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Optimal quantum-state reconstruction for cold trapped ions

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We study the physical implementation of an optimal tomographic reconstruction scheme for the case of determining the state of a multiqubit system, where trapped ions are used for defining qubits. The protocol is based on the use of mutually unbiased measurements and on the physical information described in H. Häffner *et al.* [Nature (London) **438**, 643 (2005)]. We introduce the concept of physical complexity for different types of unbiased measurements and analyze their generation in terms of one and two qubit gates for trapped ions.

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A main task in any experimental physical setup for implementing quantum computation is the ability to determine the output state of any given quantum algorithm [1]. The standard procedure applied for quantum state reconstruction of a density operator lying in a 2^N dimensional quantum system, in the case of N qubits, consists in projecting the density operator onto 3^N, completely factorized, bases in the corresponding Hilbert space [2]. All these measurements are obtained by applying rotations on single qubits (which are referred to as local operations) followed by projective measurements onto the logical basis. This was recently achieved for the case of eight qubits, with trapped ions [3]. The experiment was done by following the quantum computer architecture based on ions in a linear trap proposed by Cirac and Zoller [4]. Besides, the experimental implementations of several quantum protocols have also been reported by using trapped ions [5]. In all these cases the quality of the protocols is tested using standard tomography for quantum state determination. This scheme has also been used in the cases of considering optical setups [6] and NMR [7].

As mentioned above, in the standard measurement scheme only local operations are required to generate all the necessary projections. In each basis (setup), 2^N-1 independent measurements can be performed, so that not all the experimental outcomes obtained in different bases are linearly independent, that is, there are redundant measurements. In the case of a N-qubit system the antidiagonal elements have the larger errors. Actually, accumulated errors are not uniform; these errors depend on the number of single logic gates used for determining given elements, so that larger errors appear when single logic gates act on all the particles. Assuming that there is an error ε in the measurement of ion populations, then the accumulated error for antidiagonal elements is of the order of $\varepsilon\sqrt{2^{N-1}+2^{N-2}(2^N-1)}$. These errors may lead to a density operator which does not satisfy the positiveness condition and so the information from the experimental data must be optimized. For this purpose the maximum likelihood estimation (MLE) method [8] has been used for the improvement of the density operators in experiments with light qubits [9] as well as in experiments with matter qubits [5].

It is well known that the optimal quantum state determination is related to the concept of measurements on mutually unbiased bases (MUB) [10], which we will simply refer to as MUB tomography. Such bases possess the property of being maximally incompatible. This means that a state producing precise measurement results in one set produces maximally random results in all the others. The set of mutually unbiased projectors given by

$$P_n^{(\alpha)} = |\psi_n^{(\alpha)}\rangle\langle\psi_n^{(\alpha)}|, \quad n=1,\ldots,2^N,\ \alpha=1,\ldots,2^N+1, \tag{1}$$

where

$$\mathrm{Tr}(P_n^{(\alpha)}P_{n'}^{(\alpha')}) = \delta_{\alpha\alpha'}\delta_{nn'} + \frac{1}{2^n}(1-\delta_{\alpha\alpha'}) \tag{2}$$

and $\Sigma_{n=1}^{2^N}P_n^{(\alpha)}=I$ (I denotes the identity), defines a complete set of projection measurements, in the sense that the measured probabilities $p_{\alpha_n}=\mathrm{Tr}(P_n^{(\alpha)}\rho)$ completely determine the density operator of the system:

$$\rho = \sum_{\alpha=1}^{2^N+1}\sum_{n=1}^{2^N} p_{\alpha n}P_n^{(\alpha)} - I. \tag{3}$$

The number of MUBs for N qubits is (2^N+1), which is essentially less than 3^N. The use of MUBs can represent a considerable reduction in the time needed for performing the full state determination. For instance, recently the reconstruction of a quantum state codified in the inner states of eight ions in a linear trap was reported [3]. In this experiment the reconstruction process takes more than 10 hours, because of the measurement in 6561 different bases and a hundred of times for each one, so the number of measurements is about 656 000, which quickly grows for an increasing number of qubits. In the case of using MUB tomography, the number of measurement bases is only 257 for determining all the elements of the density operator associated with this state. This could reduce the experimental time, roughly speaking, to 25 min only, where we have considered a linear interpolation. Recently, an alternative reconstruction scheme has been proposed, which is based on using pyramidal states for single qubits [11,12]. The implementation of reconstruction using pyramidal states requires measurement of N-particle correlations [12].

In the case of MUB tomography, each coefficient $p_{\alpha n}$ in Eq. (3) has an error associated with the measurement of only one projector $P_n^{(\alpha)}$ (1). Hence, the error in each coefficient is essentially determined by the ability of projecting the system onto $P_n^{(\alpha)}$. In practice, such measurements are implemented by projecting the system onto the logical basis after performing a set of unitary transformations. Such transformations, due to their nonlocal features, can be decomposed into a sequence of single and nonlocal gates, so that the error in this reconstruction is mainly associated with the quality of these logic gates. Of course, the reconstructed density operator can be subjected to the MLE method.

The main shortcut of the MUB-tomography scheme is the experimental implementation of MUB projectors, which is related to the fact that any set of MUBs contains nonfactorizable bases. The measurements on such nonfactorizable bases require application of nonlocal gate operations, which currently can not be performed with fidelity 1. Hence, the natural question for optimizing the experimental implementation can be casted as follows: Which is the set of MUBs that requires the minimum number of nonlocal operations?

To approach this problem, we consider a complete set of MUBs where the basis factorization is denoted by the following set of natural numbers: $(k_1,k_2,\ldots,k_{\phi(N)})$, where $\phi(n)$ is the number of possible decompositions of 2^N+1 as a sum of positive numbers, such that $\Sigma_j k_j=2^N+1$. Here, the notation $(k_1,k_2,\ldots)$ means that there are k_1 completely factorized bases; k_2 bases with two particle entanglement and all the other particles are factorized, etc. Of course, for a given number of qubits, only a certain factorization structure is admitted. For instance, in the two-qubit case the only allowed structure is (3,2), which means that there are 3 completely factorized bases and 2 nonfactorized ones. In the case of 3 qubits, the bases can be described by the following notation: (n_f,n_b,n_{nf}), with $n_f+n_b+n_{nf}=2^3+1$. Here, n_f denotes the number of completely factorized bases; n_b the number of bases with bipartite entanglement, i.e., each state of such bases is factorized as $|\psi\rangle_{ij}|\varphi\rangle_k, i\neq j\neq k$, where $i,j,k=1,2,3$; and n_{nf} represents the number of nonfactorized. For 3 qubits, there are 4 different sets of MUBs [13,14], which are denoted as $(3,0,6)$, $(1,6,2)$, $(2,3,4)$ and $(0,9,0)$. Because any N-qubit entanglement operation can be decomposed into a sequence of single qubit gates and controlled-NOT gates (CNOT gates) [15], each basis (in the given set of MUBs) can be characterized by the minimum number CNOT gates.

In this work we shall concentrate on the experimental implementation of MUB tomography in the case of trapped ions as reported in Ref. [5]. In the case of $^{40}Ca^+$ trapped ions, where the qubit is codified in the ground state $|0\rangle\equiv S_{1/2}(m=-1/2)$ and the metastable $|1\rangle\equiv D_{5/2}(m=-1/2)$ state, single logic gates are implemented with a fidelity, Φ_{SG}, higher than 99%. However, nonlocal operations are less accurate; here we will assume the reported value for the CNOT gate fidelity, Φ_{CNOT}, which reaches a value up to 92(6)% [5]. Then we can characterize the *physical complexity* of each set of MUBs as a function of the number of nonlocal gates needed for implementing the projection measurements. Then, a complexity of a given MUB is characterized by the number C_j, which can be defined as

$$C_j = \ln\frac{1}{\Phi_{CNOT}^{n_j}} \propto n_j, \tag{4}$$

where n_j is the number of CNOT gates needed for generation of such a basis. The total complexity of a given set of MUBs is then

$$C = \sum_j C_j. \tag{5}$$

Thus the total complexity is proportional to the total number of CNOT gates for preparing all the required projection measurements. As a simplest example let us consider the case of 3 qubits, where there are only 4 different sets of MUBs. The bases labeled by (n_f,n_b,n_{nf}) have complexities $C\propto 0\times n_f+1\times n_b+2\times n_{nf}$. This means that the most adequate set for MUB-tomography is $(0,9,0)$. We have assumed, in this derivation, that the fidelity of the factorized bases is 1, in the sense that a sequence of single ion gates are required for their generation. We also assume that CNOT gates between neighboring ions have the same fidelity as between ions that are further apart, because CNOT gates are implemented by using the center of mass motion as a data bus. The decomposition of the corresponding bases can be generated by starting from the standard computational basis is that given in Table I.

TABLE I. Decomposition of the MUBs in the case of 3 qubits for $(0,9,0)$ structure, where $R_k^{(j)}=R_k^{(j)}(\pi/2)$, with $k=x,y,z$ are the single ions operations [16] and $\chi_{CNOT}^{(ij)}$ is the controlled-NOT gate with the ιth and jth ions as source and target, respectively.

Basis	Gate operations	Basis	Gate operations
1	$R_y^{(1)}R_x^{(3)}\chi_{CNOT}^{(23)}R_y^{(2)}$	6	$R_y^{(2)}R_z^{(3)}\chi_{CNOT}^{(13)}R_y^{(1)}$
2	$R_x^{(1)}R_y^{(3)}\chi_{CNOT}^{(23)}R_y^{(2)}$	7	$R_z^{(2)}R_x^{(3)}\chi_{CNOT}^{(12)}R_y^{(1)}$
3	$R_z^{(3)}\chi_{CNOT}^{(23)}R_y^{(2)}$	8	$R_y^{(2)}R_y^{(3)}\chi_{CNOT}^{(12)}R_y^{(1)}$
4	$R_y^{(3)}\chi_{CNOT}^{(13)}R_y^{(1)}$	9	$R_x^{(2)}\chi_{CNOT}^{(12)}R_y^{(1)}$
5	$R_x^{(2)}R_x^{(3)}\chi_{CNOT}^{(13)}R_y^{(1)}$		

In the 4-qubit case, there are 34 sets of MUBs with different factorizations, which are labeled by $(n_f,n_b,n_t,n_{bb},n_{nf})$, with $n_f+n_b+n_t+n_{bb}+n_{nf}=2^4+1$. Then, there are n_f factorized basis; n_b basis with bipartite entanglement, $|\psi\rangle_{ij}|\varphi\rangle_k|\varphi\rangle_l$; n_t basis with tripartite entanglement, $|\psi\rangle_{ijk}|\varphi\rangle_l$; n_{bb} basis with two bipartite subsystems, $|\psi\rangle_{ij}|\varphi\rangle_{kl}$; and n_{nf} nonfactorized basis, where $i\neq j\neq k\neq l$ and $i,j,k,l=1,2,3,4$. All such sets can be obtained in a regular way by applying finite phase-space methods[17]. We remark that always there exists the so-called "standard" set of MUBs, which is related to rays in the finite phase space and can be easily constructed starting with two classes of operators containing either $\hat{\sigma}_z$ or $\hat{\sigma}_x$ operators [18]. The 4 qubit case is essentially different from the above discussed 3 qubit case. The main difference consists in that now there exist two

TABLE II. The decompositions on single logic gates and controlled-NOT gates for the optimal set of MUBs $(0,0,12,2,3)$ in case of 4 qubits.

Basis	Gate operations
1	$R_x^{(4)}\chi_{\mathrm{CNOT}}^{(14)}\chi_{\mathrm{CNOT}}^{(12)}R_x^{(1)}$
2	$R_x^{(2)}\chi_{\mathrm{CNOT}}^{(31)}R_x^{(3)}\chi_{\mathrm{CNOT}}^{(41)}R_x^{(4)}$
3	$R_y^{(2)}R_x^{(1)}\chi_{\mathrm{CNOT}}^{(13)}\chi_{\mathrm{CNOT}}^{(14)}R_y^{(1)}$
4	$R_x^{(4)}R_y^{(2)}\chi_{\mathrm{CNOT}}^{(12)}R_x^{(1)}\chi_{\mathrm{CNOT}}^{(13)}R_y^{(1)}$
5	$R_y^{(1)}R_x^{(2)}\chi_{\mathrm{CNOT}}^{(34)}R_x^{(3)}\chi_{\mathrm{CNOT}}^{(24)}R_x^{(2)}$
6	$R_y^{(4)}R_x^{(3)}\chi_{\mathrm{CNOT}}^{(23)}R_x^{(2)}\chi_{\mathrm{CNOT}}^{(31)}R_x^{(3)}$
7	$R_y^{(1)}\chi_{\mathrm{CNOT}}^{(13)}\chi_{\mathrm{CNOT}}^{(12)}R_x^{(1)}$
8	$R_y^{(3)}R_y^{(1)}\chi_{\mathrm{CNOT}}^{(14)}R_x^{(1)}\chi_{\mathrm{CNOT}}^{(12)}R_y^{(1)}$
9	$R_x^{(1)}R_x^{(2)}R_x^{(4)}\chi_{\mathrm{CNOT}}^{(23)}\chi_{\mathrm{CNOT}}^{(24)}R_y^{(2)}$
10	$R_x^{(1)}\chi_{\mathrm{CNOT}}^{(34)}R_y^{(3)}\chi_{\mathrm{CNOT}}^{(14)}R_x^{(1)}$
11	$R_x^{(3)}\chi_{\mathrm{CNOT}}^{(24)}R_y^{(2)}\chi_{\mathrm{CNOT}}^{(14)}R_x^{(1)}$
12	$R_y^{(2)}R_y^{(3)}\chi_{\mathrm{CNOT}}^{(23)}\chi_{\mathrm{CNOT}}^{(24)}R_y^{(2)}$
13	$\chi_{\mathrm{CNOT}}^{(13)}R_y^{(1)}\chi_{\mathrm{CNOT}}^{(24)}R_x^{(2)}$
14	$R_y^{(2)}\chi_{\mathrm{CNOT}}^{(14)}R_y^{(1)}\chi_{\mathrm{CNOT}}^{(23)}R_y^{(2)}$
15	$R_y^{(4)}R_x^{(1)}\chi_{\mathrm{CNOT}}^{(34)}R_x^{(3)}\chi_{\mathrm{CNOT}}^{(32)}\chi_{\mathrm{CNOT}}^{(31)}R_x^{(3)}$
16	$R_x^{(3)}\chi_{\mathrm{CNOT}}^{(13)}R_x^{(1)}\chi_{\mathrm{CNOT}}^{(34)}R_x^{(3)}\chi_{\mathrm{CNOT}}^{(12)}R_y^{(1)}$
17	$R_x^{(2)}R_x^{(3)}\chi_{\mathrm{CNOT}}^{(12)}R_y^{(1)}\chi_{\mathrm{CNOT}}^{(23)}\chi_{\mathrm{CNOT}}^{(24)}R_x^{(2)}$

locally nonequivalent completely nonfactorized states. Such states are isomorphic to the so-called graph states [19–21]. The first type, A, are isomorphic to the eigenstates of the set $\{\hat{\sigma}_z\hat{\sigma}_x\hat{I}\hat{I},\ \hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x\hat{I},\ \hat{I}\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x, \hat{I}\hat{I}\hat{\sigma}_x\hat{\sigma}_z\}$. The second type, B, are isomorphic to the eigenstates of the set $\{\hat{\sigma}_z\hat{\sigma}_x\hat{I}\hat{I},\ \hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_x,\ \hat{I}\hat{\sigma}_x\hat{\sigma}_z\hat{I},\ \hat{I}\hat{\sigma}_x\hat{I}\hat{\sigma}_z\}$. Nevertheless, both types of sets in the optimal decomposition are obtained by applying 3 CNOT gates only. Taking into account that MUBs with factorizations $(1,3)$ and $(2,2)$ can be generated with 2 CNOT gates, we realize that the optimum set of MUBs corresponds to the factorization structure $(0,0,12,2,3)$, which only contains type A graph states. The complexity of such a set is $C\propto 37$ while, for instance, the standard set of MUBs, $(3,0,0,2,12)$, has a complexity $C\propto 40$.

The optimum set of MUBs corresponds to a set of 255 disjoint operators which are arranged in a table consisting of 17 lines, so that each line contains 15 commuting operators [14]. The whole table can be obtained from only 8 elements arranged in two lines of commuting operators,

$\hat{1}\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_y$	$\hat{\sigma}_x\hat{\sigma}_x\hat{1}\hat{\sigma}_x$	$\hat{1}\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x$
$\hat{1}\hat{1}\hat{\sigma}_z\hat{1}$	$\hat{\sigma}_y\hat{\sigma}_x\hat{1}\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\hat{1}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{1}\hat{\sigma}_z$

(6)

with factorization (4) and $(1,3)$ for the first and second row, respectively. All other operators of the above table can be generated by using the following rule: $A_{r,c}=A_{r,c-3}\times A_{r,c-4}$ and $A_{r,c}=A_{2,c}\times A_{1,r+c-3}$ for $r>2$, where the index r (c) labels the rth MUB (cth operator which has the basis states as eigenvectors of the rth MUB), and r (c) goes from 1 to 2^4+1 (1 to 2^4-1). Here, the sums are modulo 15. The 4 qubit case is somewhat special, because the standard table contains two bi-factorized bases $(2,2)$ and so there is no substantial difference with respect to the optimum set. This apparently does not occur for a larger number of qubits, i.e, the standard table for$N>4$ would contain 2^N-2 completely nonfactorized bases. The decompositions for the optimal set of MUBs is given in Table II.

The situation is quite different in the 5 qubit case, in which there exist at least 9000 nonisomorphic sets of MUBs with different factorizations, which are labeled by $(n_f,n_b,n_t,n_{bb},n_f,n_{bt},n_{nf})$, where n_f denotes the number of bases with four-particle entanglement, $|\psi\rangle_{ijkl}|\varphi\rangle_n$, and n_{bt} the number of bases with bipartite and tripartite entanglement, $|\psi\rangle_{ijk}|\varphi\rangle_{lm}$, with $i\neq j\neq k\neq l\neq m$ and $i,j,k,l,m=1,2,3,4,5$. In the case of 5 qubits, there are four locally nonequivalent completely nonfactorized states [22]. In this case the standard table does not contain partially factorized bases and has the structure $(3,0,0,0,0,0,30)$, and it is given by

$\hat{1}\hat{1}\hat{\sigma}_z\hat{\sigma}_z\hat{1}$	$\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\hat{1}\hat{\sigma}_z$	$\hat{1}\hat{1}\hat{1}\hat{1}\hat{\sigma}_z$	$\hat{1}\hat{\sigma}_z\hat{1}\hat{\sigma}_z\hat{1}$	$\hat{1}\hat{1}\hat{1}\hat{\sigma}_z\hat{1}$
$\hat{1}\hat{1}\hat{\sigma}_x\hat{\sigma}_x\hat{1}$	$\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{1}\hat{\sigma}_x$	$\hat{1}\hat{1}\hat{1}\hat{1}\hat{\sigma}_x$	$\hat{1}\hat{\sigma}_x\hat{1}\hat{\sigma}_x\hat{1}$	$\hat{1}\hat{1}\hat{1}\hat{\sigma}_x\hat{1}$

.

The set of MUBs corresponds to 1023 disjoint operators which are arranged in a table consisting of 33 lines, so that each line contains 31 commuting operators. All the other projectors of this table are obtained by using the following rule: $A_{r,c+18}=A_{r,c}\times A_{r,c+1}$ and $A_{r,c}=A_{1,c}\times A_{2,r+c-2}$, for $r>2$, where the sums are modulo 31. This table contains 30 nonfactorized bases among which there are three types corresponding to different graphs: 6 of type B, 18 of type C, and 6 of type D (graphs 6, 7, and 8 in Fig. 4 of Ref. [22]). It results that 4 CNOT gates are required to generate bases of type B and C; nevertheless, the minimum number of CNOT gates to obtain the type D graphs is 5. This means that one needs 126 nonlocal operations to generate the whole set of MUBs, with the corresponding complexity $C\propto 126$. The optimum set has the structure $(0,0,1,3,10,2,17)$ corresponding to a complexity $C\propto 112$, and contains one nonfactorized basis of type B and 16 bases of type C. In this case the set of operators needed for generating the whole table is

$\hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_x\hat{\sigma}_z\hat{1}\hat{\sigma}_y\hat{\sigma}_y$	$\hat{\sigma}_z\hat{\sigma}_z\hat{1}\hat{1}\hat{\sigma}_x$	$\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_z$	$\hat{\sigma}_x\hat{1}\hat{\sigma}_z\hat{\sigma}_x\hat{\sigma}_z$
$\hat{1}\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_z$	$\hat{\sigma}_z\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_z\hat{\sigma}_x$	$\hat{\sigma}_z\hat{\sigma}_z\hat{1}\hat{\sigma}_z\hat{\sigma}_z$	$\hat{\sigma}_z\hat{1}\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x$	$\hat{1}\hat{\sigma}_z\hat{\sigma}_x\hat{1}\hat{1}$

.

The rule for generating the other projectors is the same as in the standard table. For this table the decompositions on single logic gates and CNOT gates can be obtained in the same way as for the above discussed cases.

We have studied the problem of optimal tomographic reconstruction of a density operator of systems of N cold trapped ions. The optimality of a given set of MUBs is essentially defined in terms of the minimum number of required conditional operations. We have given an explicit form of operations to generate the optimal set of MUBs for the case of 3 and 4 qubits, and the factorization operations for the 5 qubit case can be obtained in the same way. In the case of larger numbers of qubits, a generic procedure allowing generation of the whole set of MUBs is also available. It basically consists in finding all the possible strings of $2^N - 1$ commuting operators (using explicit geometrical construction) with a subsequent separation onto the 2^N+1 disjoint sets [23]. Eigenstates of such commuting operators in each disjoint set generate one of the corresponding MUBs [17]. This reconstruction scheme is valid for any physical setup, where nonlocal operations between neighboring qubits have the same fidelity as between distant qubits. This is satisfied in the case of trapped ions because a CNOT gate, between any pair of ions, is implemented by using the center of mass motion as a data bus. If the physical implementation does not meet this requirement, the optimal MUBs to be used for reconstructing the state must be determined by considering the fidelities between non-neighboring qubits.

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