Continuous positive-operator-valued measurement of photon polarization

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A continuous positive operator-valued measurement (POVM) is described for the polarization state of a single photon propagating in a known spatiotemporal mode. This POVM is shown to be the maximum-likelihood quantum measurement for estimating that polarization state. Moreover, the maximum-likelihood polarization estimate derived from this POVM enables uniformly optimal measure-and-prepare state transmission via classical communication—at the $\bar{F}=2/3$ classical limit on average teleportation fidelity—for any pure state of polarization. When two continuous-POVM polarimeters are used to detect the singlet state of a pair photons that are in distinct spatial modes, the conditional probability density for one polarimeter's output given the other's encompasses the quantum interference patterns seen in standard singlet-state polarization analysis. For single-photon illumination, a $1:M$ optical splitter followed by a collection of M projective polarization measurements, whose bases are uniformly distributed over the Poincaré sphere, yields observation statistics that converge in distribution to those of the continuous-POVM polarimeter as $M\to\infty$.

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I. INTRODUCTION

The polarization state of a single photon propagating in a known (guided or unguided) spatiotemporal mode provides a convenient physical instantiation of the qubit abstraction. The prototypical single-photon polarization measurement employs wave plates, a polarizing beam splitter, and a pair of single-photon detectors to realize the projective measurement onto a particular orthonormal basis. Linear optics quantum computing [1] uses such measurements in the computational basis. Quantum key distribution systems that implement the Bennett-Brassard 1984 protocol [2] with polarization states require projective polarization measurements for both the H/V and $\pm 45°$ bases. It is a direct consequence of the no-cloning theorem [3] that no quantum measurement can perfectly determine the totally unknown polarization state of a single photon. Indeed, when a projective measurement is used to perform measure-and-prepare state transmission of such a polarization qubit via classical communication, the resulting average fidelity—assuming that qubit was drawn from a uniform distribution over the Poincaré sphere—achieves the $\bar{F}=2/3$ classical performance limit on teleportation fidelity. However, with any fixed polarization-measurement basis, the teleportation fidelity for a particular pure-state input will vary from 1/2 to 1 as that state explores all possibilities. The "optimal polarimeter," from Ref. [4], uses a 1:2 optical splitter followed by H/V and $\pm 45°$ polarization measurements. Its average measure-and-prepare teleportation fidelity, for a totally unknown single-photon input state, is again $\bar{F}=2/3$, and its teleportation fidelity for a particular pure-state input varies from 1/2 to 3/4.

The preceding polarization measurements have discrete outcomes, e.g., binary for H/V polarization analysis and quaternary for H/V plus $\pm 45°$ polarization analysis. Other work on single-photon polarimetry has also been limited to discrete outcomes, e.g., Brandt's POVM for unambiguous discrimination between a pair of nonorthogonal polarization states [5], and the prescriptions provided by Ahnert and Payne for implementing polarization POVMs [6]. In this paper, however, we will introduce a continuous POVM for single-photon polarimetry. We will show that this POVM is the maximum-likelihood quantum measurement for estimating the unknown polarization state of a single photon. Moreover, we will show that the maximum-likelihood estimate derived from this POVM enables uniformly optimal measure-and-prepare state transmission via classical communication—at the $F=2/3$ classical limit—for any pure state of polarization. We also consider what happens when two continuous-POVM polarimeters are used to detect the singlet state of a pair photons that are in distinct spatial modes. Here we will show that the probability density for one polarimeter's output, conditioned on that from the other, encompasses the quantum interference patterns seen in standard polarization analysis of this entangled state. For single-photon illumination, a $1:M$ optical splitter followed by a collection of M projective polarization measurements, whose bases are uniformly distributed over the Poincaré sphere, will be shown to yield observation statistics that converge in distribution to those of the continuous-POVM polarimeter as $M\to\infty$. For finite M, this arrangement realizes the square-root measurement for distinguishing between the $2M$ single-photon pure states associated with the M polarization bases being measured, assuming they are all equally likely to occur. Furthermore, this square-root measurement is in fact the minimum probability of error quantum receiver for that $2M$-ary quantum detection problem.

II. CONTINUOUS-POVM POLARIMETRY AND MAXIMUM-LIKELIHOOD STATE ESTIMATION

Consider a fully-polarized single photon that is propagating—for notational convenience—as a uniform plane wave in the $+z$ direction. Two equivalent representations for its polarization state are as follows. The first is the x,y representation

$$|\mathbf{i}\rangle = \alpha|x\rangle + \beta|y\rangle, \tag{1}$$

where $|x\rangle$ and $|y\rangle$ denote x-polarized and y-polarized single-photon states and α,β are complex numbers satisfying $|\alpha|^2+|\beta|^2=1$. The second is the Poincaré-sphere representation

$$\mathbf{r}=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}=\begin{bmatrix} 2\ \mathrm{Re}(\alpha^*\beta) \\ 2\ \mathrm{Im}(\alpha^*\beta) \\ |\alpha|^2-|\beta|^2 \end{bmatrix}, \tag{2}$$

where r_1, r_2, r_3 are real numbers satisfying $r_1^2+r_2^2+r_3^2=(|\alpha|^2+|\beta|^2)^2=1$. Except for a physically irrelevant absolute phase, Eq. (2) can be inverted to yield $\mathbf{i}^T\equiv[\alpha,\beta]$ as a function of $\mathbf{r}$. In particular, for any $\mathbf{i}$ we can say that

$$|\mathbf{i}\rangle\langle\mathbf{i}|=|\mathbf{i}(\mathbf{r})\rangle\langle\mathbf{i}(\mathbf{r})|, \tag{3}$$

as the aforementioned absolute phase cancels out in the projector. It is then trivial to verify, e.g., using the x,y basis, that

$$\int_{\mathcal{P}}\mathrm{d}\mathbf{r}\frac{|\mathbf{i}(\mathbf{r})\rangle\langle\mathbf{i}(\mathbf{r})|}{2\pi}=\hat{I}, \tag{4}$$

where $\mathcal{P}$ denotes the Poincaré sphere and $\hat{I}$ is the identity operator for the Hilbert space of polarization states of a $+z$-going single photon. Thus,

$$d\hat{\Pi}(\mathbf{r})\equiv\frac{|\mathbf{i}(\mathbf{r})\rangle\langle\mathbf{i}(\mathbf{r})|}{2\pi}d\mathbf{r}, \quad \text{for } \mathbf{r}\in\mathcal{P} \tag{5}$$

are a continuum of Hermitian operators that resolve the identity operator for the polarization-state Hilbert space. These operators constitute our continuous POVM for single-photon polarization measurements. When this POVM is applied to a photon whose polarization state is $|\mathbf{i}_0\rangle$, the outcome is a Poincaré-sphere vector $\mathbf{r}$. The probability density for getting a particular $\mathbf{r}$ is then

$$p(\mathbf{r}||\mathbf{i}_0\rangle)=\frac{|\langle\,\mathbf{i}_0|\mathbf{i}(\mathbf{r})\rangle|^2}{2\pi}=\frac{1+\mathbf{r}_0^T\mathbf{r}}{4\pi}, \quad \text{for } \mathbf{r}\in\mathcal{P}, \tag{6}$$

where $\mathbf{r}_0$ is the Poincaré-sphere point corresponding to $\mathbf{i}_0$.

It is easy to check that $p(\mathbf{r}||\mathbf{i}_0\rangle)$ is a proper probability density function on $\mathbf{r}\in\mathcal{P}$ for all $\mathbf{r}_0\in\mathcal{P}$, i.e., it satisfies

$$p(\mathbf{r}||\mathbf{i}_0\rangle)\geq 0 \quad \text{and} \quad \int_{\mathcal{P}}d\mathbf{r}p(\mathbf{r}||\mathbf{i}_0\rangle)=1. \tag{7}$$

It turns out that this POVM is the maximum-likelihood quantum measurement for estimating the unknown polarization of a single photon, as we now will show. The necessary and sufficient conditions that the maximum-likelihood POVM must obey are well known [7]:

$$\hat{\Upsilon}\equiv\int_{\mathcal{P}}d\mathbf{r}\ \hat{W}(\mathbf{r})d\hat{\Pi}(\mathbf{r}), \tag{8}$$

must be Hermitian, where

$$\hat{W}(\mathbf{r})\equiv\frac{|i(\mathbf{r})\rangle\langle i(\mathbf{r})|}{4\pi}, \tag{9}$$

$$[\hat{\Upsilon}-\hat{W}(\mathbf{r})]d\hat{\Pi}(\mathbf{r})=d\hat{\Pi}(\mathbf{r})[\hat{\Upsilon}-\hat{W}(\mathbf{r})]=0, \tag{10}$$

and

$$\hat{\Upsilon}-\hat{W}(\mathbf{r})\geq 0. \tag{11}$$

From Eqs. (8) and (9), plus the completeness relation for the continuous-POVM polarimeter, we find that

$$\hat{\Upsilon}=\frac{\hat{I}}{4\pi}, \tag{12}$$

from which Eqs. (10) and (11) are easily verified.

The maximum-likelihood estimate of $\mathbf{r}_0$, given that the continuous-POVM polarimeter's output is $\mathbf{r}$ is therefore

$$\hat{\mathbf{r}}_{0_{\mathrm{ML}}}\equiv\arg\max_{\mathbf{r}\in\mathcal{P}}[p(\mathbf{r}||\mathbf{i}_0\rangle)]=\mathbf{r}. \tag{13}$$

Taking $|\mathbf{i}(\hat{\mathbf{r}}_{0_{\mathrm{ML}}})\rangle$ to be the output state of a measure-and-prepare system for transmitting the qubit $|\mathbf{i}_0\rangle$ by classical communication, we find that the resulting fidelity is

$$F=\int_{\mathcal{P}}d\mathbf{r}|\langle\,\mathbf{i}(\hat{\mathbf{r}}_{0_{\mathrm{ML}}})|\mathbf{i}_0\rangle|^2p(\hat{\mathbf{r}}_{0_{\mathrm{ML}}}||\mathbf{i}_0\rangle), \tag{14}$$

$$=\int_{\mathcal{P}}d\mathbf{r}\frac{1+\mathbf{r}^T\mathbf{r}_0}{2}p(\mathbf{r}||\mathbf{i}_0\rangle), \tag{15}$$

$$=\int_{\mathcal{P}}d\mathbf{r}\frac{(1+\mathbf{r}^T\mathbf{r}_0)^2}{8\pi}=2/3, \text{for all } \mathbf{r}_0\in\mathcal{P}. \tag{16}$$

Thus, unlike the discrete-outcome polarimeters mentioned in Sec. I, continuous-POVM polarimetry enables measure-and-prepare transmission of single-photon polarization states with performance that uniformly achieves the classical limit of 2/3 on average teleportation fidelity.

III. CONTINUOUS-POVM POLARIMETRY OF THE SINGLET STATE

To explore another aspect of continuous-POVM polarimetry, suppose that a source produces a pair of photons A and B that are in the singlet state

$$|\psi\rangle_{AB}=\frac{|x\rangle_A|y\rangle_B-|y\rangle_A|x\rangle_B}{\sqrt{2}}, \tag{17}$$

and that continuous-POVM polarimeters are used to measure each photon's polarization state. Using $\mathbf{r}_A$ and $\mathbf{r}_B$ to denote the outcomes of these measurements, we have that

$$p(\mathbf{r}_A,\mathbf{r}_B||\psi\rangle_{AB})=\frac{|{}_{AB}\langle\psi|(|\mathbf{i}(\mathbf{r}_A)\rangle_A|\mathbf{i}(\mathbf{r}_B)\rangle_B)|^2}{(2\pi)^2}=\frac{1-\mathbf{r}_A^T\mathbf{r}_B}{(4\pi)^2}, \tag{18}$$

for $\mathbf{r}_A,\mathbf{r}_B\in\mathcal{P}$. By integrating Eq. (18) over $\mathbf{r}_B$, we obtain the marginal probability density function for the $\mathbf{r}_A$ measurement. As expected, from the physics of the singlet state, it is a uniform distribution

$$p(\,\mathbf{r}_A||\psi\rangle_{AB})=\frac{1}{4\pi}, \quad \text{for } \mathbf{r}_A\in\mathcal{P}. \tag{19}$$

Similarly, integrating Eq. (18) over $\mathbf{r}_A$ yields the uniform distribution for the $\mathbf{r}_B$ measurement

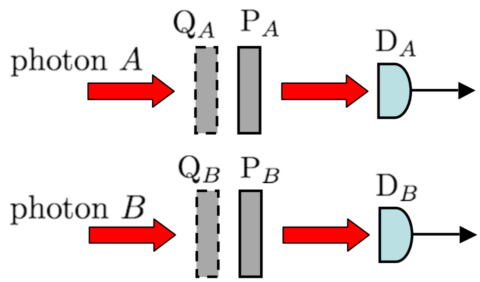

FIG. 1. (Color online) Schematic of the standard experimental apparatus for singlet-state polarization analysis. Q_A and Q_B are lossless quarter-wave plates that are absent when the linear polarization basis is employed. P_A and P_B are lossless polarizers. D_A and D_B are unity quantum efficiency single-photon detectors with no dark counts.

$$p(\mathbf{r}_B \| \psi\rangle_{AB}) = \frac{1}{4\pi}, \quad \text{for } \mathbf{r}_B \in \mathcal{P}. \tag{20}$$

Equations (18) and (19) lead to the following distribution for $\mathbf{r}_B$ conditioned on knowledge of $\mathbf{r}_A$,

$$p(\mathbf{r}_B | \mathbf{r}_A, |\psi\rangle_{AB}) = \frac{1 - \mathbf{r}_A^T \mathbf{r}_B}{4\pi}, \quad \text{for } \mathbf{r}_B \in \mathcal{P}. \tag{21}$$

Equation (21) encompasses the quantum interference fringes seen in standard singlet-state polarization analysis. To illustrate that this is so, suppose that photons A and B are first passed through lossless polarizers and then illuminate a pair of ideal single-photon detectors, as shown in Fig. 1.

With the polarizers for photons A and B set to transmit

$$\mathbf{i}_A = \cos(\theta_A)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(\theta_A)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{22}$$

and

$$\mathbf{i}_B = \cos(\theta_B)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(\theta_B)\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{23}$$

respectively, as functions of their orientation angles θ_A and θ_B, the probability that photon B will be detected, given the photon A was detected is given by

$$\text{Prob}(B \text{ detected}|A \text{ detected}) = \frac{1 - \cos[2(\theta_A - \theta_B)]}{2}. \tag{24}$$

Equation (24) shows the ideal singlet-state behavior; see Fig. 2. We get a sinusoidal fringe pattern with 100% visibility that peaks—at 100% conditional probability—when $\theta_A - \theta_B = \pm\pi/2$, and has perfect nulls when $\theta_A - \theta_B = 0, \pm\pi$. Experimental demonstration of this fringe pattern requires that a sequence of singlet states be measured—for a given (θ_A, θ_B) pair—until the empirical frequency of a B detection, conditioned on an A detection, stabilizes to a nearly constant value. This same process must then be repeated for a set of (θ_A, θ_B) values that span the fringe pattern, e.g., by keeping θ_A fixed and varying θ_B from $-\pi/2$ to $\pi/2$.

Now, using the fact that

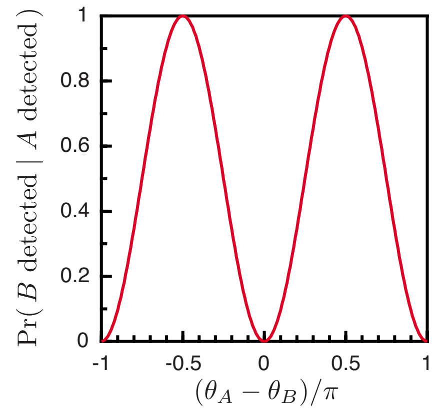

FIG. 2. (Color online) Quantum interference fringe pattern Prob(B detected|A detected) versus $(\theta_A - \theta_B)/\pi$, for standard singlet-state polarization analysis in the linear-polarization basis.

$$\mathbf{r}_A(\mathbf{i}_A) = \begin{bmatrix} \sin(2\theta_A) \\ 0 \\ \cos(2\theta_A) \end{bmatrix} \text{ and } \mathbf{r}_B(\mathbf{i}_B) = \begin{bmatrix} \sin(2\theta_B) \\ 0 \\ \cos(2\theta_B) \end{bmatrix}, \tag{25}$$

are the Poincaré-sphere vectors associated with $\mathbf{i}_A$ and $\mathbf{i}_B$, Eq. (21) gives us the following result for the continuous-POVM polarimeter's conditional probability density function

$$p(\mathbf{r}_B(\mathbf{i}_B) | \mathbf{r}_A(\mathbf{i}_A), |\psi\rangle_{AB}) = \frac{1 - \cos[2(\theta_A - \theta_B)]}{4\pi}, \tag{26}$$

which contains the same 100%-visibility sinusoidal fringe pattern as obtained from the standard singlet-state polarization analysis.

Now suppose that elliptical polarization had been chosen for the standard approach—by inserting lossless quarter-wave plates in front of each polarizer—so that the photon detections corresponded to the polarization vectors

$$\mathbf{i}_A = \cos(\theta_A)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(\theta_A)\begin{bmatrix} 0 \\ i \end{bmatrix} \tag{27}$$

and

$$\mathbf{i}_B = \cos(\theta_B)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(\theta_B)\begin{bmatrix} 0 \\ i \end{bmatrix}. \tag{28}$$

Because the singlet-state is invariant to polarization-basis transformations, this new setup still yields Prob(B detected|A detected) given by Eq. (24). Once again, the same elliptical-polarization fringe pattern is seen in the continuous-POVM polarimeter's conditional probability density function for the B polarimeter's measurement outcome to be $\mathbf{r}_B^T(\mathbf{i}_B) = [0 \ \sin(2\theta_B) \ \cos(2\theta_B)]$, conditioned on knowledge that the A polarimeter's outcome was $\mathbf{r}_A^T(\mathbf{i}_A) = [0 \ \sin(2\theta_A) \ \cos(2\theta_A)]$.

Despite the preceding agreement between the quantum interference patterns seen in standard singlet-state polarization analysis and those contained in the conditional probability densities for continuous-POVM polarimetry, there is a significant difference between the two approaches. Both, of course, require that measurements be made on a sequence of

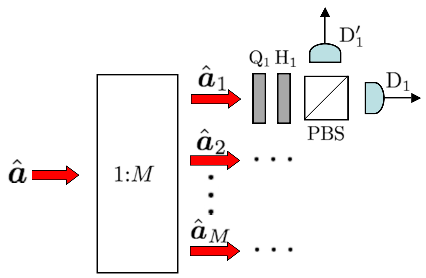

FIG. 3. (Color online) Approximate implementation of continuous-POVM polarimetry. For simplicity, only one of the polarization transformation and analysis units is shown. Q_1, quarter-wave plate; H_1, half-wave plate; PBS, polarizing beam splitter; D_1,D_1' single-photon detectors. All linear-optics elements are assumed to be lossless and the detectors are assumed to have unity quantum efficiencies and no dark counts.

singlet states to observe the quantum interference fringe. As noted earlier, the conventional approach also requires that one of the polarizer angles be varied, during data collection, to map out the full interference fringe. Because the measurement outcome from continuous-POVM polarimetry has a continuous probability density function on the Poincaré sphere, it is a practical impossibility to measure the conditional probability density function $p(\mathbf{r}_B(\mathbf{i}_B)|\mathbf{r}_A(\mathbf{i}_A),|\psi\rangle_{AB})$ for any fixed value of $\mathbf{r}_A(\mathbf{i}_A)$, owing to its conditioning event being a set of measure zero. To avoid this difficulty, we can exploit the singlet state's basis invariance by performing continuous-POVM polarimetry on the component photons of a sequence of singlets and use that data to estimate $p(\mathbf{r}_B|\mathbf{r}_A,|\psi\rangle_{AB})$ as a function of $2\Delta\theta \equiv \arccos(\mathbf{r}_A^T\mathbf{r}_B)$. In doing so we do not have to vary any polarizer settings. as was necessary in standard singlet-state polarimetry, and we use all the measurement data, i.e., every observed $(\mathbf{r}_A,\mathbf{r}_B)$ pair is included. As the number of singlet-state measurements grows without bound, this procedure will yield, from Eq. (21), an increasingly accurate estimate of the conditional probability density function

$$p(\mathbf{r}_B|\mathbf{r}_A,|\psi\rangle_{AB}) = \frac{1-\cos(2\Delta\theta)}{4\pi}, \tag{29}$$

which encompasses the entire basis-invariant quantum interference that is inherent in the singlet state.

IV. IMPLEMENTATION

We have introduced a continuous positive operator-valued measurement for the polarization state of a single photon propagating in a known spatiotemporal mode. This POVM enables uniformly-optimal measure-and-prepare state transmission at the classical limit on average teleportation fidelity. The use of two such polarimeters permits the entire basis-invariant quantum interference inherent in the singlet state to be probed simultaneously. Our final task will be to exhibit an experimental configuration that can approximate the continuous-POVM polarimeter. That setup is shown in Fig. 3. The incoming spatial mode, which contains the single photon whose polarization is to be measured, undergoes $1:M$ splitting after which standard projective polarization analysis is performed on each of the resulting output modes. The bases for these M projective measurements are chosen to be uniformly distributed over the Poincaré sphere. We now show that the measurement statistics from the Fig. 3 setup converge in distribution to those of continuous-POVM polarimetry as $M\to\infty$.

Let $\hat{\mathbf{a}}^T=[\hat{a}_x,\hat{a}_y]^T$ be the annihilation operators for the x and y polarizations of the single photon's spatiotemporal mode at the input to the Fig. 3 setup. The $1:M$ splitter yields a collection of spatiotemporal output modes whose annihilation operators $\hat{\mathbf{a}}_m^T=[\hat{a}_{m_x},\hat{a}_{m_y}]^T$, for $1\le m\le M$, obey

$$\begin{bmatrix} \hat{\mathbf{a}}_1 \\ \hat{\mathbf{a}}_2 \\ \vdots \\ \hat{\mathbf{a}}_M \end{bmatrix} = \mathbf{U}\begin{bmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{b}}_2 \\ \vdots \\ \hat{\mathbf{b}}_M \end{bmatrix}, \tag{30}$$

where $\mathbf{U}$ is a unitary matrix whose first column's entries are all $1/\sqrt{M}$ and the spatiotemporal modes associated with the annihilation operators $\{\hat{\boldsymbol{b}}_m:2\le m\le M\}$ are all in their vacuum states. The mth output from this splitter undergoes a quarter-wave-plate/half-wave-plate polarization transformation that maps the basis $(\mathbf{r}_m,-\mathbf{r}_m)$ into the x,y basis, where the $\{(\mathbf{r}_m,-\mathbf{r}_m):1\le m\le M\}$ bases are chosen to be uniformly distributed over the Poincaré sphere [8]. When the single-photon detector D_m clicks, we declare the measurement outcome to be $\mathbf{r}_m$. When the single-photon detector D_m' clicks, we declare the measurement outcome to be $-\mathbf{r}_m$. Because we have assumed ideal lossless operation in Fig. 3, and because only one photon enters this apparatus, one and only one of the photodetectors will click. All that remains is for us to examine the probability that a particular detector will click and relate that probability to the corresponding result expected from continuous-POVM polarimetry.

Suppose that the incoming photon has its polarization specified by $\mathbf{r}$ on the Poincaré sphere. The probability that this photon clicks detector D_m is

$$\mathrm{Prob}(D_m|\mathbf{r}) = \frac{1+\mathbf{r}^T\mathbf{r}_m}{2M}, \tag{31}$$

and likewise we have

$$\mathrm{Prob}(D_m'|\mathbf{r}) = \frac{1-\mathbf{r}^T\mathbf{r}_m}{2M}, \tag{32}$$

for the probability that D_m' is the detector this photon clicks. For $M\gg 1$, we may assume that the probability density for continuous-POVM polarimetry of this same photon is approximately constant over the $2\pi/M$ solid-angle regions whose centers are the $2M$ possible measurement outcomes from Fig. 3, i.e., $\{\mathbf{r}_m,-\mathbf{r}_m:1\le m\le M\}$. So, if we perform continuous-POVM polarimetry and then quantize its outcome to equal the center point of whichever of these regions occurred, we find that

$$\text{Prob}(\mathbf{r}_m|\mathbf{r}) \approx \frac{2\pi}{M} p(\mathbf{r}_m|\mathbf{r}) = \frac{1+\mathbf{r}^T\mathbf{r}_m}{2M} \tag{33}$$

and

$$\text{Prob}(-\mathbf{r}_m|\mathbf{r}) \approx \frac{2\pi}{M} p(-\mathbf{r}_m|\mathbf{r}) = \frac{1-\mathbf{r}^T\mathbf{r}_m}{2M}, \tag{34}$$

where the approximations become exact in the limit $M \to \infty$. As claimed, we have shown that the Fig. 3 setup yields measurement statistics that converge in distribution to those of continuous-POVM polarimetry.

The Fig. 3 configuration, with finite M, has some interesting properties that deserve mention. It implements the discrete POVM $\{\hat{\Pi}_m : 1 \leq m \leq 2M\}$ given by

$$\hat{\Pi}_m \equiv \begin{cases} \dfrac{|i(\mathbf{r}_m)\rangle\langle i(\mathbf{r}_m)|}{2M}, & \text{for } 1 \leq m \leq M, \\[2ex] \dfrac{|i(-\mathbf{r}_{m-M})\rangle\langle i(-\mathbf{r}_{m-M})|}{2M}, & \text{for } M+1 \leq m \leq M. \end{cases} \tag{35}$$

This discrete POVM is the minimum error probability quantum measurement for distinguishing between the $2M$ single-photon pure states $\{|\mathbf{r}_m\rangle, |-\mathbf{r}_m\rangle : 1 \leq m \leq M\}$ when they are all equally likely to occur, as can be verified by substitution into the necessary and sufficient conditions for minimum error probability $2M$-ary quantum detection given in Ref. [9]. Furthermore, this POVM is also the square-root measurement for that quantum detection problem, see, e.g., Ref. [10].

V. CONCLUSIONS

We have introduced a continuous POVM for measuring the polarization state of a single photon in a known spatiotemporal mode. This POVM is the maximum likelihood quantum measurement for estimating that unknown polarization, and the resulting maximum-likelihood estimate enables uniformly optimal measure-and-prepare state transmission via classical communication at the $\bar{F}=2/3$ classical limit on average teleportation fidelity. We have exhibited an idealized configuration—employing a $1:M$ optical splitter followed by a collection of M projective polarization measurements, whose bases are uniformly distributed over the Poincaré sphere—that yields measurement statistics which converge in distribution to those of the continuous-POVM polarimeter. This configuration is far more extravagant in its use of wave plates and detectors than the familiar two-detector discrete polarimeters that are employed, for example, in verifying the polarization entanglement produced from spontaneous parametric down-conversion. Nevertheless, it is theoretically relevant to note that the continuous-POVM polarimeter has at least a notional realization, given its preferred position as the maximum likelihood quantum measurement for single-photon polarization.

In closing, one additional point deserves mention. Our presentation has focused on the continuous-POVM polarimeter as the optimal—in maximum-likelihood sense—quantum measurement for determining the unknown polarization state of a single photon under the presumption that said photon is in a pure state. There is an established body of work [11,12] on maximum-likelihood estimation of the common density operator of an ensemble of independent, identically prepared quantum systems when arbitrary POVMs are made on a sequence of systems drawn from this ensemble. This theory of ML density-operator estimation can therefore be used, in conjunction with our continuous-POVM polarimeter, to extend our work to estimating the common mixed state of an ensemble of independent single photons.

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