# Extended photon correlation in a negative-temperature medium

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We show how two-photon correlation of a two-level amplifier in the small signal regime depends on the atomic populations and the propagation length. We predict that inverted population associated with negative temperature gives a very long correlation time even in the presence of decoherence, a useful asset for quantum communication. The correlation vanishes for very dilute atomic gas. Analytical solutions for the field operators obtained by Fourier transform and Laplace transform (with initial condition) of the time variable appear very different but yield identical numerical results except for certain parameters. The physical explanation behind the deviation is given. The presence of thermal photons is found to reduce the correlation time.

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## I. INTRODUCTION

The two-level laser amplifier was studied many years ago in the interest of pulse propagation [1]. An interesting aspect of this scheme involves a collection of inverted two-level atoms, associated with negative temperature [2]. When the inversion is established by sweeping through a delta function $\pi$ pulse, it produces a gain-swept medium with superradiant emission of unrestricted gain [3]. Semiclassical theory has been developed for both linear and nonlinear regimes [4] to study the transient dynamics of superradiant pulse. Full quantum theory [5–7] has been used to describe the initiation stage, superfluorescence. The gain swept concept is recently proposed for stand-off detection of impurities in the atmosphere [8].

Here, we take a new look into this system in the interest of quantum correlation, particularly two-photon correlation. As in our previous works [9,10] we use the Heisenberg-Langevin equations coupled to the Maxwell's equations for quantized macroscopic field, which incorporates quantum noise of the vacuum fluctuations in a transparent way. We neglect transverse diffraction by considering a rod with length $L$ and radius $w$ such that the Fresnel number $F = \pi w^2/\lambda L \simeq 1$. We focus on the small signal regime which allows for analytical solution of the field in transient regime. Our study shows that populations and optical density affect the correlation in a useful way. We also study the effect of thermal photons on the correlation.

## II. QUANTUM NOISE-PROPAGATION THEORY

The amplifier is composed of two-level atoms (Fig. 1) with finite population in the upper level. Amplification energy comes from a constant supply of inverted atoms [3]. The transition of each atom is not cycled since there is no laser field. Amplification occurs via stimulated emissions of excited atoms. In a large signal regime, the role of quantum noise becomes relatively unimportant and can be neglected, so the semiclassical approach is adequate. However, in the small signal regime, the inverted atoms generate photons from spontaneous emissions, so the quantum noise is important.

The Hamiltonian for many particles is written as

$$H = \sum_{j,\varsigma=a,b} \hbar \omega_\varsigma |\varsigma^j\rangle\langle\varsigma^j| + \sum_{\mathbf{k},\lambda} \left(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{2}\right)\hbar \nu_{\mathbf{k}} - \left[\hbar \sum_{j,\mathbf{k}} |a^j\rangle\langle b^j| \{g^j_{\mathbf{k}}\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_j} + g^j \hat{E}(z,t)\} + \text{adj}\cdot\right]. \quad (1)$$

After coarse graining the atomic parameters into continuous variables, we obtain the Heisenberg-Langevin equations (including thermal photons) with Maxwell's equations, given in the Appendix. We focus on the small signal regime where the buildup of the macroscopic field $\hat{E}(z,t)$ is sufficiently small that it does not significantly alter the populations along the sample, so $\rho_{\alpha\alpha}(z,t) \simeq \rho_{\alpha\alpha}(0,t)$ $(\alpha = a,b,c)$. Equations (A3) and (A4) become linear and can be solved either by Fourier-Laplace transform of double Laplace transform of the time and space variables.

### A. Fourier-Laplace transform method

We obtain the inhomogeneous wave equation for the macroscopic field in spectral domain,

$$\left(\mathcal{G} + \frac{\partial}{\partial z}\right)\hat{E}(z,\nu) = i\kappa \frac{\mathcal{F}\{\hat{F}^\dagger_{ab}(z,t)\}}{T^*_{ab}(\nu)}, \quad (2)$$

where $\mathcal{G}(\nu) = -\frac{\kappa g n_{ab}}{T^*_{ab}(\nu)} - i\frac{\nu}{c}$ is the gain factor in spectral domain, $\kappa = N\frac{\wp\omega_{ab}}{2c\varepsilon_o}$ with $N$ as the number density, and $n_{ab} = \rho_{aa} - \rho_{bb}$ is the inversion. The $\mathcal{F}_\nu$ denotes a Fourier transform, $\hat{F}_{ab}(z,t)$ is the noise operator of the quantum Langevin Eq. (A1), $T^*_{ab}(\nu) = \gamma_{ab} - i(\nu - \omega_{ab})$, $\gamma_{ab} = \frac{\Gamma(2\bar{n}+1)}{2} + \gamma^{ph}_{ab}$ is the decoherence

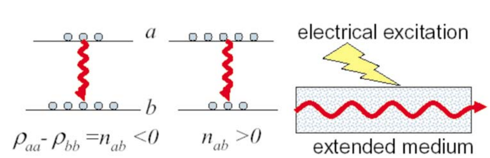


FIG. 1. (Color online) Noninverted (left) or inverted two-level system produces propagation of macroscopic field by amplified spontaneous emissions in an extended medium.

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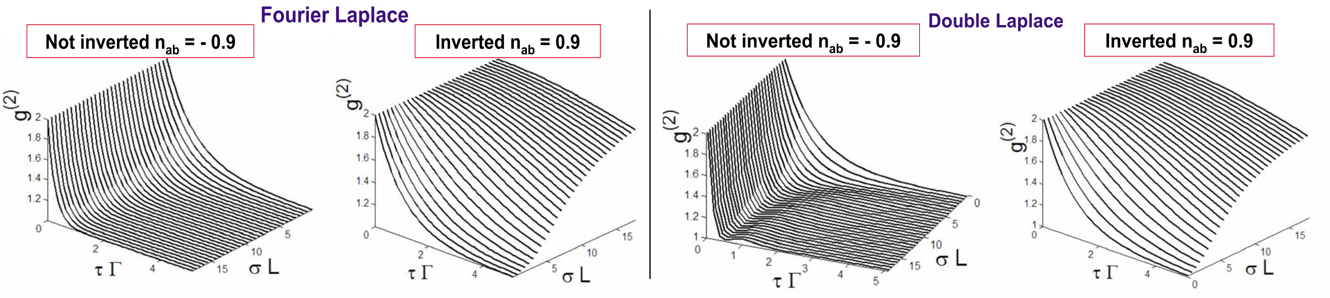


FIG. 2. (Color online) Variations of the normalized correlation $g^{(2)}$ with dimensionless delay $\tau\Gamma$ and the dimensionless length or optical density $\sigma L = NL\frac{|\wp|^2\omega_{ab}}{c\varepsilon_o\hbar\gamma_{ab}}$ of the medium for cases with noninverted ($n_{ab}<0$) and inverted ($n_{ab}>0$) populations. Note the excellent agreement of the results based on the Laplace transform and Fourier transform methods for the inverted case. The Laplace transform method gives a dip in each correlation profile for the noninverted case, especially for large optical density $\sigma L$. We assume $t=100/\Gamma$, $\bar{n}=0$, and $\gamma_{ab}^{ph}=10^6 s^{-1}$.

rate which depends on thermal photon number $\bar{n}$, dephasing rate $\gamma_{ab}^{ph}$ and spontaneous emission rate $\Gamma$.

The solution for the field at $L$ is composed of the boundary operator and the noise operator

$$\hat{E}(L,\nu) = \hat{E}(0,\nu)e^{-\mathcal{G}(\nu)L} + i\kappa\int_0^L e^{-\mathcal{G}(\nu)\xi}\frac{\mathcal{F}_\nu\{\hat{F}_{ab}^\dagger(z,t)\}}{T_{ab}^*(\nu)}dz, \quad (3)$$

where $\xi = L - z$.

Note that the adjoint $\hat{E}^\dagger(0,\nu)$ does not contribute to the solution, in contrast to the solutions of the couple parametric equations for cascade scheme [9] and double $\Lambda$ scheme [10]. The intensity is readily computed,

$$I(L) = \int_{-\infty}^{\infty} d\nu' \int_{-\infty}^{\infty} d\nu e^{i(\nu'-\nu)t}\langle\hat{E}^\dagger(0,\nu')\hat{E}(0,\nu)\rangle e^{\mathcal{S}(\nu)\sigma L} - C_\Pi\frac{2D}{2\gamma_{ab}n_{ab}}\int_{-\infty}^{\infty}[1 - e^{n_{ab}\mathcal{S}(\nu)\sigma L}]d\nu, \quad (4)$$

where $C_\Pi = \frac{2\pi}{AN}\frac{\kappa^*}{g} = \frac{\pi\hbar\omega_{ab}}{Ac\varepsilon_o}$, $A$ is the cross section of the amplifying media, $\sigma = 2\kappa g/\gamma_{ab}$ and $\mathcal{S}(\nu) = \gamma_{ab}^2/[\gamma_{ab}^2 + (\nu - \omega_{ab})^2] = -(\mathcal{G}(\nu) + \mathcal{G}^*(\nu))/\sigma$ is a Lorentzian lineshape with width $\gamma_{ab}$. The diffusion coefficient

$$2D = [2\gamma_{ab} - \Gamma(2\bar{n}+1)]\rho_{aa} + \Gamma\bar{n}\rho_{bb} \quad (5)$$

is obtained from Einstein's relation through $\langle\hat{F}_{ab}(z',t')\hat{F}_{ab}^\dagger(z,t)\rangle = \frac{2D(z,t)}{AN}\delta(t-t')\delta(z-z')$. We may take the initial values for the populations $\rho_{\alpha\alpha}(t) \simeq \rho_{\alpha\alpha}(0)$.

In the absence of external input field and negligible number of thermal photons $\bar{n} \ll 1$, the first term does not contribute since the normal ordered correlation $\langle\hat{E}^\dagger(0,\nu')\hat{E}(0,\nu)\rangle \propto \bar{n}$.

We then obtain the two-photon correlation at the output of the amplifier of length $L$:

$$G^{(2)}(L,\tau) = |B(L,\tau) + \Pi(L,\tau)|^2 + |B(L,0) + \Pi(L,0)|^2, \quad (6)$$

where the correlation due to boundary operator is

$$B(L,\tau) = \frac{\bar{n}\hbar\nu_f}{4\pi\varepsilon_o Ac}\int_{-\infty}^{\infty} e^{i\nu\tau}\exp\{n_{ab}\mathcal{S}(\nu)\sigma L\}d\nu \quad (7)$$

and the correlation due to noise operator is

$$\Pi(L,\tau) = -\frac{C_\Pi 2D}{2\gamma_{ab}n_{ab}}\int_{-\infty}^{\infty} e^{i\nu\tau}[1 - e^{n_{ab}\mathcal{S}(\nu)\sigma L}]d\nu, \quad (8)$$

where $\nu_f$ in $B(L,\tau)$ is the carrier frequency of the quantum field, which should be discerned from the Fourier frequency $\nu$.

From Eqs. (7) and (8), we find that for small length $L$, the ratio of the correlation due to noise ("$n$") and that due to boundary ("$b$") operators is

$$\frac{G^{(2)n}}{G^{(2)b}} \sim \left(\frac{2D\Gamma}{2\gamma_{ab}^2}N3\pi\lambda^2 L\right)^2 \quad (9)$$

which is similar to the result for multilevel case [11], with a value greater than unity for typical optical parameters in small signal regime.

Note that for $\bar{n}=0$, the correlation due to the noise operators vanishes if $\gamma_{ab}=\Gamma/2$ or $D=0$. This is the case for the medium composed of single atom or dilute gas where $\gamma_{ab} = \Gamma/2$, i.e., there is no collisional dephasing, $\gamma_{ab}^{ph}=0$. This result means that photons spontaneously emitted by widely separated (independent) atoms would not be correlated. For dilute gas $[\Pi(L,\tau)=0]$, finite correlation due to $B(L,\tau)$ may arise only in the presence of thermal photons. The correlation diverges (ill defined) when the levels are equally populated $n_{ab}=0$.

The correlation may be written in the normalized form

$$g^{(2)}(L,\tau) = \frac{|\langle\hat{E}^\dagger(L,t+\tau)\hat{E}(L,t)\rangle|^2}{I(L)^2} + 1, \quad (10)$$

where $\langle\hat{E}^\dagger(L,t+\tau)\hat{E}(L,t)\rangle = B(L,\tau) + \Pi(L,\tau)$. When $\bar{n}=0$ the boundary part gives no contribution, so

$$g^{(2)}(L,\tau) = \frac{\left| \int_{-\infty}^{\infty} e^{i\nu\tau}(1 - e^{n_{ab}S(\nu)\sigma L})d\nu \right|^2}{\left[ \int_{-\infty}^{\infty} (1 - e^{n_{ab}S(\nu)\sigma L})d\nu \right]^2} + 1. \quad (11)$$

In this system the correlation shows bunching; Fig. 2. This is expected since there is no driving field to create interference in the transitions. We have computed the intensity versus frequency and find narrowing in the spectrum as the medium length increases.

### B. Double Laplace transform method

We obtain the solution by *double Laplace transforms* which includes both boundary and initial conditions

$$\hat{E}(L,t) = \hat{E}(0,t-L/c) + \hat{E}(L-ct,0) + \int_0^t \Psi_1(L,\tau')\hat{E}(0,t')dt' + \int_0^L \left[ \frac{1}{c}\Psi_1(\xi,t)\hat{E}(z,0) + i\kappa\Psi_0(\xi,t)\hat{\rho}_{ba}(z,0) + \int_0^t i\kappa\Psi_0(\xi,\tau')\hat{F}^{\dagger}_{ab}(z,t')dt' \right] dz, \quad (12)$$

$$\Psi_1(z,t) = e^{-T^*_{ab}(t-z/c)}\sqrt{\frac{b}{t-z/c}}I_1(2\sqrt{b(t-z/c)}), \quad (13)$$

$$\Psi_0(z,t) = e^{-T^*_{ab}(t-z/c)}I_0(2\sqrt{b(t-z/c)}), \quad (14)$$

where $\tau'=t-t'$, $\xi=L-z$ and $b=n_{ab}g\kappa L$.

Equation (12) generalizes the result of Ref. [8] to include the quantum noise operator. In order to calculate $\langle \hat{E}^{\dagger}(L,t+\tau)\hat{E}(L,t)\rangle$ we use

$$\langle \hat{\rho}_{ab}(z',0)\hat{\rho}_{ba}(z,0)\rangle = \frac{\rho_{aa}(z,0)}{AN}\delta(z-z')$$

which is derived from $\hat{\rho}_{\beta\alpha}(z,t)=\frac{1}{AN}\Sigma_{j=1}^{n}\hat{\sigma}^{j}_{\beta\alpha}(t)\delta(z-z_j)$ and $\rho_{aa}(z,0)\delta(z-z')=\frac{1}{AN}\Sigma_{j=1}^{n}\rho^{j}_{aa}(0)\delta(z-z_j)\delta(z'-z_j)$ where $\rho_{aa}=\langle\hat{\rho}_{aa}\rangle$.

We now obtain the correlation

$$\langle \hat{E}^{\dagger}(L,t+\tau)\hat{E}(L,t)\rangle \simeq \frac{\kappa^2}{AN}\int_0^L \left[ \rho_{aa}(z,0)\mathcal{Z}(\xi,t+\tau) + \int_0^t 2D(z,t')\mathcal{Z}(\xi,t-t'+\tau)dt' \right] dz + R(z,t,\tau), \quad (15)$$

where $\mathcal{Z}(\xi,T+\tau)=\Psi^*_0(\xi,T+\tau)\Psi_0(\xi,T)$ and $2D(z,t')$ is given by Eq. (5). The transient populations are taken to be the initial values for consistency with the previous assumption used to linearize the coupled equations. Equation (15) gives the intensity $I(L,t)=\langle \hat{E}^{\dagger}(L,t)\hat{E}(L,t)\rangle$ by setting $\tau=0$.

The remaining term $R(L,t,\tau)$ that depends on the boundary conditions (the initial and boundary fields) is

$$R(L,t,\tau) = \langle \hat{E}^{\dagger}(0,t+\tau-L/c)\hat{E}(0,t-L/c)\rangle + \int_0^t\int_0^t \Psi^*_1(L,\tau''+\tau)\Psi_1(L,\tau') \times\langle \hat{E}^{\dagger}(0,t'')\hat{E}(0,t')\rangle dt'dt'' + \langle \hat{E}^{\dagger}(L-c(t+\tau),0)\hat{E}(L-ct,0)\rangle + \frac{1}{c^2}\int_0^L\int_0^L \Psi^*_1(\xi',t+\tau)\Psi_1(\xi,t) \times\langle \hat{E}^{\dagger}(z',0)\hat{E}(z,0)\rangle dz'dz. \quad (16)$$

Note that the commutator

$$[\hat{E}(0,t),\hat{E}^{\dagger}(0,t')] = (\hbar\nu_f/2\varepsilon_o\mathcal{V})\sum_{\mathbf{k}} e^{-i\nu_{\mathbf{k}}(t-t')} \to \frac{\hbar\nu_f}{2\varepsilon_o(2\pi)^3}\int 4\pi k^2 e^{-ikc(t-t')}dk$$

is zero unless $t=t'$. For classical initial or input fields which are only narrowly correlated, $\langle \hat{E}^{\dagger}(0,t)\hat{E}(0,T)\rangle=I(t)\delta(t-T)\tau_c$ and $\langle \hat{E}^{\dagger}(z,0)\hat{E}(z',0)\rangle=I(z,0)\delta(z-z')l_c$, where $\tau_c$ is the coherence time of the boundary field and $l_c\simeq c\tau_c$ is the coherence length of the initial field. Then, Eq. (16) becomes

$$R(L,t,\tau) = \delta(\tau)\tau_c[\langle \hat{I}(0,t-L/c)\rangle + \langle \hat{I}(L-ct,0)\rangle] + \tau_c\int_0^t \Psi^*_1(L,\tau'+\tau)\Psi_1(L,\tau')I(0,t')dt' + \frac{l_c}{c^2}\int_0^L \Psi^*_1(\xi,t+\tau)\Psi_1(\xi,t)I(z,0)dz. \quad (17)$$

In the absence of input or initial field we have shown rigorously that $R(L,t,\tau)=0$. Thus, the correlation Eq. (15) does not depend on the Bessel function $I_1$. In order to compare the results of the correlation from the two methods of solutions, we focus on the case without input field. Hence, the photon-photon correlation can be computed from Eq. (10). Good agreement between the Fourier-Laplace transform and double Laplace transform methods is obtained in Fig. 2(b), plotted using Eqs. (11), (6)–(8), and (15) by assuming constant populations, and hence a constant diffusion coefficient.

## III. DISCUSSIONS

We find that the correlation depends on the relative population $n_{ab}$ and propagation length $L$, but remains classical i.e., bunching. Unlike resonance fluorescence [12], the small signal amplifier in the absence of driving field does not show antibunching. The width of the correlation changes in an opposite manner for inverted and noninverted atoms as the length increases (Fig. 2).

When the populations are mostly in the ground level $n_{ab}<0$, the coherence time between photons reduces with amplifier length. On the other hand, the coherence time increases significantly with the length for the inverted ($n_{ab}>0$) case. In other words, population inversion increases the

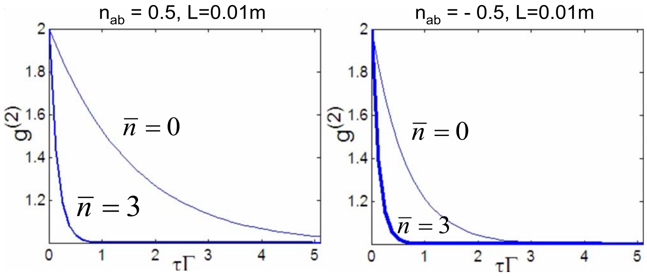


FIG. 3. (Color online) Effect of thermal photons $\bar{n}=3$ on the correlation time for both noninverted and inverted cases.

correlation time of photon pairs. This shows that a gain medium can sustain quantum correlation against decoherence, another magical effect of negative temperature [2]. This effect could be a key mechanism for producing correlated photon pairs with a long time delay for quantum communication protocols.

It is remarkable that the results based on the Fourier transform method coincide very well with those based on the Laplace transform for the inverted case [see Fig. 2(b)], even though the analytical solutions of the field operators are very different [compare Eqs. (12) and (13)]. The agreement is obtained only if we make a consistent approximation of taking the populations to be the initial values throughout the calculations for both methods.

However, for the noninverted case ($\rho_{bb} \gg \rho_{aa}$) and large optical density, the result (Fig. 2) by Laplace transform generally agrees with the Fourier transform result except that it shows a peculiar dip in the correlation. This feature is due to the oscillatory behavior of the Bessel function $I_0$ when its argument becomes imaginary for $n_{ab}<0$. The solution Eq. (11) based on Fourier transform does not contain any oscillatory function. This reflects the limitation of the Fourier transform method for solving the time-dependent problem, and the breakdown of linear approximation in neglecting excitation of the ground population by the macroscopic field.

The presence of thermal photons reduces the correlation time. The correlation decays more rapidly (Fig. 3). Thermal photons provide incoherent pumping which tend to destroy the quantum correlation between photons.

To conclude, we have presented a full quantum theory for the two-photon correlation $G^{(2)}$ of macroscopic field generated in a two-level amplifier operating in linear regime, and found interesting features in the quantum correlations based on two methods of solutions, the Laplace and Fourier transforms. We notice that more incoherent photons (which are correlated) from an inverted ensemble of atoms lead to longer correlation time.

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## APPENDIX: HEISENBERG-LANGEVIN AND PROPAGATION EQUATIONS

The coupled equations are

$$\left(\frac{\partial}{\partial t}+\Gamma(\bar{n}+1)\right)\hat{\rho}_{aa}=ig\hat{\rho}_{ab}\hat{E}+\text{adj.}+\Gamma\bar{n}\hat{\rho}_{bb}+\hat{F}_{aa}, \tag{A1}$$

$$\frac{\partial}{\partial t}\hat{\rho}_{bb}+\frac{\partial}{\partial t}\hat{\rho}_{aa}=0, \tag{A2}$$

$$\left(\frac{\partial}{\partial t}+T_{ab}\right)\hat{\rho}_{ab}=-ig^{*}\hat{E}^{\dagger}(\hat{\rho}_{bb}-\hat{\rho}_{aa})+\hat{F}_{ab}, \tag{A3}$$

where $T_{ab}=\gamma_{ab}-i\omega_{ab}$ and $\gamma_{ab}=\frac{1}{2}\Gamma(2\bar{n}+1)+\gamma_{ab}^{ph}$ with other symbols defined in the text. These equations are coupled to the propagation equation for field operator

$$\left(\frac{1}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)\hat{E}(z,t)=i\kappa\hat{\rho}_{ba}(z,t). \tag{A4}$$

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