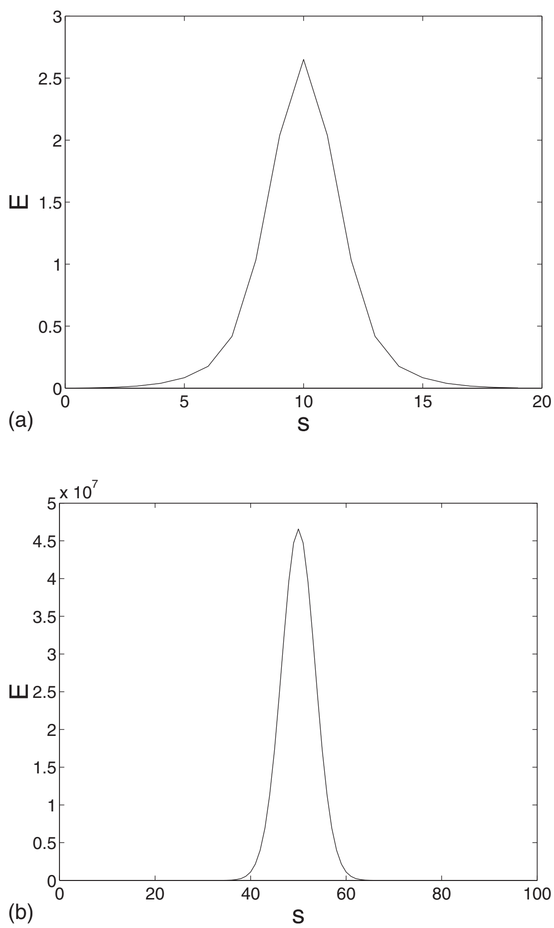

FIG. 4. Variation of $E_{\mathcal{T}}(|\psi_N(s)\rangle)$, (in units of R_N), with s, for N = (a) 20 and (b) 100 (see text).

$$E_{\mathcal{T}}(|\psi_{W+\mathrm{GHZ}}(s,\phi)\rangle) = \|\mathcal{T}^N_{|\psi_N(s)\rangle}\| - 1, \tag{18}$$

which coincides with the corresponding values of the $W(s=0)$ and GHZ$(s=1)$ states. Note that $E_{\mathcal{T}}(|\psi_{W+\mathrm{GHZ}}(s,\phi)\rangle)$ is independent of the phase ϕ, in contrast to the entanglement measure used in [28]. Figure 5 shows the dependence of $E_{\mathcal{T}}(|\psi_{W+\mathrm{GHZ}}(s,\phi)\rangle)$ on s.

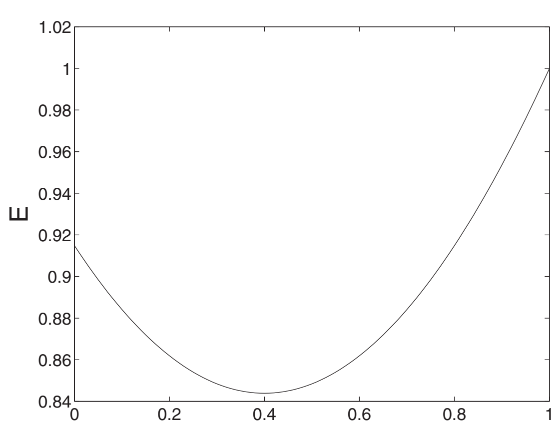

FIG. 5. Variation of $E_{\mathcal{T}}(|\psi_{W+\mathrm{GHZ}}(s,\phi)\rangle)$, expressed in units of R_N, with the superposition parameter s, for $N=3$.

IV. PROPERTIES OF $E_{\mathcal{T}}(|\psi\rangle)$

To be a valid entanglement measure, $E_{\mathcal{T}}(|\psi\rangle)$ must have the following properties [29,30].

(a) (i) *Positivity:* $E_{\mathcal{T}}(|\psi\rangle) \geq 0$ for all N-qubit pure states $|\psi\rangle$. (ii) *Discriminance:* $E_{\mathcal{T}}(|\psi\rangle)=0$ if and only if $|\psi\rangle$ is a separable (product) state.

(b) *LU invariance*: $E_{\mathcal{T}}(|\psi\rangle)$ must be invariant under local unitary (LU) operations.

(c) *Monotonicity*: local operators and classical communication do not increase the expectation value of $E_{\mathcal{T}}(|\psi\rangle)$.

We prove the above properties for $E_{\mathcal{T}}(|\psi\rangle)$. We also prove the following additional properties for $E_{\mathcal{T}}(|\psi\rangle)$:

(d) *Continuity:* $\|(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)\| \to 0 \Rightarrow |E(|\psi\rangle) - E(|\phi\rangle)| \to 0$.

(e) S*uperadditivity:* $E_{\mathcal{T}}(|\psi\rangle \otimes |\phi\rangle) \geq E_{\mathcal{T}}(|\psi\rangle) + E_{\mathcal{T}}(|\phi\rangle)$.

We need the following result, which we have proved in [24].

Proposition 0. A pure N-partite quantum state is fully separable (product state) if and only if

$$\mathcal{T}^{(N)} = \mathbf{s}^{(1)} \circ \mathbf{s}^{(2)} \circ \cdots \circ \mathbf{s}^{(N)}, \tag{19}$$

where $\mathbf{s}^{(k)}$ is the Bloch vector of the kth subsystem reduced density matrix. The symbol $\circ$ stands for the outer product of vectors defined as follows.

Let $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(M)}$ be vectors in $\mathbb{R}^{d_1^2-1}, \mathbb{R}^{d_2^2-1}, \ldots, \mathbb{R}^{d_M^2-1}$. The outer product $\mathbf{u}^{(1)} \circ \mathbf{u}^{(2)} \circ \cdots \circ \mathbf{u}^{(M)}$ is a tensor of order M (M-way array), defined by

$$t_{i_1 i_2 \cdots i_M} = \mathbf{u}^{(1)}_{i_1} \mathbf{u}^{(2)}_{i_2} \cdots \mathbf{u}^{(M)}_{i_M}, \quad 1 \leq i_k \leq d_k^2 - 1,$$

$$k = 1, 2, \ldots, M.$$

Proposition 1. Let $|\psi\rangle$ be an N-qubit pure state. Then, $\|\mathcal{T}^{(N)}_{\psi}\|=1$ if and only if $|\psi\rangle$ is a separable (product) state.

Proof. By Proposition 0, $|\psi\rangle$ is separable (product) if and only if

$$\mathcal{T}^{(N)} = \mathbf{s}^{(1)} \circ \mathbf{s}^{(2)} \circ \cdots \circ \mathbf{s}^{(N)} = \bigcirc_{k=1}^{N} s^{(k)}.$$

As shown in [31,32],

$$(\bigcirc_{k=1}^{N} s^{(k)}, \bigcirc_{k=1}^{N} s^{(k)}) = \Pi_{k=1}^{N} (s^{(k)}, s^{(k)}), \tag{20}$$

where $(.,.)$ denotes the scalar product. This immediately gives, for qubits,

$$\|\mathcal{T}^{(N)}\|^2 = (\mathcal{T}^{(N)}, \mathcal{T}^{(N)}) = \Pi_{k=1}^{N} (s^{(k)}, s^{(k)}) = \Pi_k \|s^{(k)}\|^2 = 1.$$

■

Proposition 1 immediately gives the following proposition.

Proposition 2. Let $|\psi\rangle$ be an N-qubit pure state. Then $E_{\mathcal{T}}(|\psi\rangle)=0$ if and only if $|\psi\rangle$ is a product state.

Proposition 3. Let $|\psi\rangle$ be an N-qubit pure state. Then $\|\mathcal{T}^{(N)}\| \geq 1$.

It is instructive to show this result by direct computation of $\mathcal{T}^{(N)}$ for the cases of two- and three-qubit states. First, consider a general two-qubit state

$$|\psi\rangle = a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle, \quad \sum_k |a_k|^2 = 1.$$

By direct computation we get

$$\|\mathcal{T}^{(2)}\|^2 = 1 + 8(|a_2a_3| - |a_1a_4|)^2 \geq 1. \tag{21}$$

This means, via Proposition 2, that $|\psi\rangle$ is a product state if $|a_2a_3|=|a_1a_4|$. Next, consider a three-qubit state in the general Schmidt form [33]

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \tag{22}$$

where $\lambda_i \geq 0$, $i=0,1,2,3,4$, and $\Sigma_i \lambda_i^2 = 1$.

By direct calculation of $\|\mathcal{T}^{(3)}\|$ we get

$$\|\mathcal{T}^{(3)}\|^2 \geq 1 + 12\lambda_0^2\lambda_4^2 + 8\lambda_0^2\lambda_2^2 + 8\lambda_0^2\lambda_3^2 + 8(\lambda_1\lambda_4 - \lambda_2\lambda_3)^2 \geq 1. \tag{23}$$

Here the conditions for the product state become $\lambda_1\lambda_4 = \lambda_2\lambda_3$ and $\lambda_0=0$. We now prove Proposition 3 for a general N-qubit state $|\psi\rangle$.

If $|\psi\rangle$ is not a product of N single-qubit states (i.e., $|\psi\rangle$ is not N-separable) then it is $(N-k)$-separable, $k=2,3,\ldots,N$. Viewing the N-qubit system as a system comprising $N-k$ qubits, each with Hilbert space of dimension 2, and k entangled qubits with the Hilbert space of dimension 2^k, we can apply Proposition 0 to this separable system of $N-k+1$ parts in the state $|\psi\rangle$. We get $\mathcal{T}^{(N)}_{|\psi\rangle} = (s^{(1)}) \circ (s^{(2)}) \circ \cdots \circ (s^{(N-k)}) \circ (s^{(N-k+1)})$.

This implies, as in Proposition 1, via Eqs. (20) and (6) that

$$\|\mathcal{T}^{(N)}_{|\psi\rangle}\| = \Pi_{i=1}^{N-k+1}\|(s^{(i)})\|^2 = \frac{d(d-1)}{2} > 1 \quad (d=2^k). \tag{24}$$

If $k=N$ we attach an ancilla qubit in an arbitrary state $|\phi\rangle$ and apply Proposition 0 to an $(N+1)$-qubit system in the state $|\psi\rangle \otimes |\phi\rangle$ where $|\psi\rangle$ is the N-qubit entangled state. This result, combined with Proposition 1, completes the proof. ■

Proposition 3 immediately gives the following proposition.

Proposition 4. $E_{\mathcal{T}}(|\psi\rangle) \geq 0$.

We now prove that $E_{\mathcal{T}}(|\psi\rangle)$ is nonincreasing under local operations and classical communication. Any such local action can be decomposed into four basic kinds of operations [34]: (i) appending an ancillary system not entangled with the state of the original system, (ii) performing a unitary transformation, (iii) performing measurements, and (iv) throwing away, i.e., tracing out, part of the system. It is clear that appending ancilla cannot change $\|\mathcal{T}^{(N)}\|$. We prove that $E_{\mathcal{T}}(|\psi\rangle)$ does not increase under the remaining three local operations.

Proposition 5. Let U_i $(i=1,2,\ldots,N)$ be local unitary operators acting on the Hilbert spaces of subsystems $\mathcal{H}^{(i)}$ $(i=1,2,\ldots,N)$, respectively. Let

$$\rho = (\otimes_{i=1}^N U_i)\rho'(\otimes_{i=1}^N U_i^\dagger) \tag{25}$$

for density operators ρ and ρ' acting on $\mathcal{H} = \otimes_{i=1}^N \mathcal{H}^{(i)}$ and let $\mathcal{T}^{(N)}$ and $\mathcal{T}'^{(N)}$ denote the N-partite correlation tensors for ρ and ρ', respectively. Then

$$\|\mathcal{T}'^{(N)}\| = \|\mathcal{T}^{(N)}\|, \quad \text{so that} \quad E_{\mathcal{T}}(\rho) = E_{\mathcal{T}}(\rho').$$

Proof. Let U denote a one-qubit unitary operator; then it is straightforward to show that $U\sigma_\alpha U^\dagger = \Sigma_\beta O_{\alpha\beta}\sigma_\beta$, where $[O_{\alpha\beta}]$ is a real matrix satisfying $OO^T = I = O^TO$. It is an element of the rotation group O(3). Now consider

$$\begin{aligned} t'_{i_1i_2\cdots i_N} &= \mathrm{Tr}(\rho'\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_N}) \\ &= \mathrm{Tr}[\rho(\otimes_{i=1}^N U_i)\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_N}(\otimes_{i=1}^N U_i^\dagger)] \\ &= \mathrm{Tr}(\rho U_1\sigma_{i_1}U_1^\dagger \otimes U_2\sigma_{i_2}U_2^\dagger \otimes \cdots \otimes U_N\sigma_{i_N}U_N^\dagger) \\ &= \sum_{\alpha_1\cdots\alpha_N} \mathrm{Tr}(\rho\sigma_{\alpha_1} \otimes \sigma_{\alpha_2} \otimes \cdots \otimes \sigma_{\alpha_N}) \\ &\quad \times O^{(1)}_{i_1\alpha_1}O^{(2)}_{i_2\alpha_2}\cdots O^{(N)}_{i_N\alpha_N} \\ &= \sum_{\alpha_1\cdots\alpha_N} t_{\alpha_1\cdots\alpha_N}O^{(1)}_{i_1\alpha_1}O^{(2)}_{i_2\alpha_2}\cdots O^{(N)}_{i_N\alpha_N} \\ &= (\mathcal{T}^{(N)}\times_1 O^{(1)}\times_2 O^{(2)} \times \cdots \times_N O^{(N)})_{i_1i_2\cdots i_N}, \end{aligned}$$

where $\times_k$ is the k-mode product of a tensor $\mathcal{T}^{(N)} \in \mathbb{R}^{3\times3\times\cdots3}$ by the orthogonal matrix $O^{(k)} \in \mathbb{R}^{3\times3}$ [31,32,35]. Therefore,

$$\mathcal{T}'^{(N)} = \mathcal{T}^{(N)}\times_1 O^{(1)}\times_2 O^{(2)} \times \cdots \times_N O^{(N)}.$$

By Proposition 3.12 in [31] we get

$$\|\mathcal{T}'^{(N)}\| = \|\mathcal{T}^{(N)}\times_1 O^{(1)}\times_2 O^{(2)} \times \cdots \times_N O^{(N)}\| = \|\mathcal{T}^{(N)}\|.$$

■

Proposition 6. If a multipartite pure state $|\psi\rangle$ is subjected to a local measurement on the kth qubit giving outcomes i_k with probabilities p_{i_k} and leaving the residual N-qubit pure state $|\phi_{i_k}\rangle$, then the expected entanglement $\Sigma_{i_k} p_{i_k} E_{\mathcal{T}}(|\phi_{i_k}\rangle)$ of the residual state is not greater then $E_{\mathcal{T}}(|\psi\rangle)$,

$$\sum_{i_k} p_{i_k} E_{\mathcal{T}}(|\phi_{i_k}\rangle) \leq E_{\mathcal{T}}(|\psi\rangle). \tag{26}$$

Proof. Local measurements can be expressed as the tensor product matrix $\bar{D} = \bar{D}^{(1)} \otimes \bar{D}^{(2)} \otimes \cdots \otimes \bar{D}^{(N)}$ on the expanded coherence vector $\mathcal{T}$ [36]. The expanded coherence vector $\mathcal{T}$ is the extended correlation tensor $\mathcal{T}$ (defined below) viewed as a vector in the real space of appropriate dimension. The extended correlation tensor $\mathcal{T}$ is defined by the equation

$$\rho = \frac{1}{2^N}\sum_{i_1i_2\cdots i_N=0}^{3} \mathcal{T}_{i_1i_2\cdots i_N}\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_N}, \tag{27}$$

where $\sigma_{i_k} \in \{I, \sigma_x, \sigma_y, \sigma_z\}$ is the i_kth local Pauli operator on the kth qubit $(\sigma_0 = I)$ and the real coefficients $\mathcal{T}_{i_1i_2\cdots i_N}$ are the components of the extended correlation tensor $\mathcal{T}$. Equation (2) and Eq. (27) are equivalent with $\mathcal{T}_{000\ldots0}=1$, $\mathcal{T}_{i_100\ldots0} = s^{(1)}_{i_1}, \ldots,$ $\mathcal{T}_{i_1i_2\ldots i_M00\ldots0} = \mathcal{T}^{\{1,2,\cdots M\}}_{i_1i_2\cdots i_M}, \ldots,$ and $\mathcal{T}_{i_1i_2\ldots i_N} = \mathcal{T}^{(N)}_{i_1i_2\ldots i_N}$, $i_1, i_2, \ldots, i_N \neq 0$. $\bar{D}^{(k)}$, $k=1,2,\ldots,n$, are 4×4 matrices. Without losing generality, we can assume the local measurements to be positive operator valued measures (POVMs), in which case $\bar{D}^{(k)} = \mathrm{diag}(1, D^{(k)})$ and the 3×3 matrix $D^{(k)}$ is contractive, $D^{(k)T}D^{(k)} \leq I$ [36]. The local

POVMs acting on an N-qubit state ρ correspond to the map $\rho \mapsto \mathcal{M}(\rho)$ given by

$$\mathcal{M}(\rho) = \sum_{i_1 i_2 \cdots i_N} L_{i_1}^{(1)} \otimes L_{i_2}^{(2)} \otimes \cdots \otimes L_{i_N}^{(N)} \rho L_{i_1}^{(1)\dagger} \otimes L_{i_2}^{(2)\dagger} \otimes \cdots \otimes L_{i_N}^{(N)\dagger},$$

where $L_{i_k}^{(k)}$ are the linear, positive, trace-preserving operators satisfying $\Sigma_{i_k} L_{i_k}^{(k)\dagger} L_{i_k}^{(k)} = I$ and $[L_{i_k}^{(k)}, L_{i_k}^{(k)\dagger}] = 0$. The resulting correlation tensor of $\mathcal{M}(\rho)$ can be written as

$$\mathcal{T}'^{(N)} = \mathcal{T}^{(N)} \times_1 D^{(1)} \times_2 D^{(2)} \times \cdots \times_N D^{(N)},$$

where $D^{(k)}$ is a 3×3 matrix and $D^{(k)T} D^{(k)} \leq I$.

The action of a POVM on the kth qubit corresponds to the map $\mathcal{M}_k(|\psi\rangle\langle\psi|) = \Sigma_{i_k} M_{i_k} \rho M_{i_k}^\dagger$, where $M_{i_k} = I \otimes \cdots L_{i_k}^{(k)} \otimes \cdots \otimes I$, $\Sigma_{i_k} L_{i_k}^{(k)\dagger} L_{i_k}^{(k)} = I$, and $[L_{i_k}^{(k)}, L_{i_k}^{(k)\dagger}] = 0$, with the resulting mixed state $\Sigma_{i_k} p_{i_k} |\phi_{i_k}\rangle\langle\phi_{i_k}|$, where $|\phi_{i_k}\rangle$ is the N-qubit pure state which results after the outcome i_k with probability p_{i_k}. The average entanglement of this state is

$$\sum_{i_k} p_{i_k} E_{\mathcal{T}}(|\phi_{i_k}\rangle\langle\phi_{i_k}|) = \sum_{i_k} p_{i_k} \|\mathcal{T}'^{(N)}_{|\phi_{i_k}\rangle}\| - 1 = \sum_{i_k} p_{i_k} \|\mathcal{T}^{(N)}_{|\psi\rangle} \times_k D^{(k)}\| - 1 = \sum_{i_k} p_{i_k} \|D^{(k)} T_{(k)}(|\psi\rangle)\| - 1,$$

where, by Proposition 3.7 in [31], $D^{(k)} T_{(k)}(|\psi\rangle)$ is the kth matrix unfolding [24] of $\mathcal{T}^{(N)}_{|\psi\rangle} \times_k D^{(k)}$. Therefore, from the definition of the Euclidean norm of a matrix, $\|A\| = \sqrt{\mathrm{Tr}(AA^\dagger)}$, [37] we get

$$\begin{aligned}\sum_{i_k} p_{i_k} E_{\mathcal{T}}(|\phi_{i_k}\rangle\langle\phi_{i_k}|) &= \sum_{i_k} p_{i_k} \{\mathrm{Tr}[D^{(k)} T_{(k)} \times (|\psi\rangle) T_{(k)}^\dagger(|\psi\rangle) D^{(k)T}]\}^{1/2} - 1 \\ &= \sum_{i_k} p_{i_k} \{\mathrm{Tr}[D^{(k)T} D^{(k)} T_{(k)} \times (|\psi\rangle) T_{(k)}^\dagger(|\psi\rangle)]\}^{1/2} - 1 \\ &\leq \sum_{i_k} p_{i_k} \sqrt{\mathrm{Tr}[T_{(k)}(|\psi\rangle) T_{(k)}^\dagger(|\psi\rangle)]} - 1 \\ &= \|\mathcal{T}^{(N)}_{|\psi\rangle}\| - 1 = E_{\mathcal{T}}(|\psi\rangle),\end{aligned}$$

because $D^{(k)T} D^{(k)} \leq I$, and $\Sigma_{i_k} p_{i_k} = 1$. We have also used the fact that the Euclidean norm of a tensor equals that of any of its matrix unfoldings. ■

As an example, we consider the four-qubit state [38]

$$|\psi\rangle_{ABCD} = \frac{1}{\sqrt{6}}(|0000\rangle + |0011\rangle + |0101\rangle + |0110\rangle + |1010\rangle + |1111\rangle). \quad (28)$$

A POVM $\{A_1, A_2\}$ is performed on the subsystem A, which has the form $A_1 = U_1 \mathrm{diag}\{\alpha, \beta\} V$ and $A_2 = U_2 \mathrm{diag}\{\sqrt{1-\alpha^2}, \sqrt{1-\beta^2}\} V$. Due to LU invariance of $\|\mathcal{T}^{(N)}\|$ we need only consider the diagonal matrices in which the parameters are chosen to be $\alpha = 0.9$ and $\beta = 0.2$. After the POVM, two outcomes $|\phi_1\rangle = A_1|\psi\rangle/\sqrt{p_1}$ and $|\phi_2\rangle = A_2|\psi\rangle/\sqrt{p_2}$ are obtained, with the probabilities as $p_1 = 0.5533$ and $p_2 = 0.4467$. We find

$$E_{\mathcal{T}}(|\psi\rangle) = 0.7802, \quad E_{\mathcal{T}}(|\phi_1\rangle) = 0.0725/p_1,$$

$$E_{\mathcal{T}}(|\phi_2\rangle) = 0.0436/p_2.$$

This gives

$$E_{\mathcal{T}}(|\psi\rangle) - [p_1 E_{\mathcal{T}}(|\phi_1\rangle) + p_2 E_{\mathcal{T}}(|\phi_2\rangle)] = 0.6641 > 0.$$

This is to be contrasted with the similar calculation in [38], with the same state $|\psi\rangle$ in Eq. (28) and the same POVM given above.

Proposition 7. Let $|\psi\rangle$ be an N-qubit pure state. Let ρ denote the reduced density matrix after tracing out one qubit from the state $|\psi\rangle$. Then

$$\|\mathcal{T}_\rho^{(N-1)}\| \leq \|\mathcal{T}^{(N)}_{|\psi\rangle}\|$$

with equality only when $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$, where $|\chi\rangle$ is the state of the qubit which is traced out.

Proof. We prove this for a special case whose generalization is straightforward. Let $|\psi\rangle = a|b_1 \ldots b_N\rangle + b|b_1' \ldots b_N'\rangle$, $|a|^2 + |b|^2 = 1$. Here $|b_i\rangle$ and $|b_i'\rangle$ are the eigenstates of $\sigma_z^{(i)}$ operating on the ith qubit. Now consider the set S of N-fold tensor products of qubit operators $\{\sigma_\alpha\}$, $\alpha = 1,2,3$, namely, $S = \{\sigma_{\alpha_1} \otimes \sigma_{\alpha_2} \otimes \cdots \otimes \sigma_{\alpha_N}\}$, $\alpha_1, \ldots, \alpha_N = 1,2,3$. Choosing $\alpha_1, \ldots, \alpha_N = 3$, we get $\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 |b_1 \ldots b_N\rangle = \pm |b_1 \ldots b_N\rangle$. We can choose an operator from S, denoted B, such that $B|b_1 \ldots b_N\rangle = \pm |b_1' \ldots b_N'\rangle$. If B contains $q \leq N$ σ_x operators we can replace $k \leq q$ of them by σ_y operators. We denote the resulting tensor product operator by $B_k (B_0 = B)$. We have $B_k|b_1 \ldots b_N\rangle = \pm (i)^k |b_1' \ldots b_N'\rangle$. Then,

$$\langle b_1 \ldots b_N|\sigma_3 \otimes \cdots \otimes \sigma_3|b_1 \ldots b_N\rangle = \pm 1 = \langle b_1' \ldots b_N'|\sigma_3 \otimes \cdots \otimes \sigma_3|b_1' \ldots b_N'\rangle,$$

$$\langle b_1 \ldots b_N|B|b_1' \ldots b_N'\rangle = \pm 1 = \langle b_1' \ldots b_N'|B|b_1 \ldots b_N\rangle,$$

$$\langle b_1' \ldots b_N'|B_k|b_1 \ldots b_N\rangle = \pm (i)^k,$$

$$\langle b_1 \ldots b_N|B_k|b_1' \ldots b_N'\rangle = \pm (-i)^k.$$

Now,

$$\begin{aligned} t_{\alpha_1 \ldots \alpha_N} &= \langle\psi|\sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N}|\psi\rangle \\ &= |a|^2 \langle b_1 \ldots b_N|\sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N}|b_1 \ldots b_N\rangle \\ &\quad + |b|^2 \langle b_1' \ldots b_N'|\sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N}|b_1' \ldots b_N'\rangle \\ &\quad + a^* b \langle b_1 \ldots b_N|\sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N}|b_1' \ldots b_N'\rangle \\ &\quad + a b^* \langle b_1' \ldots b_N'|\sigma_{\alpha_1} \otimes \cdots \otimes \sigma_{\alpha_N}|b_1 \ldots b_N\rangle. \end{aligned}$$

The nonzero elements of $t_{\alpha_1 \ldots \alpha_N}$ are $t_{33\ldots 3} = \pm |a|^2 \pm |b|^2$, $t_B = \pm ab^* \pm a^* b = \pm 2|a||b|\cos(\phi_a - \phi_b)$,

$$t_{B_k} = \pm (i)^k ab^* \pm (-i)^k a^* b = \begin{cases} \pm 2|a||b|\cos(\phi_a - \phi_b) & \text{if } k \text{ is even,} \\ \pm 2|a||b|\sin(\phi_a - \phi_b) & \text{if } k \text{ is odd.} \end{cases}$$

We get $\Sigma_{k=0}^{q}\binom{q}{2k}$elements with $\cos(\phi_a-\phi_b)$ and $\Sigma_{k=0}^{q}\binom{q}{2k+1}$ elements with $\sin(\phi_a-\phi_b)$. If q is odd (for the given state $|\psi\rangle$) the number of cosines and the number of sines are equal. When q is even the number of cosines exceeds by 1. Finally we get

$$\|\mathcal{T}^{(N)}_{|\psi\rangle}\|^2 = (\pm|a|^2 \pm |b|^2)^2 + 4|a|^2|b|^2\cos^2(\phi_a - \phi_b)\sum_{k=0}^{q}\binom{q}{2k} + 4|a|^2|b|^2\sin^2(\phi_a - \phi_b)\sum_{k=0}^{q}\binom{q}{2k+1}.$$

Note that, using $|a|^2+|b|^2=1$, it is easy to see that $\|\mathcal{T}^{(N)}_{|\psi\rangle}\| \geq 1$, showing that $E_{\mathcal{T}} \geq 0$. Next we consider

$$|\psi\rangle\langle\psi| = |a|^2|b_1\dots b_N\rangle\langle b_1\dots b_N| + |b|^2|b'_1\dots b'_N\rangle\langle b'_1\dots b'_N| + ab^*|b_1\dots b_N\rangle\langle b'_1\dots b'_N| + a^*b|b'_1\dots b'_N\rangle\langle b_1\dots b_N|$$

and trace out the Nth qubit to get the $(N-1)$-qubit reduced density matrix

$$\rho = |a|^2|b_1\dots b_{N-1}\rangle\langle b_1\dots b_{N-1}| + |b|^2|b'_1\dots b'_{N-1}\rangle\langle b'_1\dots b'_{N-1}| + ab^*|b_1\dots b_{N-1}\rangle\langle b'_1\dots b'_{N-1}|\langle b_N|b'_N\rangle + a^*b|b'_1\dots b'_{N-1}\rangle\langle b_1\dots b_{N-1}|\langle b'_N|b_N\rangle.$$

Now

$$\begin{aligned} t_{\alpha_1\dots\alpha_{N-1}} &= \mathrm{Tr}(\rho\sigma_{\alpha_1}\otimes\sigma_{\alpha_2}\otimes\cdots\sigma_{\alpha_{N-1}}) \\ &= |a|^2\langle b_1\dots b_{N-1}|\sigma_{\alpha_1}\otimes\cdots\otimes\sigma_{\alpha_{N-1}}|b_1\dots b_{N-1}\rangle \\ &\quad + |b|^2\langle b'_1\dots b'_{N-1}|\sigma_{\alpha_1}\otimes\cdots\otimes\sigma_{\alpha_{N-1}}|b'_1\dots b'_{N-1}\rangle \\ &\quad + a^*b\langle b_1\dots b_{N-1}|\sigma_{\alpha_1}\otimes\cdots\otimes\sigma_{\alpha_{N-1}}|b'_1\dots b'_{N-1}\rangle \\ &\quad \times\langle b_N|b'_N\rangle + ab^*\langle b'_1\dots b'_{N-1}|\sigma_{\alpha_1}\otimes\cdots \\ &\quad \otimes\sigma_{\alpha_{N-1}}|b_1\dots b_{N-1}\rangle\langle b'_N|b_N\rangle. \end{aligned}$$

We have for $N-1$ tensor product operators $\sigma_3\otimes\sigma_3\otimes\cdots\otimes\sigma_3|b_1\dots b_{N-1}\rangle = \pm|b_1\dots b_{N-1}\rangle$. We construct the operators D and D_k corresponding to B and B_k acting on $N-1$ qubits. We then get $D|b_1\dots b_{N-1}\rangle = \pm|b'_1\dots b'_{N-1}\rangle$ and $D_k|b_1\dots b_{N-1}\rangle = \pm(i)^k|b'_1\dots b'_{N-1}\rangle$ Now, the nonzero elements of $\mathcal{T}^{(N-1)}_{\rho}$ are $t_{33\dots3} = \pm|a|^2 \pm |b|^2$, $t_D = \pm ab^*\langle b_N|b'_N\rangle \pm a^*b\langle b'_N|b_N\rangle = 2|a||b||\langle b'_N|b_N\rangle|\cos(\phi_a-\phi_b-\alpha)$,

$$t_{D_k} = \pm (i)^k ab^*\langle b_N|b'_N\rangle \pm (-i)^k a^*b\langle b'_N|b_N\rangle = \begin{cases} \pm 2|a||b||\langle b'_N|b_N\rangle|\cos(\phi_a - \phi_b - \alpha) & \text{if } k \text{ is even,} \\ \pm 2|a||b||\langle b'_N|b_N\rangle|\sin(\phi_a - \phi_b - \alpha) & \text{if } k \text{ is odd.} \end{cases}$$

Finally we get

$$\|\mathcal{T}^{(N-1)}_{\rho}\|^2 = (\pm|a|^2 \pm |b|^2)^2 + 4|a|^2|b|^2|\langle b'_N|b_N\rangle|^2 \times\cos^2(\phi_a - \phi_b - \alpha)\sum_{k=0}^{q'}\binom{q'}{2k} + 4|a|^2|b|^2|\langle b'_N|b_N\rangle|^2 \times\sin^2(\phi_a - \phi_b - \alpha)\sum_{k=0}^{q'}\binom{q'}{2k+1},$$

where $q' \leq q$ is the number of σ_1 operators in D. Since $|\langle b'_N|b_N\rangle|^2 \leq 1$ we see that

$$\|\mathcal{T}^{(N-1)}_{\rho}\|^2 \leq \|\mathcal{T}^{(N)}_{|\psi\rangle}\|^2,$$

equality occurring when $|b_N\rangle = |b'_N\rangle$, in which case $|\psi\rangle = |\phi\rangle\otimes|b_N\rangle$. It is straightforward, but tedious, to elevate this proof for the general case

$$|\psi\rangle = \sum_{\alpha_1\dots\alpha_N} a_{\alpha_1\dots\alpha_N}|b_{\alpha_1}\dots b_{\alpha_N}\rangle, \quad \alpha_i = 0,1.$$

Basically we have to keep track of $\binom{r}{2}$ B type of operators, where r is the number of terms in the expansion of $|\psi\rangle$, in order to obtain all nonzero elements of $\mathcal{T}^{(N)}_{|\psi\rangle}$. When the Nth particle is traced out, the corresponding elements of $\mathcal{T}^{(N-1)}_{\rho}$ get multiplied by the overlap amplitudes, which leads to the required result. ■

Continuity of $E_{\mathcal{T}}$. We show that for N-qubit pure states $\|(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)\| \to 0 \Rightarrow |E_{\mathcal{T}}(|\psi\rangle) - E_{\mathcal{T}}(|\phi\rangle)| \to 0$.

Proof. $\|(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)\| \to 0 \Rightarrow \|\mathcal{T}^{(N)}_{|\psi\rangle} - \mathcal{T}^{(N)}_{|\phi\rangle}\| \to 0$. But $\|\mathcal{T}^{(N)}_{|\psi\rangle} - \mathcal{T}^{(N)}_{|\phi\rangle}\| \geq |\|\mathcal{T}^{(N)}_{|\psi\rangle}\| - \|\mathcal{T}^{(N)}_{|\phi\rangle}\||$. Therefore $\|\mathcal{T}^{(N)}_{|\psi\rangle} - \mathcal{T}^{(N)}_{|\phi\rangle}\| \to 0 \Rightarrow |\|\mathcal{T}^{(N)}_{|\psi\rangle}\| - \|\mathcal{T}^{(N)}_{|\phi\rangle}\|| \to 0 \Rightarrow |E_{\mathcal{T}}(|\psi\rangle) - E_{\mathcal{T}}(|\phi\rangle)| \to 0$. ■

A. Entanglement of multiple copies of a given state

LU invariance. We show that $E_{\mathcal{T}}$ for multiple copies of the N-qubit pure state $|\psi\rangle$ is LU invariant. Consider a system of $N\times k$ qubits in the state $|\chi\rangle = |\psi\rangle\otimes|\psi\rangle\otimes\cdots\otimes|\psi\rangle$ (k copies). It is straightforward to check that [24]

$$\mathcal{T}^{(N)}_{|\chi\rangle} = \mathcal{T}^{(N)}_{|\psi\rangle}\circ\mathcal{T}^{(N)}_{|\psi\rangle}\circ\cdots\circ\mathcal{T}^{(N)}_{|\psi\rangle}. \tag{29}$$

This implies, in a straightforward way, that

$$\|\mathcal{T}^{(N)}_{|\chi\rangle}\| = \|\mathcal{T}^{(N)}_{|\psi\rangle}\|^k.$$

Since by Proposition 5 $\|\mathcal{T}^{(N)}_{|\psi\rangle}\|$ is LU invariant, so is $\|\mathcal{T}^{(N)}_{|\chi\rangle}\|$.

Let $|\psi\rangle$ be an N-qubit pure state and $|\chi\rangle = |\psi\rangle\otimes|\psi\rangle$. Then $E_{\mathcal{T}}(|\chi\rangle)$ is expected to satisfy

$$E_{\mathcal{T}}(|\chi\rangle) \geq E_{\mathcal{T}}(|\psi\rangle).$$

We again use the fact that

$$\mathcal{T}^{(N)}_{|\chi\rangle} = \mathcal{T}^{(N)}_{|\psi\rangle}\circ\mathcal{T}^{(N)}_{|\psi\rangle},$$

which gives

$$\|\mathcal{T}^{(N)}_{|\chi\rangle}\| = \|\mathcal{T}^{(N)}_{|\psi\rangle}\|^2.$$

Since $\|\mathcal{T}^{(N)}_{|\psi\rangle}\| \geq 1$ we get $\|\mathcal{T}^{(N)}_{|\chi\rangle}\| \geq \|\mathcal{T}^{(N)}_{|\psi\rangle}\|$ or

$$E_{\mathcal{T}}(|\chi\rangle) \geq E_{\mathcal{T}}(|\psi\rangle).$$

Superadditivity. We have to show, for N-qubit states $|\psi\rangle$ and $|\phi\rangle$, that

$$E_{\mathcal{T}}(|\psi\rangle \otimes |\phi\rangle) \geq E_{\mathcal{T}}(|\psi\rangle) + E_{\mathcal{T}}(|\phi\rangle). \quad (30)$$

We already know that for $|\chi\rangle = |\psi\rangle \otimes |\phi\rangle$

$$\|\mathcal{T}^{(N)}_{|\chi\rangle}\| = \|\mathcal{T}^{(N)}_{|\psi\rangle}\| \|\mathcal{T}^{(N)}_{|\phi\rangle}\|.$$

Thus Eq. (30) gets transformed to

$$\|\mathcal{T}^{(N)}_{|\psi\rangle}\| \|\mathcal{T}^{(N)}_{|\phi\rangle}\| - 1 \geq \|\mathcal{T}^{(N)}_{|\psi\rangle}\| + \|\mathcal{T}^{(N)}_{|\phi\rangle}\| - 2,$$

which is true for $\|\mathcal{T}^{(N)}_{|\psi\rangle}\| \geq 1$ and $\|\mathcal{T}^{(N)}_{|\phi\rangle}\| \geq 1$. ∎

B. Computational considerations

Computation or experimental determination of $E_{\mathcal{T}}$ involves 3^N elements of $\mathcal{T}^{(N)}$ so that it increases exponentially with the number of qubits N. However, for many important classes of states, $E_{\mathcal{T}}$ can be easily computed and increases only polynomially with N. We have already computed $E_{\mathcal{T}}$ for the class of N-qubit W states, GHZ states, and their superpositions. We have also computed $E_{\mathcal{T}}$ for an important physical system like the one-dimensional Heisenberg antiferromagnet. For symmetric or antisymmetric states, $\mathcal{T}^{(N)}$ is supersymmetric, that is, the values of its elements are invariant under any permutation of its indices [24]. This reduces the problem to the computation of $\frac{1}{2}(N+1)(N+2)$ distinct elements of $\mathcal{T}^{(N)}$, which is quadratic in N.

C. Entanglement dynamics: Grover algorithm

We show that $E_{\mathcal{T}}$ can quantify the evolution of entanglement. We consider Grover's algorithm. The goal of Grover's algorithm is to convert the initial state of N qubits, say $|0\ldots0\rangle$, to a state that has probability bounded above $\frac{1}{2}$ of being in the state $|a_1\ldots a_N\rangle$, using

$$U_a|b_1 \ldots b_N\rangle = (-1)^{\Pi\delta_{a_j b_j}}|b_1 \ldots b_N\rangle$$

the fewest times possible. Grover showed that this can be done with $O(\sqrt{2^N})$ uses of U_a by starting with the state

$$\frac{1}{\sqrt{2^N}} \sum_{x=0}^{2^N-1} |x\rangle = H^{\otimes N}|0 \ldots 0\rangle,$$

where

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and then iterating the transformation $H^{\otimes N}U_a H^{\otimes N}U_a$ on this state [23]. The initial state is a product state as is the target state, but intermediate states $\psi(k)$ are entangled for $k>0$ iterations. Figure 6 shows the development of $E_{\mathcal{T}}(|\psi(k)\rangle)$ with number of iterations k, for six qubits. The values of k for which $E_{\mathcal{T}}$ vanishes are the iterations at which the probability of measuring $|a_1\ldots a_N\rangle$ is close to 1. Thus $E_{\mathcal{T}}$ can be used to quantify the evolution of an N-qubit entangled state.

V. EXTENSION TO MIXED STATES

The extension of $E_{\mathcal{T}}$ to mixed states ρ can be made via the use of the *convex roof* (or *hull*) construction as was done for

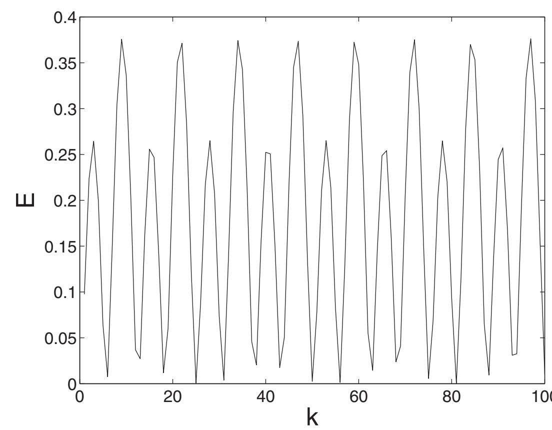

FIG. 6. Entanglement in Grover's algorithm for six qubits as a function of number of iterations.

the entanglement of formation [16]. We define $E_{\mathcal{T}}(\rho)$ as a minimum over all decompositions $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ into pure states, i.e.,

$$E_{\mathcal{T}}(\rho) = \min_{\{p_i,\psi_i\}} \sum_i p_i E_{\mathcal{T}}(|\psi_i\rangle). \quad (31)$$

The existence and uniqueness of the convex roof for $E_{\mathcal{T}}$ is guaranteed because it is a continuous function on the set of pure states [39]. This entanglement measure is expected to satisfy conditions (a), (b), and (c) given in Sec. IV and is expected to be (d) convex under discarding of information, i.e.,

$$\sum_i p_i E_{\mathcal{T}}(\rho_i) \geq E_{\mathcal{T}}\Big(\sum_i p_i\rho_i\Big). \quad (32)$$

The criteria (a)–(d) above are considered to be the minimal set of requirements for any entanglement measure so that it is an entanglement monotone [29].

Evidently, criteria (a) and (b) are satisfied by $E_{\mathcal{T}}(\rho)$ defined via the convex roof as they are satisfied by $E_{\mathcal{T}}$ for pure states. Condition (d) follows from the fact that every convex hull (roof) is a convex function [40]. We need to prove (c), which is summarized in the following proposition.

Proposition 8. If an N-qubit mixed state ρ is subjected to a local operation on the ith qubit giving outcomes k with probabilities p_k and leaving the residual N-qubit mixed state ρ_k, then the expected entanglement $\sum_k p_k E_{\mathcal{T}}(\rho_k)$ of the residual state is not greater than the entanglement $E_{\mathcal{T}}(\rho)$ of the original state,

$$\sum_k p_k E_{\mathcal{T}}(\rho_k) \leq E_{\mathcal{T}}(\rho).$$

(If the operation is simply throwing away part of the system, then there will be only one value of k, with unit probability.)

The proof follows from the monotonicity of $E_{\mathcal{T}}(|\psi\rangle)$ for pure states that is Propositions 5, 6, and 7. Bennett *et al.* prove a version of Proposition 8 in [34], which applies to any measure satisfying Propositions 5, 6, and 7. Thus the same proof applies to Proposition 8, so we skip it.

Note that any sequence of local operations comprises local operations drawn from the set of basic local operations (i)–(iv) above, so that Proposition 8 applies to any such sequence. Thus we can say that the expected entanglement of an N-qubit system, measured by $E_{\mathcal{T}}(\rho)$, does not increase under local operations.

VI. A RELATED ENTANGLEMENT MEASURE

We consider the following entanglement measure. Consider

$$\mathcal{E}_{\mathcal{T}}(|\psi\rangle) = \log_2\|\mathcal{T}^{(N)}\| = \log_2[E_{\mathcal{T}}(|\psi\rangle) + 1],$$

where $\mathcal{T}^{(N)}$ is the N-way correlation tensor occurring in the Bloch representation of $\rho=|\psi\rangle\langle\psi|$.

Proofs of Propositions 1–8 easily go through for $\mathcal{E}_{\mathcal{T}}(|\psi\rangle)$. We prove continuity as follows.

Continuity of $\mathcal{E}_{\mathcal{T}}(|\psi\rangle)$. We have to show, for two N-qubit states $|\psi\rangle$ and $|\phi\rangle$, that $\|(|\psi\rangle\langle\psi|-|\phi\rangle\langle\phi|)\|\to 0 \Rightarrow |\mathcal{E}_{\mathcal{T}}(|\psi\rangle) - \mathcal{E}_{\mathcal{T}}(|\phi\rangle)| \to 0$. We have $\|(|\psi\rangle\langle\psi|-|\phi\rangle\langle\phi|)\|\to 0 \Rightarrow \|\mathcal{T}^{(N)}_{|\psi\rangle} - \mathcal{T}^{(N)}_{|\phi\rangle}\| \to 0$. But $\|\mathcal{T}^{(N)}_{|\psi\rangle} - \mathcal{T}^{(N)}_{|\phi\rangle}\| \geq |\|\mathcal{T}^{(N)}_{|\psi\rangle}\| - \|\mathcal{T}^{(N)}_{|\phi\rangle}\||$. Further, whenever $\|\mathcal{T}^{(N)}_{|\psi\rangle}\| \geq 1$ and $|\mathcal{T}^{(N)}_{|\phi\rangle}\| \geq 1$ we have $|\|\mathcal{T}^{(N)}_{|\psi\rangle}\| - \|\mathcal{T}^{(N)}_{|\phi\rangle}\|| \geq |\log_2(\|\mathcal{T}^{(N)}_{|\psi\rangle}\|) - \log_2(\|\mathcal{T}^{(N)}_{|\phi\rangle}\|)|$. Thus $\|\mathcal{T}^{(N)}_{|\psi\rangle} - \mathcal{T}^{(N)}_{|\phi\rangle}\| \to 0 \Rightarrow |\|\mathcal{T}^{(N)}_{|\psi\rangle}\| - \|\mathcal{T}^{(N)}_{|\phi\rangle}\|| \to 0 \Rightarrow |\log_2(\|\mathcal{T}^{(N)}_{|\psi\rangle}\|) - \log_2(\|\mathcal{T}^{(N)}_{|\phi\rangle}\|)| \to 0 \Rightarrow |\mathcal{E}_{\mathcal{T}}(|\psi\rangle) - \mathcal{E}_{\mathcal{T}}(|\phi\rangle)| \to 0$.

However, $\mathcal{E}_{\mathcal{T}}(|\psi\rangle)$ has the added advantage that it is additive [while $E_{\mathcal{T}}(|\psi\rangle)$ is superadditive]. Indeed, from Sec. IV A we see that for k copies

$$\mathcal{E}_{\mathcal{T}}(|\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle) = k\mathcal{E}_{\mathcal{T}}(|\psi\rangle).$$

Similarly $\mathcal{E}_{\mathcal{T}}(|\psi\rangle \otimes |\phi\rangle) = \mathcal{E}_{\mathcal{T}}(|\psi\rangle) + \mathcal{E}_{\mathcal{T}}(|\phi\rangle)$.

The extension of $\mathcal{E}_{\mathcal{T}}(|\psi\rangle)$ to mixed states via convex roof construction is similar to that of $E_{\mathcal{T}}(|\psi\rangle)$. Thus $\mathcal{E}_{\mathcal{T}}(|\psi\rangle)$ has all the properties of $E_{\mathcal{T}}(|\psi\rangle)$, with the additional property that $\mathcal{E}_{\mathcal{T}}(|\psi\rangle)$ is additive, while $E_{\mathcal{T}}(|\psi\rangle)$ is superadditive.

VII. CONCLUSION

In conclusion, we have developed an experimentally viable entanglement measure for N-qubit pure states, which passes almost all the tests for being a good entanglement measure. This is a global entanglement measure in the sense that it does not involve partitions or cuts of the system in its definition or calculation. This measure has quadratic computational complexity for symmetric or antisymmetric states. Computational tractability is not a serious problem if N is not too large, and the measure can be easily computed for systems comprising small numbers of qubits, which can have many important applications such as teleportation of multiqubit states, quantum cryptography, dense coding, distributed evaluation of functions [41], etc. However, finding other classes of states for which $E_{\mathcal{T}}$ can be computed polynomially will be useful. It will be very interesting to seek applications of this measure to situations like quantum phase transitions [11], transfer of entanglement along spin chains [42], NOON states in quantum lithography [43], etc. Finally, we have extended our measure to mixed states and established its various properties, in particular, its monotonicity. We may also note that neither its definition nor its properties depends in an essential way on the fact that we are dealing with qubits, so that this measure can be defined and applied to a general N-partite quantum system.

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