Wave-packet entanglement

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We propose and analyze a scheme by which a many-particle system can be prepared in highly entangled wave-packet states. One of the particles is prepared initially in a quantum superposition of multiple coherent states and then coupled via a quadratic interaction Hamiltonian to a number of other particles. The system evolves into a highly entangled wave-packet state. An appropriate measure of this time-dependent entanglement is given. This scheme is applicable to a number of systems of interest in quantum-information science.

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I. INTRODUCTION

In this paper we analyze the entanglement of superpositions of coherent states via a harmonic oscillator Hamiltonian that is quadratic in the interaction between the particles and symmetric with respect to particle exchange. Quantum superpositions of coherent states are commonly referred to as "Schrödinger cat states" [1]. When the coherent states are only weakly excited, $\bar{n} \ll 1$, they are generally called "Schrödinger kitten states" [2]. We will use this nomenclature.

Many methods have been proposed for generating superpositions of coherent states. These range from traditional areas of investigation such as molecular wave-packet states [3], coherent atomic states [4], vibrational modes of an atom or ion in a trap [5,6], modes of the light field [7,8], photon-atom systems [9], optically via nonlinearity [10–12], via squeezed states and linear optics [13], etc., to the more exotic involving, for example, the superposition of a macroscopic mechanical oscillator in its vibrating and nonvibrating states [14,15] or through a Jaynes-Cummings-type model interaction where discrete levels couple to continuous levels [16,17].

Experimentally, Schrödinger cat states have been produced in atomic systems using Rydberg wave packets [18] and ion traps [19]. In optical systems only Schrödinger kitten states of the form of a superposition of two coherent states with identical amplitudes but different phases, i.e., $|\Psi\rangle = |\alpha\rangle + |\alpha e^{i\theta}\rangle$ with $|\alpha| \ll 1$ have been produced so far in the laboratory [20] (also see [21]).

In this paper we will discuss the entanglement of the above-mentioned Schrödinger cat states, i.e., the multipartite entanglement of coherent states. As in the creation of superposition states, there have been many theoretical schemes to produce multipartite entanglement. However, the bulk of the theoretical analysis of continuous variable entanglement has concerned itself with bipartite systems where the state of each party is fully described by two linearly independent coherent states [17,22,23]. Increasing the number of linearly independent coherent states in each party effectively increases the number of levels in the system, in a sense that we make more precise later. Increasing the number of levels and/or parties in a system exponentially increases the number of different types of entanglement [24,25] making computation increasingly unwieldy. To keep things simpler, the next level of complexity is generally taken to be a multipartite system in which there are still only two linearly independent coherent states for each party [26], or the system is kept bipartite but the number M of linearly independent coherent states is increased in such a way that they are symmetric in their argument $|\alpha\rangle$ [12,27]. These studies analyze entanglement in the limiting case in which the value of the coherent state $\alpha \to \infty$. In this way, one trivially obtains an orthogonal basis in which to analyze entanglement.

Another approach to continuous variable entanglement is to analyze Gaussian states, i.e., states with a Gaussian characteristic function. For these states the entanglement problem is reduced to that of a finite level system because Gaussian states are fully characterized by their first and second moments. So, when there is only a finite number of modes, the corresponding matrices of moments are also finite [28,29].

As most physical phenomena consist of multipartite systems with a countably infinite number of or even a continuum of levels, it seems worthwhile to explore the entanglement properties of such systems. Quantum systems with continuum or quasicontinuum energy spectra such as free particles, atomic or molecular Rydberg electrons, molecular vibrational and rotational levels, and photons are generally produced in wave-packet states [30,31]. Additionally, superpositions of coherent states, also known as wave packets, can usefully store information, which is useful for quantum information [29]. In this paper we will generalize the beam-splitter model already discussed in [23,27] to arbitrary number of parties N and solve it for all time. As in [30,31] we will assume an arbitrary number M of symmetric linearly independent coherent states. We will also eliminate the assumption of large coherent state value α by using an orthogonal basis that takes advantage of the symmetry of the linearly independent coherent states in each party. In our analysis, α can take on any complex value.

II. MODEL HAMILTONIAN

Harmonic oscillator models for interactions help to clarify the types of interactions that lead to entanglement. It is easily shown that time-dependent interactions that do not contain terms of the form $\hat{a}_p^\dagger \hat{a}_q^\dagger$ do not lead to entanglement between

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coherent states. In this paper we will model weak-coupling interactions between systems that are in a stable configuration using a coupled harmonic oscillator Hamiltonian, symmetric with respect to particle exchange, in the rotating-wave approximation. The time-dependent interaction term will have the form

$$\hat{H} = \hbar\omega[\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + \kappa f(t)(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})], \tag{1}$$

where $\hat{a}$ and $\hat{b}$ are destruction operators for two different particles, while κ and $f(t)$ denote the strength and time dependence of the interaction, respectively. This leads to the following Heisenberg equations of motion in the rotating-wave approximation,

$$i\hbar\dot{\hat{a}} = \hbar\omega\hat{a} + \hbar\omega\kappa f(t)\hat{b},$$

$$i\hbar\dot{\hat{b}} = \hbar\omega\hat{b} + \hbar\omega\kappa f(t)\hat{a}. \tag{2}$$

The time dependence of $\hat{a}$ and $\hat{b}$ is then

$$\hat{a}(t) = e^{-i\omega t}\{\hat{a}(0)\cos[\Theta(t)] - i\hat{b}(0)\sin[\Theta(t)]\},$$

$$\hat{b}(t) = e^{-i\omega t}\{-i\hat{a}(0)\cos[\Theta(t)] + \hat{b}(0)\sin[\Theta(t)]\}, \tag{3}$$

where

$$\Theta(t) = \kappa\int_0^t dt'\,\omega f(t') \tag{4}$$

is the "area" of the interaction pulse, which is constant for t after the end of the pulse.

Thus, if $|\Psi(0)\rangle = |\alpha,\beta\rangle$ in a simultaneous eigenstate of $\hat{a}(0)$ and $\hat{b}(0)$, then it is also an eigenstate of $\hat{a}(t)$ and $\hat{b}(t)$. As an eigenstate we then have the relation in the Heisenberg picture,

$$\hat{a}(t)|\alpha,\beta\rangle = \lambda_a(t)|\alpha,\beta\rangle,$$

$$\hat{b}(t)|\alpha,\beta\rangle = \lambda_b(t)|\alpha,\beta\rangle, \tag{5}$$

so that in the Schrödinger picture

$$|\Psi(t)\rangle = |\lambda_a(t)\rangle \otimes |\lambda_b(t)\rangle, \tag{6}$$

where $|\lambda_a(t)\rangle$ and $|\lambda_b(t)\rangle$ are coherent states with time-dependent amplitudes

$$\lambda_a(t) = e^{-i\omega t}\{\alpha\cos[\Theta(t)] - i\beta\sin[\Theta(t)]\},$$

$$\lambda_b(t) = e^{-i\omega t}\{-i\alpha\sin[\Theta(t)] + \beta\cos[\Theta(t)]\}. \tag{7}$$

With the model specified above, any interaction of particles prepared in coherent states results in the final states of the particles also being in coherent states. It is known that for such a bilinear interaction, entanglement at the output requires nonclassical states at the input. Several authors (see, for example, [32]) have pointed out that a squeezed state input will produce entanglement at the output. In this paper we will explore another quantum state whose input results in entanglement and that is the Schrödinger cat state.

III. MULTIPARTITE-MULTISUMMAND HAMILTONIAN

We examine the evolution of nonentangled Schrödinger cat states that evolve into entangled Schrödinger cat states using a multiparty-multisummand generalization of Eq. (1) in which one party is coupled symmetrically to many. The system Hamiltonian is then given by

$$H = \hbar\omega\left(\sum_{p=0}^{N-1}\hat{a}_p^\dagger\hat{a}_p + \kappa f(t)\sum_{p=1}^{N-1}(\hat{a}_0^\dagger\hat{a}_p + \hat{a}_p^\dagger\hat{a}_0)\right), \tag{8}$$

where we use the notation p to denote different particles that are coupled symmetrically to particle 0. This Hamiltonian is only a slight generalization of Eq. (1). To see this, we transform to a new basis as follows:

$$\hat{a}_p = \sqrt{\frac{1}{N-1}}\sum_{q=1}^{N-1}\exp\left(\frac{-2\pi iqp}{N-1}\right)\hat{b}_q, \quad 1 \le p \le N-1, \tag{9}$$

so that now the Hamiltonian takes the form

$$H = \hbar\omega\left(\hat{a}_0^\dagger\hat{a}_0 + \sum_{p=1}^{N-1}\hat{b}_p^\dagger\hat{b}_p + \kappa f(t)\sqrt{N-1}(\hat{a}_0^\dagger\hat{b}_{N-1} + \hat{b}_{N-1}^\dagger\hat{a}_0)\right). \tag{10}$$

Using the result of Eq. (3) we can write

$$\hat{a}_0(t) = e^{-i\omega t}\{\hat{a}_0(0)\cos[\Theta_N(t)] - i\hat{b}_{N-1}(0)\sin[\Theta_N(t)]\},$$

$$\hat{b}_{N-1}(t) = e^{-i\omega t}\{-i\hat{a}_0(0)\cos[\Theta_N(t)] + \hat{b}_{N-1}(0)\sin[\Theta_N(t)]\},$$

$$\hat{b}_q(t) = e^{-i\omega t}\hat{b}_q(0), \quad 1 \le q \le N-1, \tag{11}$$

where

$$\Theta_N(t) = \Theta(t)\sqrt{N-1}. \tag{12}$$

We can then find the $\hat{a}(t)$ operators by taking the inverse Fourier transform

$$\hat{b}_q = \sqrt{\frac{1}{N-1}}\sum_{p=1}^{N-1}\exp\left(\frac{2\pi iqp}{N-1}\right)\hat{a}_p, \quad 1 \le q \le N-1. \tag{13}$$

Explicitly the result is

$$\hat{a}_0(t) = e^{-i\omega t}\cos[\Theta_N(t)]\hat{a}_0 - i\sqrt{\frac{1}{N-1}}e^{-i\omega t}\sin[\Theta_N(t)]\sum_{q=1}^{N-1}\hat{a}_q(0), \tag{14}$$

while for $1 \le p \le N-1$, we have

$$\hat{a}_p(t) = e^{-i\omega t}\left[-i\sin[\Theta_N(t)]\frac{\hat{a}_0(0)}{\sqrt{N-1}} + \cos[\Theta_N(t)]\sum_{q=1}^{N-1}\left(1 + \sum_{m=2}^{N-1}e^{2\pi im(q-p)/(N-1)}\right)\frac{\hat{a}_q(0)}{N-1}\right]. \tag{15}$$

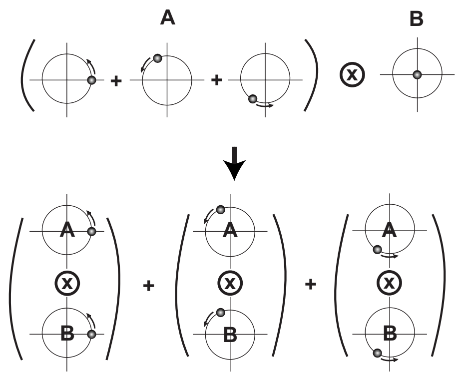

FIG. 1. Evolution of two particles from a product state to entangled Schrödinger cat states. Initially particle A is in a nonclassical Schrödinger cat state, that is, a superposition of coherent states localized at three different points in phase space, while B is in its ground state. After the interaction the particles are in an entangled superposition of the three localizations.

IV. EXAMPLE OF WAVE-PACKET ENTANGLEMENT

Our example is similar to the one introduced by van Enk in [27]. We simplify the coefficients but generalize his example to arbitrary N and solve for all time. We begin by choosing an initial state where one oscillator is excited and this oscillator is in a coherent superposition of localized coherent states, i.e., it is a nonclassical Schrödinger cat state. The other $N-1$ oscillators are in their ground states. The initial state is

$$|\Psi(0)\rangle = \mathcal{N}\sum_{\sigma=0}^{M-1}[|\alpha e^{2\pi i\sigma/M}\rangle \otimes |0\rangle^{\otimes(N-1)}] \equiv \mathcal{N}\sum_{\sigma=0}^{M-1}|\psi_\sigma\rangle, \tag{16}$$

where M,N are integers >1. For the case $N=2$, this corresponds to our first example with $\beta=0$. Notice that the phases of the summation terms in the first particle, $|\alpha e^{2\pi i\sigma/M}\rangle$, are equally spaced around a circle of radius α.

A diagram illustrating the evolution of the initial state under this Hamiltonian is shown in Fig. 1 for the case of two particles. At an initial time $t=0$ the overall state of the system is nonentangled. The first particle is in a nonclassical superposition of three coherent states, but the second party is in the ground state. Under evolution of the Hamiltonian, interaction between the parties induces the second party to evolve into a superposition of coherent states as well. The resulting overall state is not a separable product of two superposition states, but rather an entangled state of the two particles.

Applying the operators $\hat{a}_p(t)$ to this state gives us the following relation:

$$\hat{a}_p(0)|\psi_\sigma\rangle = \begin{cases} \alpha e^{2\pi i\sigma/M}|\psi_\sigma\rangle, & \text{when } p=0, \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Therefore,

$$\hat{a}_p(t)|\psi_\sigma\rangle = \lambda_p(t)e^{2\pi i\sigma/M}|\psi_\sigma\rangle, \tag{18}$$

where

$$\lambda_p(t) = \begin{cases} \alpha e^{-i\omega t}\cos[\Theta_N(t)], & \text{when } p=0, \\ \dfrac{-\alpha}{\sqrt{N-1}}e^{-i\omega t}i\sin[\Theta_N(t)], & \text{otherwise,} \end{cases} \tag{19}$$

and $|\lambda_p(t)e^{2\pi i\sigma/M}\rangle$ is a coherent state with time-dependent amplitude. One then switches from the Heisenberg picture to the Schrödinger picture to obtain $|\Psi(t)\rangle$. The result is

$$|\Psi(t)\rangle = \mathcal{N}\sum_{\sigma=0}^{M-1}|\psi_\sigma(t)\rangle = \mathcal{N}\sum_{\sigma=0}^{M-1}\bigotimes_{p=0}^{N-1}|\lambda_p(t)e^{2\pi i\sigma/M}\rangle. \tag{20}$$

Notice that this reduces to Eq. (7) for the case of $\beta=0$ and $M=1$. The above solution for $\lambda_p(t)$ shows that all properties, including entanglement, will be periodic in the interaction area. Each term in the superposition remains a product of coherent states, but the sum is in general an entangled superposition.

Interesting in this case is that we start with only one excited nonclassical state, and we oscillate between this state and an entangled state of all the oscillators as a function of the "pulse area" of the interaction, $\Theta(t)$.

V. ORTHOGONAL BASIS CALCULATION

In a continuous variable system there is a lot of freedom in choosing a basis, and the choice of basis will determine the level system for each party. The choice of basis is determined by what is entangled [33]. For example, for the coherent states, one can choose the Fock basis, which is orthogonal, and look at photon number entanglement. The difficulty with such a basis is that the level system becomes infinite.

In our case, the most natural choice of basis is to choose a basis that spans the same subspace as the set of linearly independent coherent states. In our model, which has a finite number of oscillators, this allows us, after suitable transformations, to analyze the entanglement using a standard entanglement measure.

In order to transform our states into a more manageable form, we first recognize that corresponding to each particle, we have a finite sum of coherent states, so that the dimension of our Hilbert space is finite. The state vectors in Eq. (20) form a complete but nonorthogonal basis, which we abbreviate as $|W_\sigma\rangle$, i.e.,

$$|W_1\rangle = |\lambda_p(t)\rangle, \quad |W_2\rangle = |\lambda_p(t)e^{2\pi i/M}\rangle, \ldots,$$
$$|W_M\rangle = |\lambda_p(t)e^{2\pi i(M-1)/M}\rangle. \tag{21}$$

Next, we recognize that the coherent states in Eq. (21) are symmetrically displaced in phase around the unit circle. This symmetry, due to our choice of generalized Schrödinger cat states, is utilized to our advantage by transforming to an orthonormal basis as follows:

$$|W_\sigma\rangle = \frac{1}{M}\sum_{\varsigma=0}^{M-1}\mathcal{N}_\varsigma \exp(2\pi i\sigma\varsigma/M)|V_\varsigma\rangle, \tag{22}$$

where the orthonormal basis is given by

$$|V_\varsigma\rangle = \frac{1}{\mathcal{N}_\varsigma}\sum_{\sigma=0}^{M-1}\exp(-2\pi i\sigma\varsigma/M)|\lambda_p(t)e^{2\pi i\sigma/M}\rangle. \quad (23)$$

We can verify that these vectors are indeed orthogonal,

$$\langle V_\mu|V_\nu\rangle = \frac{\sum_{\sigma=0}^{M-1} e^{2\pi i\sigma\mu/M}\langle\lambda_p(t)e^{2\pi i\sigma/M}|}{\mathcal{N}_\mu} \times\frac{\sum_{\varsigma=0}^{M-1} e^{-2\pi i\varsigma\nu/M}|\lambda_p(t)e^{2\pi i\varsigma/M}\rangle}{\mathcal{N}_\nu} = \frac{M}{\mathcal{N}_\mu\mathcal{N}_\nu}\delta_{\mu\nu}\sum_{\varpi=0}^{M-1} e^{-2\pi i\varpi\nu/M}\langle\lambda_p(t)|\lambda_p(t)e^{2\pi i\varpi/M}\rangle, \quad (24)$$

where $\varpi=\varsigma-\sigma$. In Eq. (24) we have used the symmetry of the superposition, i.e., we have used the fact that

$$\langle\lambda_p(t)|\lambda_p(t)e^{2\pi i\varsigma/M}\rangle = \langle\lambda_p(t)e^{2\pi i\sigma/M}|\lambda_p(t)e^{2\pi i(\sigma+\varsigma)/M}\rangle. \quad (25)$$

Thus, we see that the normalization factors $\mathcal{N}_\sigma$ are given by

$$\mathcal{N}_\sigma^2 = M\sum_{\varsigma=0}^{M-1}(e^{-2\pi i\varsigma/M})^\sigma \exp\{|\lambda_p(t)|^2[\exp(2\pi i\varsigma/M)-1]\}. \quad (26)$$

More generally, we can take account of the fact that the normalization factors also depend on the particle, by introducing the notation $\mathcal{N}_{\sigma_p}$. When σ_p is denoted by a concrete integer we use the notation $\mathcal{N}_{(\sigma_p,p)}$ to denote the particle number, as in $\mathcal{N}_{(0,1)}$ where $p=1$ and $\sigma_p=0$.

Using the orthogonal basis we can rewrite $|\Psi\rangle$ in this notation as

$$|\Psi(t)\rangle = \mathcal{N}\sum_{\sigma_0\cdots\sigma_{N-1}=0}^{M-1}[(\delta_{(\Sigma_{p=0}^{N-1}\sigma_p)\bmod M,0})(\mathcal{N}_{\sigma_0}\cdots\mathcal{N}_{\sigma_{N-1}}) \times(|V_{\sigma_0}\rangle\otimes\cdots\otimes|V_{\sigma_{N-1}}\rangle)]. \quad (27)$$

VI. SPECIAL CASES

In the bipartite case Eq. (27) reduces to

$$|\Psi(t)\rangle = \mathcal{N}(\mathcal{N}_{(0,0)}\mathcal{N}_{(0,1)}[|V_{(0,0)}\rangle\otimes|V_{(0,1)}\rangle] + \sum_{\sigma_0=1}^{M-1}[\mathcal{N}_{(\sigma_0,0)}\mathcal{N}_{(M-\sigma_0,1)}][|V_{(\sigma_0,0)}\rangle\otimes|V_{(M-\sigma_0,1)}\rangle]), \quad (28)$$

which is the Schmidt decomposition. This should not be too surprising, for as $|\alpha|\to\infty$, the collection of $|W_\sigma\rangle$ in Eq. (21) becomes an orthogonal set. In this case, Eq. (20) shows $|\Psi(t)\rangle$ to be a Greenberger-Horne-Zeilinger (GHZ) state, and multipartite GHZ states always admit a Schmidt decomposition [34].

To see the advantage of this formalism in another example, we consider the case of arbitrary particle number N where $M\to\infty$. We can expand Eq. (26) as follows:

$$\left(\frac{N_\sigma}{M}\right)^2 = \frac{1}{M}\sum_{\varsigma=0}^{M-1}(e^{-2\pi i\varsigma/M})^\sigma e^{|\lambda_p|^2[\exp(2\pi i\varsigma/M)-1]}$$
$$= \frac{e^{-|\lambda_p|^2}}{M}\sum_{\varsigma=0}^{M-1}(e^{-2\pi i\varsigma/M})^\sigma\sum_{\gamma=0}^{\infty}\frac{[|\lambda_p|^2\exp(2\pi i\varsigma/M)]^\gamma}{\gamma!}$$
$$= \frac{e^{-|\lambda_p|^2}}{M}\sum_{\gamma=0}^{\infty}\frac{|\lambda_p|^{2\gamma}}{\gamma!}\sum_{\varsigma=0}^{M-1}(e^{-2\pi i\varsigma/M})^{\sigma-\gamma}$$
$$= \frac{e^{-|\lambda_p|^2}}{M}\sum_{\gamma=0}^{\infty}\frac{|\lambda_p|^{2\gamma}}{\gamma!}M\delta_{(\sigma-\gamma)\bmod M,0}. \quad (29)$$

Thus $\gamma=mM+\sigma$ for some non-negative integer m,

$$\left(\frac{\mathcal{N}_\sigma}{M}\right)^2 = e^{-|\lambda_p|^2}\sum_{m=0}^{\infty}\frac{|\lambda_p|^{2(mM+\sigma)}}{(mM+\sigma)!}. \quad (30)$$

Then for very large M, compared to $|\lambda_p|$ and σ, we see that only the first term in the series survives, so that

$$\left(\frac{\mathcal{N}_\sigma}{M}\right)^2 \sim e^{-|\lambda_p|^2}\frac{|\lambda_p|^{2\sigma}}{\sigma!}. \quad (31)$$

This quantity is maximum at the integer $\sigma_{\max}$ that is closest to $|\lambda_p|^2$. On the other hand, this quantity is quite small for very large σ, i.e., σ values that are close to large M. Now Eq. (27) requires that for each term $|V_{\sigma_0}\otimes\cdots\otimes V_{\sigma_{N-1}}\rangle$ the subscripts σ_p satisfy

$$\sum_{p=0}^{N-1}\sigma_p = qM \quad (32)$$

for some non-negative integer q. This, and the largeness of M, guarantees that for $q\neq 0$, at least one of the σ_p in Eq. (27) will be very large, and hence the product $(\mathcal{N}_{\sigma_0}\cdots\mathcal{N}_{\sigma_{N-1}})/M^N$ will be very small. The situation is different when $q=0$. In this case all the $\sigma_p=0$. So for large M, i.e., $M\gg\alpha\geq|\lambda_p|$, the main contribution to Eq. (27) is the term $(|V_{0,0}\rangle\otimes\cdots\otimes|V_{0,N-1}\rangle)$.

To illustrate this point, we again refer to the simple bipartite case in Eq. (28). When $q\neq 0$, then whenever σ_0 is small, $\sigma_1=M-\sigma_0$ is large, and vice versa. Another possibility is that both σ_0 and σ_1 are about equal to $M/2$. In this case both σ's are large for large M. In all of these cases the product $\mathcal{N}_{\sigma_0}\mathcal{N}_{\sigma_1}$, with $\sigma_0\neq 0$, is always small compared to the term $\mathcal{N}_{(0,0)}\mathcal{N}_{(0,1)}$. So Eq. (28) reduces to the product state

$$|\Psi(t)\rangle \sim \mathcal{N}\{\mathcal{N}_{(0,0)}\mathcal{N}_{(1,0)}[|V_{(0,0)}\rangle\otimes|V_{(0,1)}\rangle]\}. \quad (33)$$

Thus, for large M, and arbitrary particle number N, the state always becomes disentangled. Consequently, entanglement does not increase linearly with the number of terms M in the superposition that makes up the Schrödinger cat state. In fact, as we shall see in our numerical calculations below, as the number of terms M increases, the entanglement eventually decreases sharply. This is due to the eventual overlap of the superposition terms in phase space.

VII. USING THE BARNUM-KNILL-ORTIZ-VIOLA ENTANGLEMENT MONOTONE

We remarked above that our Hilbert space is finite, and in fact partitioned as follows:

$$\mathcal{H} = \bigotimes_{p=0}^{N-1} \mathcal{H}_p, \tag{34}$$

where each $\mathcal{H}_p$ is M dimensional. The state $|\Psi(t)\rangle$ derived in Eq. (20) is a multilevel-multipartite state in this Hilbert space. It is also a pure state and, as such, its entanglement can be studied through a measure of entanglement (hereafter referred to as BKOV) introduced by Barnum, Knill, Ortiz, and Viola [35]. The BKOV measure of entanglement is defined by choosing an appropriate Lie algebra $\mathfrak{g}$ of operators with regard to the symmetry of the system. In our case, respecting the partition of our Hilbert space given by Eq. (34), our Lie algebra is given by

$$\mathfrak{g} = \bigoplus_{p=0}^{N-1} \mathrm{su}(M)_p. \tag{35}$$

The BKOV of a normalized state $|\Psi\rangle$ is defined by

$$P_{\mathfrak{g}} = A \sum_{d=1}^{D} \langle \Psi | \hat{E}_d | \Psi \rangle^2, \tag{36}$$

where D is the dimension of $\mathfrak{g}$, the set $\{\hat{E}_d\}$ is a Hermitian basis for $\mathfrak{g}$, and A is a normalization factor chosen so that when $|\Psi\rangle$ is a product state, the purity is equal to 1. In our case $D=N(M^2-1)$ and $A=M^N/[N(M-1)]$. We choose our Hermitian basis for $\mathfrak{g}$ by choosing a basis for each $\mathrm{su}(M)_p$ corresponding to each party, or in this paper, corresponding to each particle. This basis consists of off-diagonal elements of the form

$$\frac{1}{\sqrt{2}}(|m\rangle\langle n| + \mathrm{H.c.}), \quad \frac{1}{\sqrt{2}}(i|m\rangle\langle n| + \mathrm{H.c.}),$$

$$0 \le n \ne m \le M-1, \tag{37}$$

and diagonal elements of the form

$$\frac{1}{\sqrt{J+J^2}}\left(\sum_{m=0}^{J-1} |m\rangle\langle m| - J|J\rangle\langle J|\right), \quad 1 \le J \le M-1. \tag{38}$$

Each such basis term is then tensored with a normalized identity on the remaining parties. The normalization is chosen with respect to the trace inner product, i.e., $\langle \hat{E}_d, \hat{E}_d \rangle = \mathrm{Tr}(\mathbf{E_d} \mathbf{E_d^\dagger}) = 1$. In words, their inner product as vectors is equal to their trace as matrices.

As an example, in the bipartite two-level case our Lie algebra would be $\mathrm{su}(2) \oplus \mathrm{su}(2)$. A normalized basis for su(2) is then the three Pauli matrices divided by $\sqrt{2}$. So the basis for our Lie algebra consists of six operators, three for each party. They would be

$$\sqrt{\frac{1}{2}}\sigma_x \otimes \sqrt{\frac{1}{2}}\mathbf{I}, \quad \sqrt{\frac{1}{2}}\sigma_y \otimes \sqrt{\frac{1}{2}}\mathbf{I}, \quad \sqrt{\frac{1}{2}}\sigma_z \otimes \sqrt{\frac{1}{2}}\mathbf{I},$$

$$\sqrt{\frac{1}{2}}\mathbf{I} \otimes \sqrt{\frac{1}{2}}\sigma_x, \quad \sqrt{\frac{1}{2}}\mathbf{I} \otimes \sqrt{\frac{1}{2}}\sigma_y, \quad \sqrt{\frac{1}{2}}\mathbf{I} \otimes \sqrt{\frac{1}{2}}\sigma_z, \tag{39}$$

where $\mathbf{I}$ denotes the 2×2 identity matrix.

Notice that because a product state has purity equal to 1, maximum BKOV purity is equivalent to minimum entanglement. We note as well that BKOV is an entanglement monotone and has been shown to be equivalent to the Meyer-Wallach measure of entanglement for the case of N-party qubit systems when the Lie algebra is chosen to be $\oplus_{p=0}^{N-1}\mathrm{su}(2)$ [36,37]. An objection to the Meyer-Wallach measure, and hence to the BKOV measure, is that it cannot differentiate between completely entangled states such as GHZ states and states that are entangled but separable into subsystems, such as tensor products of Bell states [38]. For our purposes, where we are only interested in the evolution between a GHZ-type continuous variable state and a completely disentangled state, the BKOV measure suffices.

VIII. NUMERICAL CALCULATIONS OF BKOV ENTANGLEMENT IN THE ORTHOGONAL BASIS

We first note that from Eqs. (24), (26), and (27), we see that time enters in $\langle V_{\mu_p} | V_{\nu_p} \rangle$, $\mathcal{N}_{(\sigma,p)}$, and $\mathcal{N}$ only through $\Theta_N(t)$. That is, the $\exp(-i\omega t)$ term in Eq. (19) disappears. Therefore, there is no $\exp(-i\omega t)$ term in Eq. (36) and the BKOV depends on time only through $\Theta_N(t)$. We can thus express the entanglement of our states entirely in terms of the pulse of the interaction.

Using the BKOV measure we illustrate in Figs. 2 and 3 the entanglement we obtain as a function of pulse area, M, N, and α.

When $|\alpha| \le 1$ the entanglement is suppressed. This is the case in which the different wave packets overlap since their variance is $\sqrt{\alpha}$ and indicates that the generation of the so-called Schrödinger kitten states are not useful for entanglement. On the other hand, when $|\alpha| \ge 1$ the entanglement is quite pronounced provided M is not too large. In fact, for large α increasing M increases the entanglement. However, making M too large will overlap the different wave packets, and we return effectively to the case of the Schrödinger kitten states. As $\alpha \to \infty$ the state $|\Psi(t)\rangle$ tends to a state that is GHZ for all area except when the BKOV is identically equal to 1. This is illustrated in Fig. 2(b).

In Fig. 3 we plot the entanglement as measured by the BKOV measure as a function of the renormalized area $\Theta_N(t)/(2\pi)$ for different values of α and two different values of N. We note in this case that the amount of entanglement is periodic in the area Θ with period given by $\Theta_N = \pi$ when $N \ge 3$.

When $\Theta_N = \pi/2$ or $\Theta_N = 3\pi/2$ there is also a local BKOV maximum in the graphs, i.e., a local minimum in entangle-

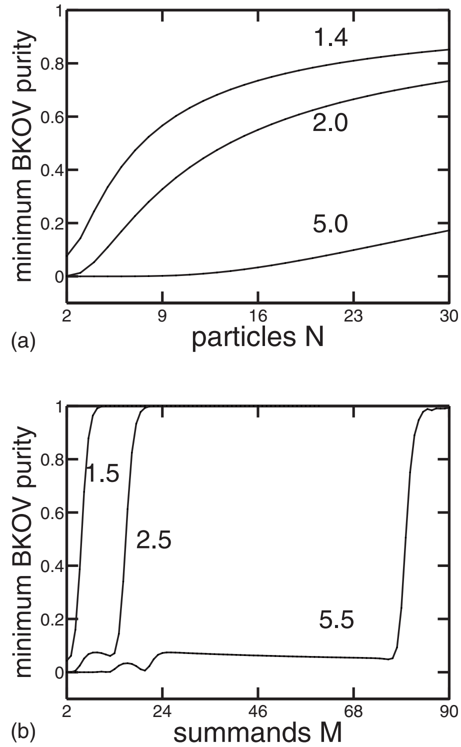

FIG. 2. Minimum possible BKOV purity for different values of α as a function of (a) N particles for $M=2$ summand terms and (b) M summand terms and $N=2$ particles.

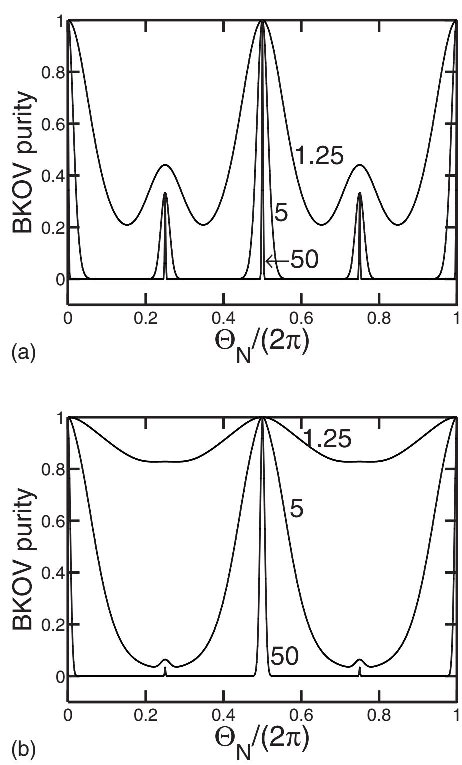

FIG. 3. BKOV purity as a function of renormalized area $\Theta_N/(2\pi)$ [see Eq. (19)] for $M=2$ summand terms, $\alpha=1.25$, 5, and 50, and (a) $N=3$ particles and (b) $N=30$ particles.

ment. These occur when the first oscillator is unexcited and the entanglement in this case is entirely in the mode $\hat{b}_0$ of Eq. (10). That is, the residual entanglement is in the remaining $N-1$ oscillators. When $N=2$ (not shown here) there is no residual entanglement since $N-1=1$. In this case, when the first oscillator is unexcited, there is no entanglement. In this case the local BKOV maximum becomes a global maximum. Thus, entanglement is symmetric as a function of Θ in the first and second oscillators and the period of entanglement as a function of Θ is halved, i.e., $\Theta_N=\Theta=\frac{\pi}{2}$.

We can summarize the above as follows: At the time $t=0$, $\Theta_N=0$, and we are in the initial state corresponding to maximum excitation of the first particle, but with the other particles remaining in their respective ground states. Therefore, there is no entanglement in the system since all of the particles, but one, are in their ground state. The BKOV is thus at a maximum, i.e., equal to 1. When the area $\Theta_N=\pi/2$ or $\Theta_N=3\pi/2$, we have no excitation of the first particle but maximum excitation of the remaining particles. In this case the total entanglement of the system is from entanglement among the remaining particles (for $N>2$). There is no entanglement with the first particle since it is in its ground state.

The total amount of entanglement for $\Theta_N=\pi/2$ and $\Theta_N=3\pi/2$ depends on the magnitude of λ_p and N. For small N the entanglement increases with N, since there is no entanglement when $N=2$. For larger N the entanglement decreases due to the $\sqrt{N-1}$ factor in the denominator of λ_p in Eq. (19). Increasing the value of $|\alpha|$ counters this effect. This is illustrated in Figs. 3(a) and 3(b). In the case of $\alpha=1.25$ the entanglement decreases from $N=3$ to $N=30$. For $\alpha=50$ the situation is reversed and the entanglement increases, while for $\alpha=5$ the situation is mixed.

IX. NEW QUANTUM OPERATOR

Finally, we add that our orthogonal basis construction can be used to define a new quantum operator that is analogous to the displacement operator

$$\hat{D}(\alpha)|0\rangle = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})|0\rangle = |\alpha\rangle \tag{40}$$

that generates a coherent state from the ground state, or the squeezing operator that generates a squeezed state from the ground state. This new quantum operator $\hat{\mathcal{C}}$ generates a superposition of coherent states, that we refer to in this paper as a Schrödinger cat, from the ground state. Specifically, we have

$$\hat{\mathcal{C}}|0\rangle = \exp\left(\frac{|\alpha|^2}{2}\right)\frac{\mathcal{N}_\sigma}{M}|\alpha,M,\sigma\rangle, \tag{41}$$

where

$$|\alpha,M,\sigma\rangle = \frac{1}{\mathcal{N}_\sigma}\sum_{s=0}^{M-1} e^{-2\pi i s\sigma/M}|\alpha e^{2\pi i s/M}\rangle, \tag{42}$$

i.e., $|\alpha,M,\sigma\rangle$ is a basis vector as given by Eq. (23).

With a bit of whimsy, we call $\hat{\mathcal{C}}$ our cat operator. It is derived from a function $\mathcal{C}_M^\sigma$ defined by

$$\mathcal{C}_M^{\sigma}(x) = \frac{1}{M}\sum_{\varsigma=0}^{M-1} e^{-2\pi i\varsigma\sigma/M}\exp(e^{2\pi i\varsigma/M}x)$$
$$= e^{x}\left(\frac{\mathcal{N}_{\sigma}(x)}{M}\right)^{2} = \sum_{m=0}^{\infty}\frac{x^{mM+\sigma}}{(mM+\sigma)!}, \qquad (43)$$

where the second equality follows from Eq. (26) and the third equality follows from Eq. (30) with x substituted for $|\lambda_p|^2$. Then

$$\hat{\mathcal{C}} \equiv \mathcal{C}_M^{\sigma}(\alpha\hat{a}^{\dagger}). \qquad (44)$$

The cat operator formalism allows us to write complicated superposition states in a compact manner. For instance, $\mathcal{C}_M^{\sigma}(\alpha^m(\hat{a}^{\dagger})^m)|0\rangle$ gives us a complicated infinite sum of Fock or coherent states in which only every mth Fock state is retained in the sum. Also, in analogy to the displacement and squeezing operators, this operator formalism also allows us to analyze moments and variances of the entangled Schrödinger cat states, allowing noise analysis for these states (see [39]).

X. CONCLUSION

In this paper we have shown that generalized Schrödinger cat states are very useful in entangling wave-packet states of many systems and that the resulting entanglement can be analyzed conveniently via an orthonormal basis. This basis is discrete allowing a quantitative treatment of the time-dependent entanglement of the coupled systems. An analytic scheme involving what we call "cat operator" provides a set of tools that are powerful in analyzing the entangled states.

[1] C. C. Gerry and R. Grobe, Phys. Rev. A **75**, 034303 (2007).
[2] A. Biswas and G. S. Agarwal, Phys. Rev. A **75**, 032104 (2007).
[3] S. Ghosh, A. Chiruvelli, J. Banerji, and P. K. Panigrahi, Phys. Rev. A **73**, 013411 (2006).
[4] J. Recamier, O. Castanos, R. Jauregui, and A. Frank, Phys. Rev. A **61**, 063808 (2000).
[5] I. H. Deutsch and P. S. Jessen, Phys. Rev. A **57**, 1972 (1998).
[6] Ö. E. Müstecaplioğlu, Phys. Rev. A **68**, 023811 (2003).
[7] M. Brune, S. Haroche, J. M. Raimond, L. Davidovich, and N. Zagury, Phys. Rev. A **45**, 5193 (1992).
[8] C. C. Gerry, A. Benmoussa, and K. Elalouani, J. Mod. Opt. **53**, 1485 (2006).
[9] G. Agarwal, P. Lougovski, and H. Walther, J. Mod. Opt. **52**, 1397 (2005).
[10] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).
[11] M. Genovese and C. Novero, Phys. Rev. A **61**, 032102 (2000).
[12] K. Tara, G. S. Agarwal, and S. Chaturvedi, Phys. Rev. A **47**, 5024 (1993).
[13] C. J. Villas-Bôas, F. R. de Paula, R. M. Serra, and M. H. Y. Moussa, Phys. Rev. A **68**, 053808 (2003).
[14] W. Marshall, C. Simon, R. Penrose, and D. Bouwmeester, Phys. Rev. Lett. **91**, 130401 (2003).
[15] D. Kleckner and D. Bouwmeester, Nature (London) **444**, 75 (2006).
[16] X. Wang and B. C. Sanders, Phys. Rev. A **65**, 012303 (2001).
[17] H. Jeong, Phys. Rev. A **72**, 034305 (2005).
[18] M. W. Noel and C. R. Stroud, Jr., Phys. Rev. Lett. **77**, 1913 (1996).
[19] C. Monroe, D. Meekhof, B. King, and D. Wineland, Science **272**, 1131 (1996).
[20] A. Ourjoumtsev, R. Tualle-Brouri, J. Laurat, and P. Grangier, Science **312**, 83 (2006).
[21] It has been pointed out that Schrödinger cat states are particularly susceptible to dephasing. However, we will show in a later paper that collections of Schrödinger cat states possess decoherence free subspaces.
[22] B. C. Sanders, Phys. Rev. A **45**, 6811 (1992).
[23] A. Luis, Phys. Rev. A **64**, 054102 (2001).
[24] N. Linden, S. Popescu, and A. Sudbery, Phys. Rev. Lett. **83**, 243 (1999).
[25] D. A. Meyer and N. R. Wallach, in *The Mathematics of Quantum Computation*, edited by R. K. Brylinski and G. Chen (Chapman and Hall CRC Press, Boca Raton, FL, 2002), pp. 77–97.
[26] H. Jeong and N. B. An, Phys. Rev. A **74**, 022104 (2006).
[27] S. J. van Enk, Phys. Rev. Lett. **91**, 017902 (2003).
[28] R. Simon, in *Quantum Information with Continuous Variables*, edited by S. L. Braunstein and A. K. Pati (Kluwer Academic, Dordrecht, The Netherlands, 2003), pp. 155–172.
[29] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).
[30] D. J. Tannor, *Introduction to Quantum Mechanics: A Time Dependent Perspective* (University Science Books, Sausalito, 2007).
[31] J. Parker and C. R. Stroud, Jr., Phys. Rev. Lett. **56**, 716 (1986).
[32] M. S. Kim, W. Son, V. Buzek, and P. L. Knight, Phys. Rev. A **65**, 032323 (2002).
[33] S. J. van Enk and O. Hirota, Phys. Rev. A **64**, 022313 (2001).
[34] P. van Loock, Fortschr. Phys. **50**, 1177 (2002).
[35] H. Barnum, E. Knill, G. Ortiz, and L. Viola, Phys. Rev. A **68**, 032308 (2003).
[36] D. Meyer and N. Wallach, J. Math. Phys. **43**, 4273 (2002).
[37] R. Somma, G. Ortiz, H. Barnum, E. Knill, and L. Viola, Phys. Rev. A **70**, 042311 (2004).
[38] P. Love, A. van Brink, A. Y. Smirnov, M. Amin, M. Grajcar, E. Il'ichev, A. Izmalkov, and A. Zagoskin, Quantum Inf. Process. **6**, 187 (2007).
[39] These operators have many interesting properties that we will explore in a later paper.