Quantum phase transition in the generalized Dicke model: Inhomogeneous coupling and universality

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The Dicke model generalized to the case where the dependence of the coupling constant on individual atoms is taken into account is investigated. It is shown that the generalized Dicke model exhibits a quantum phase transition from the normal phase to the superradiant phase in the thermodynamic limit, as well as the standard Dicke model. The mean photon number, the atomic inversion, the lowest excitation energy, and the entanglement entropy between the field and the atoms are evaluated analytically or numerically, and the critical behavior is examined. As a result, it turns out that the critical behavior in the generalized model is essentially the same as that in the standard model. This implies universality in the Dicke model.

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I. INTRODUCTION

The quantum phase transition (QPT), which occurs at zero temperature and is due to a qualitative change of the ground state, is one of the most interesting phenomena in quantum many-body systems [1]. The QPT has also recently attracted much attention from the viewpoint of quantum information theory (entanglement) [2–4].

The Dicke (superradiance) model in quantum optics [5] provides a simple example of the QPT [6] and has recently been well studied theoretically [7–16]. The Dicke model describes a system consisting of bosonic modes such as cavity modes and N identical two-level atoms interacting with the modes. In most cases, a single mode is assumed and the dependence of the coupling constant on the individual atoms is neglected [17,18]. Hereafter, we call such a model the standard Dicke model. The standard Dicke model exhibits a QPT from the normal phase, where the ground state mainly consists of the vacuum state of the mode and the atomic ground states, to the superradiant phase, where the mean boson number and the atomic inversion in the ground state are finite [6]. However, since experimental realization of the standard Dicke model is extremely difficult, such an intriguing QPT has never been observed so far.

Several generalizations of the model have been reported [9,10,12,14]. In this paper, we present a study of another type of generalized Dicke model: the dependence of the coupling constant on the individual atoms is taken into account, where the coupling constants are assumed to be real [19–21]. Such a model describes the situation where a standing-wave cavity is used and the spatial dimension of the atomic ensemble is large compared to the cavity resonance wavelength [18,22]. (In contrast, the standard Dicke model assumes that the size of the atomic ensemble is small compared to the wavelength [12,23,24].) The use of a standing-wave cavity and an extended atomic ensemble is substantially advantageous to the experimental realization of the model [23,25]. To our knowledge, however, it has been unclear whether such a generalized model exhibits a QPT. We show that the generalized Dicke model does exhibit a QPT from the normal phase to the superradiant phase in the thermodynamic limit and examine its critical behavior. The result implies *universality* in the Dicke model [26].

The Dicke-model QPT occurs when the coupling constant becomes comparable to the geometric mean of the excitation energies for the cavity mode and the two-level atoms. This condition is very difficult to satisfy for optical systems. Some proposals to overcome this problem have been reported [15,16]. The present generalization also allows one to use a standing-wave cavity for realization of the effective model proposed in Ref. [15], where a traveling-wave (ring) cavity was assumed [27]. Thus, the present generalization makes it more feasible to realize the effective model of Ref. [15].

II. GENERALIZED DICKE MODEL

The Hamiltonian of the generalized Dicke model studied here is given by ($\hbar=1$)

$$H=\omega a^{\dagger}a+\omega_0\sum_{j=1}^{N}\sigma_z^{(j)}+\sum_{j=1}^{N}\frac{\lambda_j}{\sqrt{N}}(a+a^{\dagger})(\sigma_-^{(j)}+\sigma_+^{(j)}), \quad (1)$$

where ω and ω_0 are the excitation energies for the cavity mode and the two-level atoms, respectively; N is the number of the atoms; $\sigma_z^{(j)}$ and $\sigma_{\pm}^{(j)}$ are the pseudospin operators for the jth atom satisfying the commutation relations $[\sigma_z^{(j)},\sigma_{\pm}^{(j')}]=\pm\sigma_{\pm}^{(j)}\delta_{j,j'}$ and $[\sigma_+^{(j)},\sigma_-^{(j')}]=2\sigma_z^{(j)}\delta_{j,j'}$; and λ_j is the coupling constant between the cavity mode and the jth atom, which has been assumed to be real. Here the distribution of $\{\lambda_j\}$ can be arbitrarily assumed. If all the coupling constants are equal to one another, this model becomes the standard Dicke model. Note that the rotating-wave approximation is not applied as well as in the standard Dicke model [6,7]. The system consisting of a standing-wave cavity mode and an atomic ensemble whose size is large compared to the cavity resonance wavelength can be described by the present model.

The present system has so-called parity symmetry as well as the standard Dicke-model system. The parity operator Π is defined as $\Pi=\exp[i\pi(a^{\dagger}a+J_z+N/2)]$ (see below for the

defintion of J_z). The eigenvalues of Π are ± 1. Since $[H,\Pi]=0$, the system is symmetric with respect to Π.

III. MULTIMODE HOLSTEIN-PRIMAKOFF REPRESENTATION

The theoretical approach to the standard Dicke model is based on the Holstein-Primakoff representation for the collective atomic operators defined as $J_z \equiv \Sigma_j \sigma_z^{(j)}$ and $J_\pm \equiv \Sigma_j \sigma_\pm^{(j)}$ [6,7,28]. Note, however, that the present Hamiltonian cannot be expressed with the collective operators, and consequently the theoretical treatment becomes difficult [20,21]. The present approach is based on the *multimode* Holstein-Primakoff representation. We first divide the atoms into many groups by their coupling constants. Here, for simplicity, we assume that the probability density of the distribution of $\{\lambda_j\}$ is finite from $\lambda_{\min}$ to $\lambda_{\max}$. The other more complicated distributions can be treated in a similar manner. We divide the atoms into $M+1$ groups by their coupling constants as follows:

$$\text{group } 0\text{: } \lambda_{\min} < \lambda_j < \lambda_{\min} + \frac{1}{2M}(\lambda_{\max} - \lambda_{\min}), \quad (2a)$$

$$\text{group } M\text{: } \lambda_{\max} - \frac{1}{2M}(\lambda_{\max} - \lambda_{\min}) < \lambda_j < \lambda_{\max}, \quad (2b)$$

$$\text{group } m\text{: } \lambda_{\min} + \frac{2m-1}{2M}(\lambda_{\max} - \lambda_{\min}) < \lambda_j < \lambda_{\min} + \frac{2m+1}{2M}(\lambda_{\max} - \lambda_{\min}) \quad (m = 1, \ldots, M-1). \quad (2c)$$

In the thermodynamic limit ($N \to \infty$), the number of the atoms belonging to the mth group, which is denoted by N_m, becomes large.

Next, we apply the Holstein-Primakoff representation to each group as follows:

$$J_+^{(m)} \equiv \sum_{\text{group } m} \sigma_+^{(j)} = b_m^\dagger \sqrt{N_m - b_m^\dagger b_m}, \quad J_-^{(m)} \equiv J_+^{(m)\dagger},$$

$$J_z^{(m)} \equiv \sum_{\text{group } m} \sigma_z^{(j)} = b_m^\dagger b_m - \frac{N_m}{2}, \quad (3)$$

where $[b_m, b_{m'}^\dagger] = \delta_{m,m'}$ and $[b_m, b_{m'}] = 0$ $(m, m' = 0, 1, \ldots, M)$.

Here we approximate the coupling constants of the atoms belonging to the mth group as

$$\lambda_j \simeq \Lambda_m \equiv \lambda_{\min} + \frac{m}{M}(\lambda_{\max} - \lambda_{\min}). \quad (4)$$

This approximation may be good if M is sufficiently large [29]. As a result, the present Hamiltonian becomes

$$H \simeq \omega a^\dagger a + \omega_0 \sum_{m=0}^{M} \left(b_m^\dagger b_m - \frac{N_m}{2} \right) + \sum_{m=0}^{M} \Lambda_m \sqrt{\frac{N_m}{N}} (a + a^\dagger) \times \left(b_m^\dagger \sqrt{1 - \frac{b_m^\dagger b_m}{N_m}} + \sqrt{1 - \frac{b_m^\dagger b_m}{N_m}} b_m \right). \quad (5)$$

IV. MEAN PHOTON NUMBER, ATOMIC INVERSION, AND CRITICAL POINT

To obtain the expectation values of a and b_m in the ground state, we displace the bosonic operators as $a \to a + \alpha$ and $b_m \to b_m - \beta_m$. We expand the square roots in the Hamiltonian and neglect the higher-order terms with respect to a and b_m. α and β_m are determined so that the first-order terms with respect to a and b_m in the Hamiltonian vanish. As a result, α and $-\beta_m$ become the expectation values of a and b_m, respectively, in the ground state [6]. The equations for α and β_m are as follows:

$$0 = \omega \alpha^* - \sum_{m=0}^{M} \Lambda_m \sqrt{\frac{N_m}{N}} \sqrt{1 - \frac{|\beta_m|^2}{N_m}} (\beta_m + \beta_m^*), \quad (6)$$

$$0 = -\omega_0 \beta_m^* + \Lambda_m \sqrt{\frac{N_m}{N}} \sqrt{1 - \frac{|\beta_m|^2}{N_m}} (\alpha + \alpha^*) \times \left(1 - \frac{\beta_m^* (\beta_m + \beta_m^*)}{2(N_m - |\beta_m|^2)} \right). \quad (7)$$

From these equations, it is found that α and β_m are real. By eliminating β_m from Eqs. (6) and (7), we obtain the following equation for α:

$$\alpha \left[1 - \sum_{m=0}^{M} \frac{4\Lambda_m^2 \frac{N_m}{N}}{\omega \sqrt{\omega_0^2 + 16\Lambda_m^2 \frac{\alpha^2}{N}}} \right] = 0. \quad (8)$$

Here we introduce $\bar{\lambda}$ and λ_c as

$$\bar{\lambda} \equiv \sqrt{\sum_{m=0}^{M} \Lambda_m^2 \frac{N_m}{N}}, \quad \lambda_c \equiv \frac{\sqrt{\omega \omega_0}}{2}. \quad (9)$$

From Eqs. (6)–(8), the following turns out: when $\bar{\lambda} \le \lambda_c$, there is only a trivial solution $\alpha = \beta_m = 0$; when $\bar{\lambda} > \lambda_c$, two nontrivial solutions exist. Thus, it turns out that the generalized Dicke model exhibits a QPT from the normal phase to the superradiant phase, as well as the standard Dicke model. In contrast to the standard Dicke model, the nontrivial solutions cannot be evaluated analytically. These are obtained by numerically solving Eqs. (6)–(8).

In the superradiant phase, the Hamiltonian corresponding to each nontrivial solution does not commute with the parity operator Π. This means that the parity symmetry of the ground state is spontaneously broken in the superradiant phase [6].

In the present model, the critical point is at $\bar{\lambda} = \lambda_c$. Note that λ_c is exactly the same as the critical point in the standard

Dicke model, where the critical point is at $\lambda_j=\lambda=\lambda_c$. Thus, this result is a natural generalization of that in the standard model.

Around the critical point ($|\bar{\lambda}-\lambda_c|/\lambda_c \ll 1$), we can analytically obtain the mean photon number $\bar{n}$ and the atomic inversion $\bar{J}_z$ in the superradiant phase as follows [30]:

$$\bar{n} \equiv |\alpha|^2 \simeq 2N\frac{\bar{\lambda}^2}{\omega^2}\frac{\lambda_c^4}{\sum_{m=0}^{M}\Lambda_m^4\frac{N_m}{N}}\left(1-\frac{\lambda_c^2}{\bar{\lambda}^2}\right), \tag{10}$$

$$\bar{J}_z + N/2 \equiv \sum_{m=0}^{M}|\beta_m|^2 \simeq \frac{4\bar{\lambda}^2}{\omega^2}\bar{n}. \tag{11}$$

On the other hand, those in the standard Dicke model ($\lambda_j=\lambda$) are

$$\bar{n} = 2N\frac{\lambda^2}{\omega^2}\left(1-\frac{\lambda_c^2}{\lambda^2}\right), \tag{12}$$

$$\bar{J}_z + N/2 = \frac{4\lambda^2}{\omega^2}\bar{n}. \tag{13}$$

Since $\Sigma_{m=0}^{M}\Lambda_m^4\frac{N_m}{N} \geq \lambda_c^4$ in the superradiant phase, the mean values in the generalized model with $\bar{\lambda}$ are equal to or smaller than those in the standard model where all the coupling constants are equal to $\bar{\lambda}$.

V. EXCITATION ENERGY

Next, we diagonalize the Hamiltonian and evaluate the lowest excitation energy. By using α and β_m determined above and moving to a position-momentum representation defined as

$$x = \frac{a+a^\dagger}{\sqrt{2\omega}}, \quad p_x = i\sqrt{\frac{\omega}{2}}(a^\dagger - a), \tag{14}$$

$$y_m = \frac{b_m+b_m^\dagger}{\sqrt{2\tilde{\omega}_m}}, \quad p_m = i\sqrt{\frac{\tilde{\omega}_m}{2}}(b_m^\dagger - b_m), \tag{15}$$

the Hamiltonian becomes

$$H = \frac{1}{2}(x \;\; \vec{y}^T)A\begin{pmatrix} x \\ \vec{y}\end{pmatrix} + \frac{1}{2}(p_x \;\; \vec{p}^T)\begin{pmatrix} p_x \\ \vec{p}\end{pmatrix}, \tag{16}$$

where unimportant constants have been dropped and

$$\tilde{\omega}_m \equiv \omega_0 + 2\Lambda_m\sqrt{\frac{N_m}{N}}\frac{\alpha\beta_m}{\sqrt{N_m(N_m-\beta_m^2)}}, \tag{17}$$

$$\vec{y}^T = (y_0 \cdots y_M), \quad \vec{p}^T = (p_0 \cdots p_M), \tag{18}$$

$$A \equiv \begin{pmatrix} \omega^2 & \vec{\gamma}^T \\ \vec{\gamma} & \mathrm{diag}(\omega_0'^2, \ldots, \omega_M'^2)\end{pmatrix}, \tag{19}$$

$$\omega_m'^2 \equiv \tilde{\omega}_m^2 + 2\tilde{\omega}_m\Lambda_m\sqrt{\frac{N_m}{N}}\frac{\alpha\beta_m}{\sqrt{N_m(N_m-\beta_m^2)}}\frac{2N_m-\beta_m^2}{N_m-\beta_m^2}, \tag{20}$$

$$\vec{\gamma} \equiv \begin{pmatrix} 2\sqrt{\omega\tilde{\omega}_0}\Lambda_0\sqrt{\frac{N_0}{N}}\frac{N_0-2\beta_0^2}{\sqrt{N_0(N_0-\beta_0^2)}} \\ \vdots \\ 2\sqrt{\omega\tilde{\omega}_M}\Lambda_M\sqrt{\frac{N_M}{N}}\frac{N_M-2\beta_M^2}{\sqrt{N_M(N_M-\beta_M^2)}}\end{pmatrix}. \tag{21}$$

$\mathrm{diag}(\cdots)$ denotes a diagonal matrix. Using the orthogonal matrix U satisfying $U^TAU=\mathrm{diag}(\Omega_x^2,\Omega_0^2,\ldots,\Omega_M^2)$, we move to another position-momentum representation

$$\begin{pmatrix} x \\ \vec{y}\end{pmatrix} = U\begin{pmatrix} x' \\ \vec{y}'\end{pmatrix}, \quad \begin{pmatrix} p_x \\ \vec{p}\end{pmatrix} = U\begin{pmatrix} p_x' \\ \vec{p}'\end{pmatrix}. \tag{22}$$

Then,

$$H = \Omega_x c_x^\dagger c_x + \sum_{m=0}^{M}\Omega_m c_m^\dagger c_m, \tag{23}$$

where

$$x' = \frac{c_x+c_x^\dagger}{\sqrt{2\Omega_x}}, \quad p_x' = i\sqrt{\frac{\Omega_x}{2}}(c_x^\dagger - c_x), \tag{24}$$

$$y_m' = \frac{c_m+c_m^\dagger}{\sqrt{2\Omega_m}}, \quad p_m' = i\sqrt{\frac{\Omega_m}{2}}(c_m^\dagger - c_m). \tag{25}$$

Thus, it turns out that the energies are given by $n_x\Omega_x + \Sigma_{m=0}^{M}n_m\Omega_m$ (n_x and n_m are nonnegative integers) and the ground state $|G\rangle$ satisfies $c_x|G\rangle=0$ and $c_m|G\rangle=0$.

In the normal phase, the excitation energies Ω_x and Ω_m ($m=0,\ldots,M$) can be analytically evaluated as

$$\{\Omega_x,\Omega_0,\Omega_1,\ldots,\Omega_M\} = \{\omega_-,\omega_+,\omega_0,\ldots,\omega_0\}, \tag{26}$$

where

$$\omega_\pm^2 = \frac{\omega^2+\omega_0^2}{2} \pm \frac{1}{2}\sqrt{(\omega^2-\omega_0^2)^2+16\bar{\lambda}^2\omega\omega_0}. \tag{27}$$

ω_- is the lowest excitation energy. It should be noted that $\omega_\pm$ are exactly the same as the excitation energies in the standard Dicke model ($\lambda_j=\lambda$) if $\bar{\lambda}$ is replaced by λ [6]. The lowest excitation energy ω_- vanishes when $\bar{\lambda}\to\lambda_c$. This induces the QPT in the present system. Since $\omega_-(\bar{\lambda}\to\lambda_c)\propto|\lambda_c-\bar{\lambda}|^{1/2}$, the critical exponent for the energy is 1/2. The characteristic length l_c is defined as $l_c=1/\sqrt{\omega_-}$ [1,6]. l_c diverges as $|\lambda_c-\bar{\lambda}|^{-1/4}$. Thus, the critical exponent for the characteristic length is $-1/4$. These results are exactly the same as those in the standard Dicke model.

In contrast to the standard Dicke model, it is difficult to analytically evaluate the excitation energies in the superradiant phase. We present numerical results for the lowest excitation energy and the critical exponents in the superradiant phase in Sec. VII.

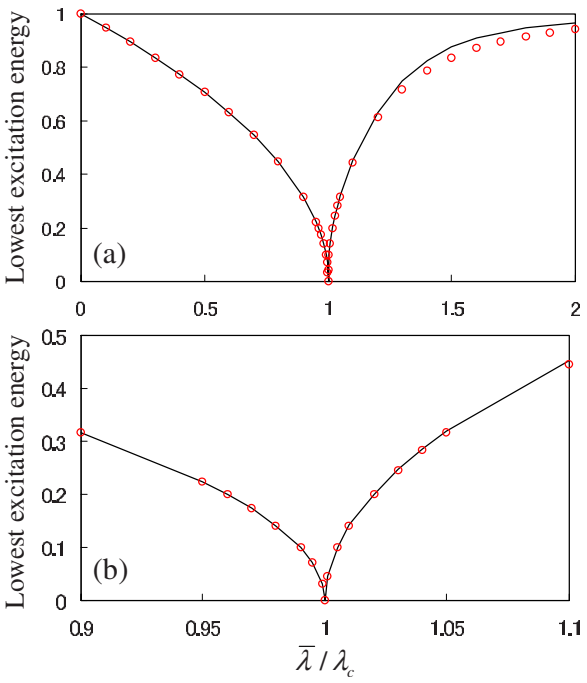

FIG. 1. (Color online) Lowest excitation energy. Circles: numerical calculation results in the generalized model. Solid line: analytic result in the standard model where all the coupling constants are equal to $\bar{\lambda}$. (b) is an enlargement of (a) around the critical point $\bar{\lambda}=\lambda_c$. The Hamiltonian is on scaled resonance $\omega=\omega_0=1$

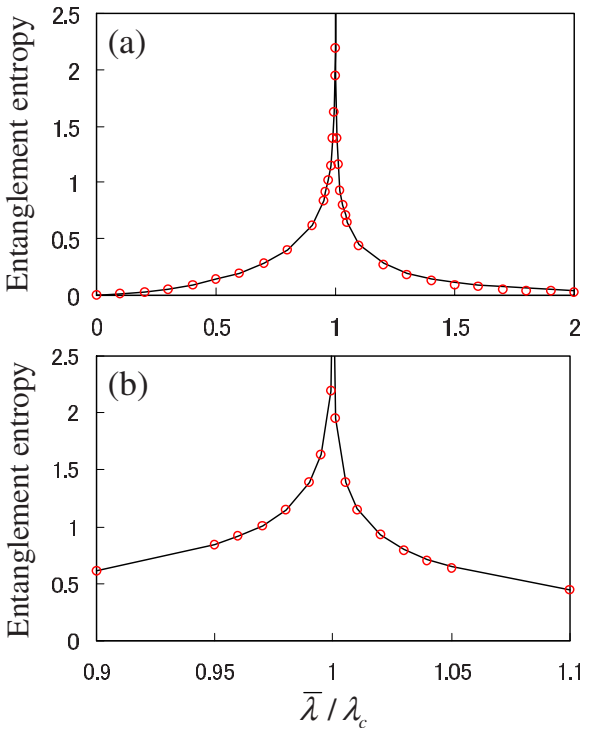

FIG. 2. (Color online) Entanglement entropy between the field and the atoms. Circles: numerical calculation results in the generalized model. Solid line: analytic result in the standard model where all the coupling constants are equal to $\bar{\lambda}$. (b) is an enlargement of (a) around the critical point $\bar{\lambda}=\lambda_c$.

VI. ENTANGLEMENT ENTROPY BETWEEN FIELD AND ATOMS

Here we evaluate the entanglement entropy, which is a standard measure of entanglement [7,31], between the field and the atoms in the ground state. From $c_x|G\rangle=0$ and $c_m|G\rangle=0$, the ground-state wave function can be obtained. Using this, the reduced density matrix of the field for the ground state is obtained as

$$\rho_F(x_1,x_2) \propto \exp\left[-\frac{d_{xx}}{2}(x_1^2+x_2^2)+\frac{\vec{d}^T D^{-1}\vec{d}}{4}(x_1+x_2)^2\right], \tag{28}$$

where

$$\begin{pmatrix} d_{xx} & \vec{d}^T \\ \vec{d} & D \end{pmatrix} = U\,\mathrm{diag}(\Omega_x,\Omega_0,\ldots,\Omega_M)U^T. \tag{29}$$

The entanglement entropy is defined as the von Neumann entropy of ρ_F. The entanglement entropy $S(\rho_F)$ is

$$S(\rho_F)=\frac{\zeta\coth\frac{\zeta}{2}}{2\ln 2}-\log_2\left(2\sinh\frac{\zeta}{2}\right), \tag{30}$$

$$\cosh\zeta=\frac{2d_{xx}}{\vec{d}^T D^{-1}\vec{d}}-1. \tag{31}$$

In the case of the standard Dicke model, the entanglement entropy diverges logarithmically at the critical point [7]. In the present case, however, it may be difficult to obtain such an analytic result. Instead, we present a numerical result for the entanglement entropy in Sec. VII.

VII. NUMERICAL CALCULATIONS

Finally, we evaluate numerically the lowest excitation energy and the entanglement entropy assuming the following distribution of $\{\lambda_j\}$:

$$P(\lambda)=\frac{1}{\pi}\left(1-\frac{\lambda^2}{\lambda_{\max}^2}\right)^{-1/2}, \tag{32}$$

where $\lambda_{\min}=-\lambda_{\max}$ and $P(\lambda)d\lambda$ denotes the probability that $\lambda<\lambda_j<\lambda+d\lambda$. This corresponds to the case where the cavity is a one-dimensional standing-wave cavity and the atomic ensemble is uniformly distributed and is large compared to the cavity resonance wavelength. The results are compared with those in the standard Dicke model where all the coupling constants are equal to $\bar{\lambda}$. The parameters were set as $\omega=\omega_0=1$ and $M=400$.

The circles in Fig. 1 show the lowest excitation energy calculated numerically. The solid line is the analytic result in

the standard model where all the coupling constants are equal to $\bar{\lambda}$. These agreement is fairly good. This means that the lowest excitation energy (and the characteristic length) in the generalized model is well reproduced by the standard model where all the coupling constants are equal to $\bar{\lambda}$, and consequently the critical exponents in the generalized and standard models are equal to each other.

The circles in Fig. 2 show the entanglement entropy between the field and the atoms calculated numerically. The solid line is the analytic result in the standard model where all the coupling constants are equal to $\bar{\lambda}$. This agreement is also good. This means that the entanglement entropy in the generalized model diverges logarithmically at the critical point, as well as in the standard model.

VIII. CONCLUSION

We have investigated a generalized Dicke model where the dependence of the coupling constant on individual atoms is taken into account. It has been shown that the generalized model exhibits a QPT from the normal phase to the superradiant phase in the thermodynamic limit. The critical point is at $\bar{\lambda}=\lambda_c$, where $\bar{\lambda}$ is an average of the coupling constants and λ_c is the same as the critical point in the standard model. Interestingly, it has also been found that the critical behavior in the generalized model is well reproduced by the standard model where all the coupling constants are equal to $\bar{\lambda}$. This result implies universality in the Dicke model in the sense that the critical behavior is independent of the details of the coupling and is characterized by the single parameter $\bar{\lambda}$.

The present result also opens the possibility that a standing-wave cavity and an extended atomic ensemble can be used for the Dicke-model QPT. Consequently, the experimental realization of the Dicke-model QPT becomes more feasible.

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