Bose-Einstein condensate as a nonlinear Ramsey interferometer operating beyond the Heisenberg limit

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We show that a dynamically evolving two-mode Bose-Einstein condensate (TBEC) with an adiabatic, time-varying Raman coupling maps exactly onto a Ramsey interferometer that includes a nonlinear medium. Assuming a realistic quantum state for the TBEC that has been achieved experimentally, we find that the measurement uncertainty of the "path-difference" phase shift scales as the standard quantum limit $(1/\sqrt{N})$, where N is the number of atoms, while that for the interatomic scattering strength scales as $1/N^{7/5}$, overcoming the conventional Heisenberg limit of $1/N$.

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High-precision quantum interferometry is one of the most important tools of metrology enabling one to infer certain properties through the measurement of the phase shift ϕ of the input quantum state. It has been shown in previous studies that when one uses the coherent state as the input state of an idealized interferometer, it is possible to achieve measurement uncertainties approaching the shot noise or the standard quantum limit (SQL) $\Delta\phi \sim 1/\sqrt{N}$, where N is the number of particles conjugate to the phase variable ϕ. Much of the recent work in the literature concerns schemes to overcome the SQL to approach the Heisenberg limit $\Delta\phi \sim 1/N$, by using carefully chosen input states such as the squeezed state or the Schrödinger's cat state instead of the coherent state for light in optical interferometry and Bose-Einstein condensates (BECs) in matter-wave interferometry [1]. The Heisenberg limit was found to be the ultimate limit regardless of how we engineer the quantum state.

Such precision interferometry should be attainable with matter waves as well as with light. Though with current experimental limitations of matter-wave systems (for multicomponent BEC $\sim 5\times 10^5$ atoms and a lifetime of ~ 20 s), matter-wave interferometry is not yet competitive with optical interferometry, the goal of this paper is to investigate new theoretical limits of a possible matter-wave interferometry scheme. In this paper we shall refer to the $1/N$ scaling as the "conventional" Heisenberg limit to make it clear that the terminology originates from studies of interferometry with linear probes.

Very recently it was shown using a more general parameter estimation theory that measurement uncertainty of the order $1/N^k$, where k is the number of parameter-sensitive terms, is possible [2]. This formal result demonstrated that the conventional Heisenberg limit is in fact the case with $k=1$ and BECs with two-body collisions ($k=2$) may be able to achieve up to $\Delta\phi \sim 1/N^2$ accuracy in measurements of atom-atom interactions through a modulation of the scattering length using a Feshbach resonance or by density variation due to gravitational gradients. This rather surprising gain in measurement accuracy is an inherent property of the probe with a nonlinear generator for the phase shift, and *not* a consequence of quantum squeezing. A squeezed state, independent of the quadrature, cannot surpass the conventional Heisenberg limit as long as the probe is linear. This formal work [2], however, did not show how one may achieve such limits in real physical systems.

A nonlinear interferometer, as opposed to a normal (linear) interferometer, includes nonlinear medium in one or both arms. Nonlinear interferometers have been studied previously [3,4] but with the nonlinearity being used to generate squeezing rather than as a probe. In this paper, we show that direct temporal evolution of a TBEC such as Na atoms in the $|F=1, \ M_F=\pm 1\rangle$ hyperfine states trapped in an optical dipole trap with Raman coupling such as that already realized experimentally [5,6] maps *precisely* onto a nonlinear Ramsey interferometer with which one can realize measurement accuracy better than the conventional Heisenberg limit. In particular, we consider a natural state for TBEC, the coherent spin state (CSS), which is known to give only SQL under usual circumstances. We find that even with CSS (i.e., no special presqueezing) as our input, the conventional Heisenberg limit can be overcome in the measurement of atom-atom interactions. It was shown very recently [7] that the measurement of the interatomic scattering length of a system of ultracold spin-$\frac{1}{2}$ atoms evolving under a nonlinear Hamiltonian gives results which surpass the conventional Heisenberg limit when an optimal entangled state generated from a separate time-dependent Hamiltonian [8] is used; in our work, such quantum state engineering is not necessary.

A quantum interferometer can be described in terms of the angular momentum operators as a transformation operator

$$\hat{\mathcal{I}} = \hat{\mathcal{B}}_- \hat{\mathcal{P}}(\phi) \hat{\mathcal{B}}_+ = e^{-i\phi \hat{J}_y}. \tag{1}$$

The 50:50 beam splitter and the phase shifter are given by $\hat{\mathcal{B}}_\pm = \exp(\pm i\pi \hat{J}_x/2)$ and $\hat{\mathcal{P}}(\phi) = \exp(i\phi \hat{J}_z)$, where $\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$, $\hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)$, $\hat{J}_{+(-)} = \hat{a}^\dagger_{1(2)}\hat{a}_{2(1)}$, and $\hat{J}_z = \frac{1}{2}(\hat{a}^\dagger_1\hat{a}_1 - \hat{a}^\dagger_2\hat{a}_2)$, with $\hat{a}_1$ and $\hat{a}_2$ being the two annihilation operators for the two input modes into interferometer. For the TBEC with a Raman coupling like that considered here, the two annihilation operators $\hat{a}_1$ and $\hat{a}_2$ correspond to the atoms in the two hyperfine states. The time evolution operator $\hat{U}(t)$ for this system defined by $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$ is [9]

$$\hat{U}(t) = \hat{R}^\dagger e^{-i\hat{H}' t}\hat{R}, \tag{2}$$

where $\hat{R} = e^{-\pi(\hat{J}_- - \hat{J}_+)/4}$ and $H' = 2\Omega \hat{J}_z - \frac{q}{2}\hat{J}_z^2$. Ω is the tunneling coupling and q is the strength of the scattering interaction between the bosons. As shown earlier [9], the detuning of the laser from the transition between the two species is set to be zero to make the Hamiltonian diagonal in the $\hat{J}_z$ representation. This also prevents the generation of an additional geometric phase on top of the dynamical phase.

The overall action of the time evolution operator $\hat{U}(t)$, Eq. (2), can clearly be mapped onto a nonlinear Ramsey interferometer with the transformation operator

$$\hat{\mathcal{I}} = \hat{\mathcal{B}}_{-}\hat{\mathcal{P}}(\phi_1')\hat{\mathcal{S}}(\phi_2')\hat{\mathcal{B}}_{+} = e^{-i\phi_1'\hat{J}_x - i\phi_2'\hat{J}_x^2/2}. \quad (3)$$

$\hat{\mathcal{B}}_{-}=\hat{R}^\dagger \equiv \exp(-i\pi\hat{J}_y/2)$ and $\hat{\mathcal{B}}_{+}=\hat{R}$ are the two 50:50 beam splitters, while $\hat{\mathcal{P}}(\phi_1')=e^{-i\phi_1'\hat{J}_z}$ and $\hat{\mathcal{S}}(\phi_2')=e^{-i\phi_2'\hat{J}_z^2/2}$ represent, respectively, the path-difference phase shifter and the nonlinear medium. The phase variables can be written as $\phi_1'=2\Omega t+\phi_1$ and $\phi_2'=qt+\phi_2$, where we have explicitly written out external phase shift ϕ_i to be measured on top of the time-dependent phase. The nonlinear phase shift ϕ_2 can be induced via a Feshbach resonance on the two-component BEC. Since $q=4\pi\hbar^2(a_{AA}+a_{BB}-a_{AB})/2m$, where a_{AA}, a_{BB}, and a_{AB} denote the two intraspecies scattering lengths and the interspecies scattering length, respectively, the nonlinear interferometry can be used to detect changes in the intraspecies or interspecies scattering lengths.

As mentioned above, for our input state we shall consider an SU(2) atomic coherent state or CSS, $|\theta,\phi\rangle$, which is a reasonable quantum state representing a TBEC [9,10]. It is noted that $\hat{R}(t)|\theta,\phi\rangle=\Sigma_{m=-j}^{j}\mathcal{R}_m^j(\theta+\lambda,\phi-\Delta t)|j,m\rangle$, where $\mathcal{R}_m^j(\theta,\phi)$ is defined $\mathcal{R}_m^j(\theta,\phi) = \binom{2j}{j+m}^{1/2}\cos^{j+m}(\frac{\theta}{2})\sin^{j-m}(\frac{\theta}{2})e^{i(j-m)\phi}$. Since the azimuthal angle ϕ simply shifts the origin, we shall only consider CSS with $\phi=0$ in this paper. Exotic input states such as the NOON or the Yurke state [4] will be considered elsewhere as they are currently not yet practical in the context of TBEC.

The simplest possible scenario is to measure the path-difference phase shift ϕ_1 while applying a magnetic field to tune $\phi_2=0$ via the Feshbach resonance: i.e., no nonlinear perturbations to the Hamiltonian. This is the standard Ramsey interferometry which has been studied extensively. The fact that a TBEC is used instead of the thermal atoms simply provides clean signals owing to the inherent long range coherence of a condensate. We will not consider this case any further in this paper. What is more interesting is the case of finite q. Here the presence of the nonlinear component modifies the interferometric outcome ϕ_1, and brings to the forefront the question of the uncertainty associated with measuring the scattering length or ϕ_2.

First, we analyze TBEC as a nonlinear Ramsey interferometer in the idealized situation where the measurement of the phase is carried out as a projective measurement onto a phase state. We estimate the measurement uncertainty using the Cramers-Rao inequality in such cases. Then a more practical scheme, measurement of the atom number difference as a function of the phase shifts ϕ_1 and ϕ_2, is considered along with the corresponding measurement uncertainties. The fundamental limit to the phase shift measurements can be calculated by first defining the positive valued operator measure $\hat{E}(\phi)$ such that the probability density of the corresponding

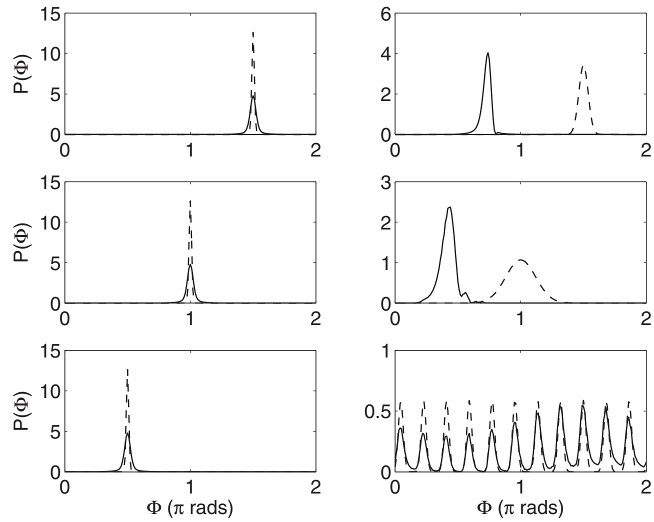

FIG. 1. Probability density $P(\Phi)$ for the initial CSS $|\theta=\pi/4,\phi=0\rangle$ (solid line) and $|\theta=\pi/2,\phi=0\rangle$ (dashed line) at different times $\Omega t=0.75\pi, 2.5\pi, 60.25\pi$ (top, middle, and bottom rows, respectively). Left column: $q=0$; right column: $q\neq 0$.

measurement result is $P(\phi)=\mathrm{Tr}[\hat{\rho}\hat{E}(\phi)]$, where $\hat{\rho}$ is the density matrix for the system. As in Ref. [11], we define the normalized phase state $|j,\Phi\rangle=(2j+1)^{-1/2}\Sigma_{m_x=-j}^{j}e^{im_x\Phi}|j,m_x\rangle_x$ so that $\hat{E}(\Phi)d\Phi=(2j+1)|j,\Phi\rangle\langle j,\Phi|d\Phi/2\pi$, and for an arbitrary input state $|\psi\rangle$, the probability density of the measurement result is $P(\Phi)=\frac{2j+1}{2\pi}|\langle\psi|\hat{\mathcal{I}}^\dagger|j,\Phi\rangle|^2$.

With a CSS input, the phase measurement gives a probability density distribution

$$P(\Phi) = \frac{1}{2\pi}\left|\sum_{m_x,m_z=-j}^{j} e^{i(\Phi-2\Omega t)m_x}e^{iqtm_x^2/2}\mathcal{R}_{m_z}^j(\theta,\phi)d_{m_z,m_x}^j(\pi/2)\right|^2,$$

where

$$d_{m_z,m_x}^j(\pi/2)={}_z\langle j,m_z|j,m_x\rangle_x$$

is the Wigner d matrix:

$$d_{m_z,m_x}^j(\pi/2)={}_z\langle j,m_z|e^{-i\pi\hat{J}_y/2}|j,m_x\rangle_z = \frac{1}{2^{m_z}}\left[\frac{(j-m_z)!(j+m_z)!}{(j-m_x)!(j+m_x)!}\right]^{1/2}P_{j-m_z}^{(m_z-m_x,m_z+m_x)}\quad(x=0)$$

for $m_z-m_x>-1$ and $m_z+m_x>-1$. $P^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomials. Symmetries give $d_{m_z,m_x}^j=(-1)^{m_z-m_x}d_{m_x,m_y}^j = d_{-m_x,-m_z}^j$. We plot the probability density in Fig. 1 at various times starting from the initial states $|\theta=\pi/4\rangle$ (solid line) and $|\theta=\pi/2\rangle$ (dashed line). The initial Dicke state $|\theta=0\rangle$ is not considered as it is orthogonal to the projective measurement on the phase. In order to highlight the effect of nonlinearity on the measurement of ϕ_1, we plot in the left column the case of $q=0$ for comparison with the corresponding results with $q\neq 0$ in the right column. We choose $q=3/N$, which corresponds to the Josephson regime [9]. The presence of nonlinearity generally degrades the performance of the interferometer as evidenced by the increase in the width of the

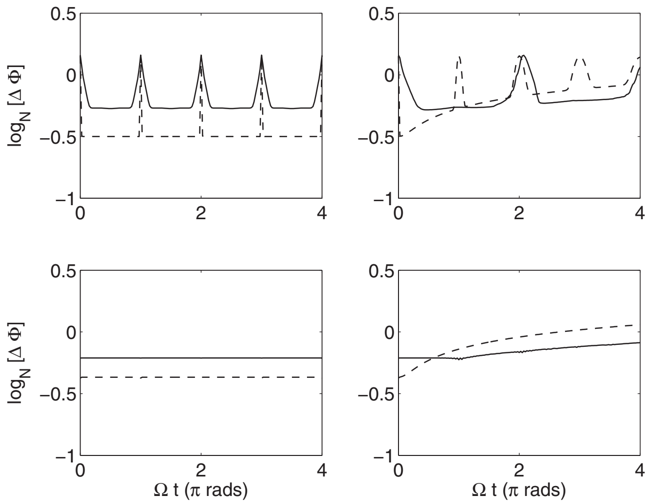

FIG. 2. Time evolution of $\log_N[\Delta\Phi]$ for the initial CSS $|\theta=\pi/4,\phi=0\rangle$ (solid line) and $|\theta=\pi/2,\phi=0\rangle$ (dashed line). Top row: direct calculation from the probability density $P(\Phi)$; Bottom row: Cramers-Rao lower bound. Left column: $q=0$; right column $q\neq 0$.

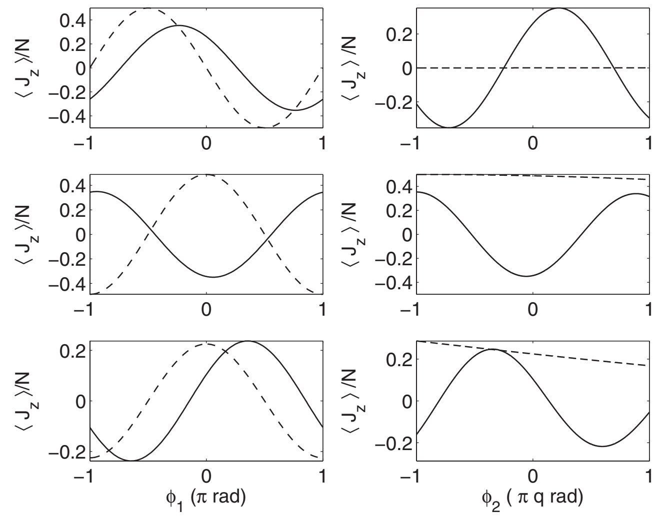

FIG. 3. $\langle\hat{J}_z\rangle/N$ as a function of the changes in the phase shifts ϕ_1 (left column) and ϕ_2 (right column) for the initial CSS $|\theta=0,\phi=0\rangle$ (dashed line) and $|\theta=\pi/4,\phi=0\rangle$ (solid line) at different times $\Omega t=\pi/4,\pi,6\pi$ (top, middle, and bottom rows, respectively).

probability distribution. It is also notable that the probability density becomes multiple peaked after a long time. This may be interpreted as the generation of a superposition state due to nonlinearity as studied by Yurke and Stoler [12]. It is clear that to use the TBEC as an effective interferometer based on projective measurement onto phase states, q needs to be minimized and the time of measurement must be kept relatively short.

The uncertainty in phase measurement can be studied using the standard techniques of probability theory, particularly the Cramers-Rao lower bound (CRLB). The CRLB establishes the lower bound on the phase shift estimate where the phase uncertainty scales as $\Delta\Phi=1/\sqrt{F_n}$, where F_n is the Fisher information defined by

$$F_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{d}{d\Phi}\ln P(\Phi)\right]^2 P(\Phi)d\Phi. \tag{4}$$

In Fig. 2, we plot the quantity $\log_N[\Delta\Phi]$, where N is the total number of atoms and $\Delta\Phi$ is the uncertainty in phase. We used the standard deviation $\Delta\Phi$ calculated directly from the probability distribution $P(\Phi)$ (Fig. 1) and the CRLB, where the CRLB effectively gives a time-averaged value of the directly calculated uncertainty. It is noted that in all these figures $\Delta\Phi\geq 1/N^{1/2}$, the standard quantum limit. It is also noted that although the uncertainty associated with the $|\theta=\pi/2\rangle$ state is lower than that of the $|\theta=\pi/4\rangle$ state for $q=0$ it quickly loses this advantage with $q>0$, indicating sensitivity to dephasing due to interatomic collisions. The $|\theta=\pi/4\rangle$ state is therefore a more robust state for interferometry in the presence of nonlinearity.

Next, instead of projective measurement onto a phase state, we consider projective measurement of the atom number difference. The total number of atoms measured indicates the number of "input" atoms, while the atom number difference, $\langle\hat{J}_z(\phi_1',\phi_2')\rangle\equiv\langle\psi(0)|\hat{\mathcal{I}}^\dagger(\phi_1',\phi_2')\hat{J}_z\hat{\mathcal{I}}(\phi_1',\phi_2')|\psi(0)\rangle$, allows us to infer the phase shift and is equivalent to measuring the number of atoms at each of the output ports of a typical Mach-Zehnder interferometer. An analytic expression for $\langle\hat{J}_z\rangle$ is given by [9]:

$$\langle\hat{J}_z(\phi_1',\phi_2')\rangle=-\sum_{m=-N/2}^{N/2-1}\mathcal{D}(\theta,m)\tan^{-1}\left(\tfrac{\theta-\pi/2}{2}\right)\times\cos\left[\phi_1'-\phi_2'\left(m+\tfrac{1}{2}\right)\right]$$

where we have defined $\mathcal{D}(\theta,m)=C^N_{N/2+m+1}(\frac{N}{2}+m+1)\cos^{2N}(\frac{\theta-\pi/2}{2})\tan^{N-2m}(\frac{\theta-\pi/2}{2})$.

Figure 3 shows $\langle\hat{J}_z\rangle/N$ as a function of the changes in the phase shifts ϕ_1 and ϕ_2 for the initial states $|\theta=0,\ \phi=0\rangle$ and $|\theta=\pi/4\rangle$ at different times $\Omega t=\pi/4,\pi,6\pi$. In contrast to the earlier phase state projection method, the initial CSS $|\theta=\pi/2\rangle$ is known as a "self-trapping" state in the new context of projective number measurement, and gives trivial results. Since the interferometry is carried out at fixed times, we see in Fig. 3 clear sinusoidal fringes, without the "collapses and revivals" typical of temporal evolution. Even when measuring ϕ_2 we see clear fringes for a range of values around $-3q,\ldots,3q$ for the initial state $|\theta=\pi/4\rangle$. This is possible because, for this choice of θ, the factor $\mathcal{D}(\theta=\pi/4,\ m)$ is narrow enough to limit the interfering effect of summing up the cosine terms. On the other hand, $\mathcal{D}(\theta=0,\ m)$ is wider and the resulting interference fringes do not allow for a sensitive detection of small variations in ϕ_2.

Finally, we consider the phase resolution for this scheme, which is given by $[\Delta\phi_k]^2=[\Delta\hat{J}_z]^2/|\partial\langle\hat{J}_z\rangle/\partial\phi_k|^2$, $k=1,2$, where, as found in Ref. [9], the variance is $[\Delta\hat{J}_z]^2=\langle\hat{J}_z^2\rangle-\langle\hat{J}_z\rangle^2$ with

$$\langle \hat{J}_z^2(t)\rangle = \frac{1}{4}\sum_{m=-N/2}^{N/2-1} \mathcal{D}(\theta,m)\tan^{-2}(\theta/2-\pi/4)(N/2-m)$$
$$+\mathcal{D}(\theta,m)(N/2+m+1)$$
$$+\sum_{m=-N/2}^{N/2-2}\Big[\tfrac{1}{2}\mathcal{D}(\theta,m)\tan^{-2}(\theta/2-\pi/4)$$
$$\times(N/2-m-1)\Big]\cos[2\phi_1'-2\phi_2'(m+1)].$$

Since the denominator involves a function of the form $\sin[\phi_1'-\phi_2'(m+\frac{1}{2})]$, the quantity $[\Delta\phi_k]^2$ is minimized for the values of $\phi_1'-\phi_2'(m+\frac{1}{2})=\pm\pi/2$. This indicates that the measurement accuracy is dependent on the measurement values, where results such as $\phi_1'=\pm\pi/2$ and $\phi_2'=0$ give optimum results. In particular, for a large number of atoms N one can approximate the coefficient $\mathcal{D}(\theta,m)$ by $\sqrt{N/\pi}e^{-(2m-N\sin\theta)^2/N}$ and replace the sums by integrals $\int\mathcal{D}(\theta,x)dx\sim N$ and $\int x\mathcal{D}(\theta,x)dx\sim N^2$. This leads to

$$[\Delta\phi_k]^2\sim\frac{\alpha N}{(\beta N+\delta_{k,2}\gamma N^2)^2}, \tag{5}$$

where $k=1,2$, and $\alpha,\ \beta,\ \gamma$ are constants and $\delta_{k,2}$ is the Kronecker delta function. For $k=1$ one has $\Delta\phi_1\sim 1/N^{1/2}$, i.e., the standard quantum limit in accuracy for the measurement of ϕ_1. On the other hand, it is remarkable that with $k=2$, i.e., measurement of the phase shift due to the interatomic interactions, $\Delta\phi_2\sim 1/N^{3/2}<1/N$, implying that, although not reaching the theoretical limit of $1/N^2$ [2], such measurement for this CSS input state has uncertainty below the conventional Heisenberg limit. We have verified this estimate numerically; the quantity $\log_N\Delta\phi_k$ calculated as a function of the initial angle of the CSS, θ at the optimal values of ϕ_1 and ϕ_2 is plotted in Fig. 4. The solid and the dashed line represent the uncertainty in the measurement of ϕ_1 and ϕ_2, respectively. It is clear that the best result is obtained for $\theta=0$ for the measurement of ϕ_1 and $\theta=\pi/4$ for the measurement of ϕ_2. In the bottom panel, we plot the result as a function of atom numbers for these chosen values of θ. It shows that the result is independent of the number of atoms and, on average, $\Delta\phi_1\sim 1/N^{1/2}$ and $\Delta\phi_2\sim 1/N^{7/5}$, which is indeed very close to the above estimate.

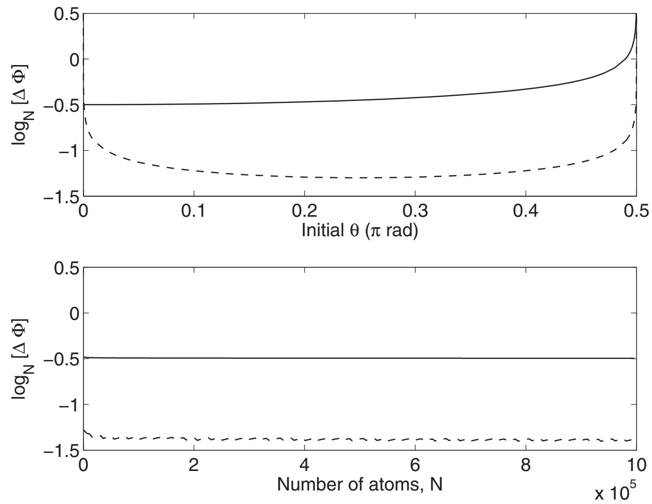

FIG. 4. Top: $\log_N[\Delta\phi_k]$ plotted as a function of the angle θ of the initial CSS with $N=1000$. Solid line: $k=1$. Dashed line: $k=2$. Bottom: same quantity plotted as a function of the number of atoms N. Solid line: $k=1$ with $\theta=0$. Dashed line: $k=2$ with $\theta=\pi/4$.

In summary, we have shown that a TBEC with a Josephson-type coupling directly maps onto a nonlinear Ramsey interferometer. The system is already experimentally available and the state we consider is the realistic coherent spin state rather than some exotic quantum state. It was found that projective phase measurement reaches the standard quantum limit in accuracy while projective number measurement of the phase shifts due to interatomic interactions was found able to overcome the conventional Heisenberg limit, suggesting new implications for quantum metrology.

Note added. Recently, a closely related result was found independently in Ref. [13].

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