

## Quantum measurements without macroscopic superpositions

Dominique Spehner<sup>1,\*</sup> and Fritz Haake<sup>2</sup>

<sup>1</sup>*Institut Fourier, 100 rue des Maths, 38402 Saint-Martin d'Hères, France*

<sup>2</sup>*Fachbereich Physik, Universität Duisburg-Essen, 47048 Duisburg, Germany*

(Received 16 October 2007; published 20 May 2008)

We study a class of quantum measurement models. A microscopic object is entangled with a macroscopic pointer such that each eigenvalue of the measured object observable is tied up with a specific pointer deflection. Different pointer positions mutually decohere under the influence of a bath. Object-pointer entanglement and decoherence of distinct pointer readouts proceed simultaneously. Mixtures of macroscopically distinct object-pointer states may then arise without intervening macroscopic superpositions. Initially, object and apparatus are statistically independent while the latter has pointer and bath correlated according to a metastable local thermal equilibrium. We obtain explicit results for the object-pointer dynamics with temporal coherence decay in general neither exponential nor Gaussian. The decoherence time does not depend on details of the pointer-bath coupling if it is smaller than the bath correlation time, whereas in the opposite Markov regime the decay depends strongly on whether that coupling is Ohmic or super-Ohmic.

DOI: [10.1103/PhysRevA.77.052114](https://doi.org/10.1103/PhysRevA.77.052114)

PACS number(s): 03.65.Ta, 03.65.Yz

### I. INTRODUCTION

The interpretation and theoretical description of measurements on quantum systems have been under intense debate since the birth of quantum mechanics [1,2]. In the last three decades the major role played by environment-induced decoherence in a measurement process has been fully acknowledged thanks to the works of Zeh, Zurek, and others (see [3–6], and references therein). A renewal of interest for quantum detection and decoherence came in the last decade with new developments in quantum information. It is desirable to better understand the relation between quantum and classical information and how one can convert one into another. Moreover, good control over all sources of decoherence is required for quantum-information processing. On the experimental side, measurements can be used either to extract information on quantum states or to monitor quantum systems (quantum trajectories [7], quantum Zeno effect [8,9]). Experimental data are now available for the decoherence time in microwave cavities [10], in trapped ions [11], in solid-state devices such as quantum dots [12] and superconducting tunnel junction nanocircuits [13,14], for fullerene molecules decohered by collisions with a background gas [15], and for beams of electrons decohered by Coulomb interaction with a semiconducting plate [16]. These and other experiments call for studies of concrete models for quantum measurements. Various models have been investigated so far (see, e.g., [17] and the interesting statistical physics models of Refs. [18,19]), but a satisfactory treatment of decoherence resulting from the many-body interactions in the measurement apparatus is still lacking.

A measurement on a quantum system consists in letting this system (called “object” in the following) interact with a measurement apparatus in such a way that some information about the state of the object is transferred to the apparatus. As already recognized by Bohr, even though the composite

(object and apparatus) system has to be described by quantum theory, some part of the apparatus (called the “pointer” in the following) must be capable of classical behavior. The interaction builds up a one-to-one correspondence between the eigenvalues  $s$  of the measured observable  $S$  (supposed here to have a discrete spectrum) and macroscopically distinguishable pointer states (characterized, e.g., by sharply defined pointer positions separated by macroscopic distances). In addition to this object-apparatus coupling, the measurement must involve some “superselection rules” destroying the coherences between the pointer states [3–5,20]. Most previous discussions in the literature consider these two processes separately: A first step (“premeasurement”) exclusively treats the unitary evolution entangling object and pointer. For an object initially uncorrelated with the apparatus and in a linear superposition  $|\psi_S\rangle = \sum_s c_s |s\rangle$  of eigenstates of  $S$ , this entanglement produces a superposition of macroscopically distinguishable object-pointer states  $|\Psi_{\text{ent}}\rangle = \sum_s c_s |s\rangle \otimes |\psi_P^s\rangle$ , where  $|\psi_P^s\rangle$  is the pointer state tied up with the eigenvalue  $s$  (for simplicity we provisionally assume that this state is pure). The “Schrödinger cat” state  $|\Psi_{\text{ent}}\rangle$  is taken as the initial state for a second dynamical process, decoherence, which leads to superselection rules; there, the quantum correlations between object and apparatus are transformed into classical correlations, as the superposition of object-pointer states is degraded to a statistical mixture of the same states according to  $|\Psi_{\text{ent}}\rangle\langle\Psi_{\text{ent}}| \rightarrow \sum_s |c_s|^2 |s\rangle\langle s| \otimes |\psi_P^s\rangle\langle\psi_P^s|$ . For such a sequential treatment to make physical sense, the duration of the entanglement process must be short compared with the decoherence time  $t_{\text{dec}}$  associated with the latter transformation. However, it is known that  $t_{\text{dec}}$  is extremely short for macroscopic superpositions. The present paper is devoted to the more realistic situation where entanglement and decoherence proceed simultaneously. If the characteristic time for entanglement is larger than  $t_{\text{dec}}$ , macroscopic superpositions decohere to mixtures faster than entanglement can create them. The measurement process then yields the final mixture without involving a Schrödinger cat state at any previous moment, the object-pointer initial product state being directly transformed as

\*spehner@ujf-grenoble.fr

$$|\psi_S\rangle\langle\psi_S| \otimes |\psi_P^0\rangle\langle\psi_P^0| \rightarrow \sum_s |c_s|^2 |s\rangle\langle s| \otimes |\psi_P^s\rangle\langle\psi_P^s|. \quad (1)$$

Here  $|\psi_S\rangle = \sum_{s,c_s} |s\rangle$ ,  $|\psi_P^0\rangle$ , and  $|\psi_P^s\rangle$  refer to the object initial state, the pointer initial state, and the pointer state tied up with  $s$ , respectively.

Our model is a three-partite model and consists of the quantum object to be measured, a single “pointer” degree of freedom of the apparatus singled out by its strong coupling to the object and by affording a macroscopic range of “read-out” values and a “bath” comprising all other degrees of freedom of the apparatus. A pointer-bath coupling is responsible for decoherence. Overcoming limitations of many previous approaches, we (i) allow for object-pointer entanglement and decoherence of distinct pointer readouts to proceed simultaneously, (ii) cope with initial correlations between pointer and bath by considering them initially in a metastable local thermal equilibrium, and (iii) go beyond the Markovian treatment of decoherence. The physical relevance of point (i) has been discussed above. This simultaneity of entanglement and decoherence and the possibility of having decoherence much faster than entanglement have been considered in [18,19]. Let us now comment on (ii) and (iii). Most models studied so far (in particular in [18,19]) are based on the assumption that the pointer and bath are initially statistically independent. Taking instead the whole apparatus to be initially in a local thermal equilibrium seems more realistic. The Markov approximation mentioned in (iii) consists in neglecting memory effects for the (reduced) object-pointer dynamics. It assumes the decoherence time  $t_{\text{dec}}$  to be larger than the bath correlation time, a condition not satisfied in some experiments [12–14]. Since decoherence for macroscopic and even mesoscopic superpositions is faster than bath relaxation [21], this approximation is clearly unjustified if such superpositions arise during the object-pointer evolution.

We shall assume a certain ordering of time scales. One of these, denoted by  $T_S$ , is the characteristic time for the evolution of the measured observable  $S$  under the Hamiltonian of the object. A second (classical) time scale  $T_P$  characterizes significant changes in position of the pointer under its proper Hamiltonian (i.e., in the absence of coupling with the object). The initial temperature  $T$  of the apparatus sets a time scale  $\hbar\beta = \hbar / (k_B T)$ , referred to below as the thermal time ( $k_B$  is the Boltzmann constant). Finally, the object and pointer are put in contact during a time  $t_{\text{int}}$ . For a macroscopic pointer, the limit  $\hbar\beta \ll T_P$  seems difficult to avoid. Similarly, the decoherence and object-pointer interaction times  $t_{\text{dec}}$  and  $t_{\text{int}}$  are small compared with  $T_P$ . During an ideal measurement, the measured observable  $S$  may change but weakly under the full (object+apparatus) Hamiltonian  $H$ —i.e.,  $e^{iH/\hbar} S e^{-iH/\hbar} \simeq S$  for  $0 \leq t \leq t_{\text{int}}, t_{\text{dec}}$ . Only under this condition can an eigenstate of the measured observable  $S$  be left almost unchanged by the measurement. One has to require that (i) the object-pointer interaction Hamiltonian producing the entanglement commute with  $S$  (see [3–5]) and (ii)  $T_S$  be much larger than  $t_{\text{int}}$  and  $t_{\text{dec}}$ . It is thus legitimate to assume

$$t_{\text{int}}, t_{\text{dec}} \ll T_S, \quad t_{\text{int}}, t_{\text{dec}}, \hbar\beta \ll T_P. \quad (2)$$

As far as we are aware, this separation of time scales in ideal measurements has not been fully exploited in previous works

except in Ref. [18]. Unlike in the latter reference, given (2) we shall not need a further hypothesis on the bath correlation time  $T_B$  and its relation with  $t_{\text{int}}$  and  $t_{\text{dec}}$ .

A further key input in what follows is the quantum central-limit theorem (QCLT) [22,23] which implies Gaussian statistics (Wick theorem) for the bath coupling agent in the pointer-bath interaction. This will allow us to study a broad class of pointers and baths, following the approach of Ref. [21]. The harmonic oscillator bath linearly coupled to the pointer [24,25] is one member of this class, but more general (nonharmonic) baths as well as nonlinear couplings in the position  $X$  of the pointer will be also considered. It turns out that the decoherence time  $t_{\text{dec}}$  may be considerably reduced by allowing such nonlinear couplings.

The paper is organized as follows. The model and its different time scales are introduced in Sec. II. We discuss the separation of the time scales (2) and the ensuing simplification of the object-pointer dynamics in Sec. III. Section IV contains a separate study of the two dynamical processes producing the entanglement of the object with the pointer and the loss of coherences between well-separated pointer readings. That section is pedagogical in character and serves to fix the notation and to introduce the relevant time scales; readers familiar with the theory of quantum measurement might want to skip the section save for Secs. IV C and IV D. Our principal results are presented in Sec. V, discussed in Sec. VI, and finally derived in Sec. VII. Our conclusions are drawn in Sec. VIII. Appendix A and Appendix B are devoted to an example for a measurement apparatus and to a technical derivation of an approximation for the pointer-bath thermal state. We review in Appendix C the general properties of two-point correlation functions used in Secs. V–VII. Finally, we discuss the QCLT and its consequences (Wick theorem for the bath correlation functions) in Appendix D. Let us point out that a short report of our results can be found in [26].

Before going on, some remarks may be permitted to put our paper in perspective. Remaining within the frame of quantum mechanics and its probabilistic interpretation, we are concerned with unitary evolution of the composite system object+apparatus. We discard information about the (dynamics of) the microscopic degrees of freedom of the apparatus (“bath”) and their entanglement with pointer and object by tracing out the bath (see Sec. IV B and [3,6,27]). We so obtain a reduced object-pointer density operator with an irreversible evolution. Pointer and object end up in the mixed state (1) wherein the different pointer states  $|\psi_P^s\rangle$  correspond to macroscopically distinguishable positions. Such states have quantum uncertainties in position and momentum much smaller than the scales of macroscopic readings. Therefore, the irreversible process (1) leaves a pointer position revealing an eigenvalue of the measured object variable. Over many runs of the measurement, the outcome  $|\psi_P^s\rangle$  arises with probability  $|c_s|^2$ . Similar behavior arises for all processes where initial microscopic fluctuations evolve toward macroscopically distinct outcomes. A nice example is provided by superfluorescence where light pulses with substantial shot-to-shot fluctuations grow from initial quantum uncertainties [28–30]. Let us also recall that quantum mechanics is not compatible with the idea that the specific

outcome of a single run is predetermined by some unknown but “real” property of the object (such a property being independent of the measurement apparatus) [31]. Competing hidden-variable theories which indulge in such more “intuitive” notions of reality than quantum mechanics have been experimentally falsified [32,33] in ever larger classes, most recently even in nonlocal variants [34], while the respective quantum predictions were invariably confirmed.

## II. MODEL

As many authors [3–6,17–19] we consider a three-partite system: the object of measurement is some microscopic system ( $S$ ); a single-degree-of-freedom macroscopic pointer ( $\mathcal{P}$ ) will allow readouts; finally, a bath ( $\mathcal{B}$ ) with many ( $N \gg 1$ ) degrees of freedom serves to decohere distinct pointer readings. We shall have to deal with the following dynamical variables: for  $S$ , the observable  $S$  to be measured; for  $\mathcal{P}$ , the position  $X$  and momentum  $P$ ; and for  $\mathcal{B}$ , a certain coupling agent  $B$  given by a sum of  $N$  self-adjoint operators  $B_\nu$  acting on single degrees of freedom of the bath. The pointer is coupled to  $S$  and  $\mathcal{B}$  via the Hamiltonians

$$H_{\mathcal{P}S} = \epsilon SP, \quad H_{\mathcal{P}B} = BX^\alpha, \quad B = N^{-1/2} \sum_{\nu=1}^N B_\nu, \quad (3)$$

where  $\epsilon$  is a coupling constant and  $\alpha$  a positive integer. The object-pointer coupling  $H_{\mathcal{P}S}$  is chosen so as to (i) not change the measured observable  $S$  (i.e.,  $[H_{\mathcal{P}S}, S]=0$ ); (ii) be capable of shifting the pointer position by an amount proportional to  $S$ , in such a way that each eigenvalue  $s$  of  $S$  becomes tied up with a specific pointer reading; (iii) be a strong coupling ( $\epsilon$  is large), so that different eigenvalues  $s \neq s'$  eventually become associated with pointer readings separated by large distances. The pointer-bath interaction  $H_{\mathcal{P}B}$  is chosen for the most efficient decoherence of distinct pointer positions [21]. Depending on the value of  $\alpha$ , nonlinear ( $\alpha > 1$ ) as well as linear ( $\alpha = 1$ ) couplings will be considered [35]. The additivity of the bath coupling agent  $B$  in single-degree-of-freedom contributions  $B_\nu$  having zero mean and positive variance with respect to the bath thermal state will allow us to invoke the quantum central-limit theorem when taking the limit  $N \rightarrow \infty$ . The factor  $N^{-1/2}$  in front of the sum in (3) is introduced for convergence purposes (the same scaling with  $N$  is familiar to the classical CLT); note that the pointer-bath coupling constants are incorporated within the operators  $B_\nu$ .

The free evolutions of  $S$ ,  $\mathcal{P}$ , and  $\mathcal{B}$  are generated by the respective Hamiltonians  $H_S$ ,  $H_{\mathcal{P}}$ , and  $H_B$ . We do not have to specify  $H_S$ . The pointer Hamiltonian  $H_{\mathcal{P}} = P^2/2M + V(X)$  has a potential  $V(x)$  with a local minimum at  $x=0$ , so that  $V'(0)=0$  and  $V''(0) > 0$ . The bath Hamiltonian  $H_B$  is like  $B$  a sum of Hamiltonians acting on single degrees of freedom,  $H_B = \sum_\nu H_{B,\nu}$ . We thus disregard couplings between different degrees of freedom of the bath. The Hamiltonian of the full system  $S+\mathcal{P}+\mathcal{B}$  is  $H = H_S + H_{\mathcal{P}} + H_B + H_{\mathcal{P}S} + H_{\mathcal{P}B}$ . An example of a physical system realizing the apparatus  $\mathcal{P}+\mathcal{B}$  is given in Appendix A.

We now proceed to describing the initial states allowed for. It is appropriate to require initial statistical independence

between object and apparatus. The initial density operator  $\rho_S$  of the object may represent a pure or a mixed state. Two types of initial conditions for the apparatus will be considered. The first one, to be referred to as partial equilibrium, is a product state in which  $\mathcal{P}$  has some density operator  $\rho_{\mathcal{P}}$  and  $\mathcal{B}$  is at thermal equilibrium with the Gibbs density operator  $\rho_B^{(eq)} = Z_B^{-1} \exp(-\beta H_B)$ , wherein  $\beta = (k_B T)^{-1}$  is the inverse temperature. For this first initial state all three subsystems are statistically independent. In the second (more realistic) initial state, the apparatus is in thermal equilibrium according to the density operator  $\rho_{\mathcal{P}B}^{(eq)} = Z_{\mathcal{P}B}^{-1} \exp(-\beta(H_{\mathcal{P}} + H_B + H_{\mathcal{P}B}))$ . The two initial states of  $S+\mathcal{P}+\mathcal{B}$  are

$$\rho(0) = \rho_S \otimes \rho_{\mathcal{P}} \otimes \rho_B^{(eq)}, \quad \text{partial-equilibrium apparatus,} \quad (4a)$$

$$\rho(0) = \rho_S \otimes \rho_{\mathcal{P}B}^{(eq)}, \quad \text{equilibrium apparatus.} \quad (4b)$$

We further specify the partial-equilibrium state (4a) by requiring that the probability density  $\langle x | \rho_{\mathcal{P}} | x \rangle$  to find the pointer at position  $x$  has a single peak of width  $\Delta x = \Delta$  centered at  $x=0$ . A momentum uncertainty  $\Delta p = 2\pi\hbar/\lambda$  defines a second length scale  $\lambda$ . A macroscopic pointer has both  $\Delta$  and  $\lambda$  negligibly small against any macroscopic readout scale  $\Delta_{\text{class}}$

$$\lambda \leq 4\pi\Delta \ll \Delta_{\text{class}}, \quad (5)$$

where the first inequality is the uncertainty principle. We shall also require that

$$\frac{\lambda\Delta}{2\pi\hbar} = \frac{\Delta x}{\Delta p} \approx [MV''(0)]^{-1/2}, \quad (6)$$

which means that the state  $\rho_{\mathcal{P}}$  is not highly squeezed in momentum or in position. As a concrete example we may consider a Gaussian pointer density matrix

$$\langle x | \rho_{\mathcal{P}} | x' \rangle = \frac{1}{\sqrt{2\pi\Delta^2}} e^{-(x+x')^2/(8\Delta^2)} e^{-2\pi^2(x-x')^2/\lambda^2}. \quad (7)$$

If  $\mathcal{P}$  is initially in a pure state, then  $\text{tr}_{\mathcal{P}} \rho_{\mathcal{P}}^2 = \int dx dx' \langle x | \rho_{\mathcal{P}} | x' \rangle^2 = 1$ , which implies that this state has the minimum uncertainty product  $\Delta x \Delta p = \hbar/2$ —i.e.,  $\lambda = 4\pi\Delta$ .

The Gaussian density (7) also arises if  $\mathcal{P}$  is in a Gibbs state  $\rho_{\mathcal{P}}^{(eq)} = Z_{\mathcal{P}}^{-1} e^{-\beta H_{\mathcal{P}}}$  provided that the potential  $V(x)$  is confining and  $\beta$  is small enough. To see this, we note that the pointer observables  $X$  and  $P$  evolve noticeably under the Hamiltonian  $H_{\mathcal{P}}$  on a *classical time scale*  $T_{\mathcal{P}}$ , which is much larger than all other (quantum) time scales in the model. In particular,  $T_{\mathcal{P}}$  is much larger than the thermal time,  $T_{\mathcal{P}} \gg \hbar\beta$ . As a result, the matrix elements  $\langle x | \rho_{\mathcal{P}}^{(eq)} | x' \rangle$  of  $\rho_{\mathcal{P}}^{(eq)}$  can be approximated by  $Z_{\mathcal{P}}^{-1} e^{-\beta[V(x)+V(x')]/2} e^{-2\pi^2(x-x')^2/\lambda_{\text{th}}^2}$ , wherein  $\lambda_{\text{th}} = 2\pi\hbar(\beta/M)^{1/2}$  is the thermal de Broglie wavelength. The reader may recognize in this expression the short-time behavior of the quantum propagator  $\langle x | e^{-iH_{\mathcal{P}}t/\hbar} | x' \rangle$  for  $t = -i\hbar\beta$  (see, e.g., [36]). Since the potential  $V(x)$  has a local minimum at  $x=0$ , it can be approximated near the origin by a quadratic potential,  $V(x) \approx V(0) + x^2 V''(0)/2$ . Therefore, for small  $x$  and  $x'$ ,  $\langle x | \rho_{\mathcal{P}}^{(eq)} | x' \rangle$  has the Gaussian form (7) with  $\Delta = \Delta_{\text{th}} = [\beta V''(0)]^{-1/2}$  and  $\lambda \approx \lambda_{\text{th}}$ . It is important to bear in

mind the separation of length scales  $\lambda_{\text{th}} \ll \Delta_{\text{th}} \ll \Delta_{\text{class}}$ . Inasmuch as the pointer classical time scale  $T_{\mathcal{P}}$  may be defined as  $T_{\mathcal{P}} = [M/V''(0)]^{1/2}$ , the fact that  $\lambda_{\text{th}}$  is much smaller than  $\Delta_{\text{th}}$  is equivalent to  $T_{\mathcal{P}} \gg \hbar\beta$ . To fix ideas, for  $T_{\mathcal{P}} = 1$  s,  $M = 1$  g,  $\Delta_{\text{class}} = 1$  cm, and a temperature of 1 K, the above-mentioned length scales differ by more than eight orders of magnitude. Hence (5) and (6) are well satisfied if  $\rho_{\mathcal{P}} = \rho_{\mathcal{P}}^{(\text{eq})}$ .

All of these illustrations, including the Gaussian (7), are meant to give an intuitive picture. What we shall need in actual fact is the quasiclassical nature of the pointer initial state, as implied by (5) and (6), together with the single-peak character of the initial density of pointer positions.

Let us point out an essential difference between our model and the interacting spin model of [19]. Unlike in this reference,  $\mathcal{S}$  is strongly coupled to a single degree of freedom (the pointer  $\mathcal{P}$ ) of the apparatus, e.g., with its total momentum  $P$  in a given direction (see Appendix A). The coupling of  $\mathcal{S}$  with the other apparatus degrees of freedom (the bath  $\mathcal{B}$ , for us) is assumed to be much weaker and can therefore be neglected, as will be seen in Sec. VI C. Given the separation of time scales (2) and our choice of a quasiclassical pointer initial state, the pointer Hamiltonian  $H_{\mathcal{P}}$  only plays a role in providing an amplification mechanism, as we shall see in Secs. III B and IV D. Hence allowing  $\mathcal{P}$  to have two or three degrees of freedom, instead of one, would make the notation more cumbersome without changing significantly the results. Our results below should also remain valid if the bath consists of interacting degrees of freedom (like in a spin chain) provided that the (spin-spin) correlations  $\langle B_{\mu} B_{\nu} \rangle$  in the bath thermal state decay more rapidly than  $1/|\mu - \nu|$  as  $|\mu - \nu| \rightarrow \infty$ . In fact, the validity of the QCLT can be extended in this context [23]. Decoherence via coupling with a bath of interacting spins and random matrix models for the coupling and bath have been considered in [19,37,38].

We shall study the dynamics of the reduced state of  $\mathcal{S} + \mathcal{P}$  (object and pointer). That state is defined by a density operator  $\rho_{\mathcal{P}\mathcal{S}}(t)$  obtained by tracing out the bath degrees of freedom in the state of  $\mathcal{S} + \mathcal{P} + \mathcal{B}$ ,

$$\rho_{\mathcal{P}\mathcal{S}}(t) = \text{tr}_{\mathcal{B}}[e^{-itH/\hbar} \rho(0) e^{itH/\hbar}]. \quad (8)$$

Here and in what follows,  $\text{tr}_j$  refers to the partial trace over the Hilbert space of  $j = \mathcal{S}, \mathcal{P}$ , or  $\mathcal{B}$ . When tracing out the bath we admit the inability to acquire information about it [27].

### III. SEPARATION OF TIME SCALES

#### A. Time scales of object, pointer, and bath

Let us denote by  $\tilde{S}(t)$  the time-evolved observable  $S$  in the absence of the coupling  $H_{\mathcal{P}\mathcal{S}}$ —i.e., for the dynamics implemented by the “free Hamiltonian”  $H_{\mathcal{S}}$ . Similarly, let  $\tilde{X}(t)$  and  $\tilde{B}(t)$  be the time-evolved observables  $X$  and  $B$  when both couplings  $H_{\mathcal{P}\mathcal{S}}$  and  $H_{\mathcal{P}\mathcal{B}}$  are turned off: namely,

$$\tilde{O}_j(t) = e^{itH_j/\hbar} O_j e^{-itH_j/\hbar},$$

$$O_j = S, X, \text{ or } B, \quad j = \mathcal{S}, \mathcal{P}, \text{ or } \mathcal{B}. \quad (9)$$

One may associate with the time evolution of  $\tilde{X}(t)$ ,  $\tilde{B}(t)$ , and  $\tilde{S}(t)$  four distinct time scales. The time scale  $T_{\mathcal{P}}$

$= [M/V''(0)]^{1/2}$  has been already introduced in Sec. II; it is the time scale for significant evolution of  $\tilde{X}(t)$  [or, equivalently, of  $\tilde{P}(t) = M d\tilde{X}/dt$ ] when the pointer is in the initial state  $\rho_{\mathcal{P}}$ . The Gaussian form (7) for  $\rho_{\mathcal{P}}$  and the no-squeezing condition (6) make sure that  $T_{\mathcal{P}}$  is indeed a classical time.

The bath correlation time  $T_{\mathcal{B}}$  is defined with the help of the  $n$ -point correlation functions

$$h_n(t_1, \dots, t_n) = \text{tr}_{\mathcal{B}}[\tilde{B}(t_1) \cdots \tilde{B}(t_n) \rho_{\mathcal{B}}^{(\text{eq})}]. \quad (10)$$

For simplicity we assume

$$\text{tr}_{\mathcal{B}}(B \rho_{\mathcal{B}}^{(\text{eq})}) = 0. \quad (11)$$

Since the bath has infinitely many degrees of freedom,  $h_n(t_1, \dots, t_n)$  decays to zero as  $|t_m - t_l|$  goes to infinity. We define  $T_{\mathcal{B}}(t_{\mathcal{B}})$  as the largest (smallest) time constant characterizing the variations of  $h_n$ . It follows from the QCLT of Ref. [22] that for a bath coupling agent  $B$  and Hamiltonian  $H_{\mathcal{B}}$  which are sums of  $N$  independent contributions coming from single degrees of freedom, the  $n$ -point functions (10) satisfy the bosonic Wick theorem in the limit  $N \gg 1$ . This means that  $h_n$  vanishes if  $n$  is odd and is given if  $n$  is even by the sums of products of two-point functions,

$$h_n(t_1, \dots, t_n) = \sum_{\text{pairing of } \{1, \dots, n\}} h_2(t_{i_1}, t_{j_1}) \cdots h_2(t_{i_{n/2}}, t_{j_{n/2}}). \quad (12)$$

That manifestation of the QCLT amounts to Gaussian statistics for the bath correlation functions. It follows from (12) that  $T_{\mathcal{B}}(t_{\mathcal{B}})$  can be defined more simply as the largest (smallest) time scale associated with the variations of  $h_2(t_1, t_2) = h_2(t_1 - t_2)$  as function of  $t = t_1 - t_2$ . More precisely,  $h_2(t) \simeq 0$  whenever  $|t| \gg T_{\mathcal{B}}$  and  $h_2(t) \simeq h_2(0)$  whenever  $|t| \ll t_{\mathcal{B}}$ . Note that with  $\mathcal{B}$  in thermal equilibrium, the thermal time  $\hbar\beta$  figures among the decay rates of  $h_2$  and thus  $t_{\mathcal{B}} \leq \hbar\beta \leq T_{\mathcal{B}}$ .

The time scale  $T_{\mathcal{S}}$  is defined in analogy to  $t_{\mathcal{B}}$ , so as to signal significant variation of the object  $n$ -point functions  $\text{tr}_{\mathcal{S}}[\tilde{S}(t_1) \cdots \tilde{S}(t_n) \rho_{\mathcal{S}}]$ . Let us stress that  $T_{\mathcal{S}}$  can be larger than the typical inverse Bohr frequency  $\hbar/|E - E'|$  of  $\mathcal{S}$  (here  $E$  and  $E'$  are two eigenvalues of  $H_{\mathcal{S}}$ ). For instance, if  $S$  is (or commutes with) the energy  $H_{\mathcal{S}}$ , then  $T_{\mathcal{S}} = \infty$ .

#### B. Simplified dynamics and initial state

We assume that the object and pointer observables  $\tilde{S}(t)$ ,  $\tilde{X}(t)$ , and  $\tilde{P}(t)$  do not evolve noticeably under the “free” Hamiltonian  $H_{\mathcal{S}} + H_{\mathcal{P}}$  during the time span of the measurement, so that  $t_{\text{int}}, t_{\text{dec}} \ll T_{\mathcal{S}}, T_{\mathcal{P}}$ . We show now that thanks to this separation of time scales, the impact of  $H_{\mathcal{S}}$  and  $H_{\mathcal{P}}$  on the dynamics can be fully accounted for at times  $t \ll T_{\mathcal{S}}, T_{\mathcal{P}}$  by modifying the initial states (4a) and (4b) according to

$$\rho(0) \rightarrow e^{-it(H_{\mathcal{S}} + H_{\mathcal{P}})/\hbar} \rho(0) e^{it(H_{\mathcal{S}} + H_{\mathcal{P}})/\hbar}. \quad (13)$$

With that slippage of the initial condition accounted for, one makes a small error by otherwise dropping  $H_{\mathcal{S}}$  and  $H_{\mathcal{P}}$  from the total Hamiltonian  $H$  in the object-pointer state (8).

Actually, for times  $t$  short compared with  $T_{\mathcal{S}}$  and  $T_{\mathcal{P}}$ , the full evolution operator can be approximated by

$$e^{-itH/\hbar} \simeq e^{-it(H_B+H_{PS}+H_{PB})/\hbar} e^{-it(H_S+H_P)/\hbar}, \quad |t| \ll T_S, T_P. \quad (14)$$

To justify that simplification we express this evolution operator in the interaction picture with respect to  $H_0=H_S+H_P$  as

$$e^{itH_0/\hbar} e^{-itH/\hbar} = \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau [H_B + \epsilon \tilde{S}(\tau) \tilde{P}(\tau) + B \tilde{X}(\tau)^\alpha] \right\}; \quad (15)$$

here,  $\mathcal{T}$  denotes the time ordering and  $\tilde{S}(\tau)$ ,  $\tilde{P}(\tau)$ , and  $\tilde{X}(\tau)$  are given by (9). Note that for  $|t| \ll T_S, T_P$  these operators are almost constant in time between  $\tau=0$  and  $\tau=t$  and may thus be replaced in (15) by  $S$ ,  $P$ , and  $X$ . In other words, the right-hand side of (15) can be approximated by  $\exp\{-it(H_B + H_{PS} + H_{PB})/\hbar\}$ , whereupon (14) is obtained by taking the adjoint and by setting  $t \rightarrow -t$ .

More specific remarks are in order for each of our two initial states (4a) and (4b). We first comment on the partial equilibrium (4a). Due to its assumed quasiclassical nature, the pointer state  $\rho_P$  is weakly modified by the substitution (13) in the range of time under study. Deferring the justification of that statement to Appendix B we shall use  $e^{-itH_P/\hbar} \rho_P e^{itH_P/\hbar} \simeq \rho_P$  for  $t \ll T_P$ . As regards the object of measurement,  $\mathcal{S}$ , the free time evolution of the object in

$$\rho_S^0(t) = e^{-itH_S/\hbar} \rho_S e^{itH_S/\hbar} \quad (16)$$

cannot be neglected, even for  $t \ll T_S$ . [For instance, for  $S = H_S$  one has  $T_S = \infty$  and  $\langle s | \rho_S^0(t) | s' \rangle = e^{-it(s-s')/\hbar} \langle s | \rho_S | s' \rangle$  is not close to  $\langle s | \rho_S | s' \rangle$  for all finite times  $t$  if  $s \neq s'$ .] However, it suffices for our purposes to notice that the diagonal elements  $\langle s | \rho_S^0(t) | s \rangle \simeq \langle s | \rho_S | s \rangle$  remain nearly unaffected by the free evolution when  $t \ll T_S$ . Actually,  $\text{tr}[\tilde{S}(t) \rho_S] = \sum_s s \langle s | \rho_S^0(t) | s \rangle$  has to approximate  $\text{tr}(S \rho_S) = \sum_s s \langle s | \rho_S | s \rangle$  in this limit by definition of  $T_S$ .

When allowing the apparatus to start out from thermal equilibrium according to (4b), we shall take advantage of the ‘‘high-temperature’’ condition  $\hbar\beta \ll T_P$  discussed in the Introduction. We argue in Appendix B that under this condition (which to violate for a macroscopic pointer would be a nearly impossible task) and for a weak enough pointer-bath coupling satisfying  $\eta_{\text{th}} = h_2(0)^{1/2} \Delta_{\text{th}}^\alpha \beta \ll 1$ , the Gibbs state of the apparatus can be approximated by

$$\rho_{PB}^{(\text{eq})} \simeq \frac{1}{Z_{PB}} e^{-\beta H_P/2} e^{-\beta(H_B+H_{PB})} e^{-\beta H_P/2}, \quad \hbar\beta \ll T_P. \quad (17)$$

Moreover,  $e^{-itH_P/\hbar} \rho_{PB}^{(\text{eq})} e^{itH_P/\hbar} \simeq \rho_{PB}^{(\text{eq})}$  as long as  $t, \hbar\beta \ll T_P$ .

In conclusion, for both initial states (4a) and (4b), the substitution (13) amounts to replacing  $\rho_S$  by  $\rho_S^0(t)$  in the object-pointer initial state.

## IV. ENTANGLEMENT AND DECOHERENCE SEPARATED

### A. Entanglement of object and pointer

Before studying the dynamics generated by the total Hamiltonian  $H$ , it is instructive to discuss what happens if we discard the free dynamics of  $\mathcal{S}$ ,  $\mathcal{P}$ , and  $\mathcal{B}$  as well as the

pointer-bath interaction. For the initial state (4a), the bath can then be ignored and the object-pointer entanglement produced by the interaction  $H_{PS} = \epsilon SP$  becomes particularly easy to describe. Recalling that  $P$  is the generator of space translations we have  $e^{i\epsilon SP/\hbar} |s, x\rangle = |s, x - t\epsilon s\rangle$ , where  $|s, x\rangle$  is the joint eigenstate of  $S$  and  $X$  with eigenvalues  $s$  and  $x$ , normalized as  $\langle s, x | s', x' \rangle = \delta_{ss'} \delta(x - x')$ . Hence an initial product state of  $\mathcal{S} + \mathcal{P}$  becomes at time  $t$

$$\begin{aligned} \rho_{PS}(t) &= e^{-itH_{PS}/\hbar} \rho_S \otimes \rho_P e^{itH_{PS}/\hbar} \\ &= \sum_{s, s'} \langle s | \rho_S | s' \rangle |s\rangle \langle s'| \\ &\quad \times \int dx dx' \langle x_s(t) | \rho_P | x'_s(t) \rangle |x\rangle \langle x'|, \end{aligned} \quad (18)$$

with

$$x_s(t) = x - t\epsilon s, \quad x'_s(t) = x' - t\epsilon s', \quad (19)$$

and, for the Gaussian initial state (7),

$$\begin{aligned} \langle x_s(t) | \rho_P | x'_s(t) \rangle &= \frac{1}{\sqrt{2\pi\Delta^2}} e^{-[x + x' - t\epsilon(s + s')]^2 / (8\Delta^2)} \\ &\quad \times e^{-2\pi^2 [x - x' - t\epsilon(s - s')]^2 / \lambda^2}. \end{aligned} \quad (20)$$

It is now well to put forth a specification: throughout the present paper we assume for simplicity that  $S$  has a *discrete and nondegenerate spectrum*. Moreover, if the Hilbert space of  $\mathcal{S}$  has infinite dimension, we restrict ourselves to initial states of the object satisfying  $\langle s | \rho_S | s' \rangle = 0$  if  $s$  and  $s'$  belong to a part of the spectrum containing arbitrarily close eigenvalues (near an accumulation point).

In the state (18), the diagonal ( $s=s'$ ) matrix elements of the object state  $\rho_S$  are multiplied by the pointer density matrix  $\rho_P$  shifted by  $t\epsilon s$  in position space, as given by (20) for  $s=s'$ . The interaction has thus tied up each eigenstate  $|s\rangle$  of  $S$  with a pointer state which has position  $x \simeq t\epsilon s$  with uncertainty  $\Delta$  and momentum  $p \simeq 0$  with uncertainty  $2\pi\hbar/\lambda$ . In position representation, each of these pointer states has a peak at  $x = t\epsilon s$ . The different peaks are separated by at least by the distance  $t\epsilon\delta s$ , where  $\delta s$  is the minimum of  $|s - s'|$  over all pairs  $(s, s')$  of nondegenerate eigenvalues such that  $\langle s | \rho_S | s' \rangle \neq 0$ . In order to be able to infer the value of  $s$  from the position of the pointer, one must wait until all peaks are well resolved. That resolvability begins at the *entanglement time*

$$t_{\text{ent}} = \frac{\Delta}{\epsilon\delta s}. \quad (21)$$

At that time, the reduced pointer density operator  $\rho_P(t) = \text{tr}_S[\rho_{PS}(t)]$  has a Wigner function as represented in Fig. 1(b). Much later yet, the separation between the peaks reaches a macroscopic value  $\Delta_{\text{class}}$  at the time

$$t_{\text{class}} = \frac{\Delta_{\text{class}}}{\epsilon\delta s} \gg t_{\text{ent}}, \quad (22)$$

allowing for a ‘‘reading’’ of the result by a classical observer.

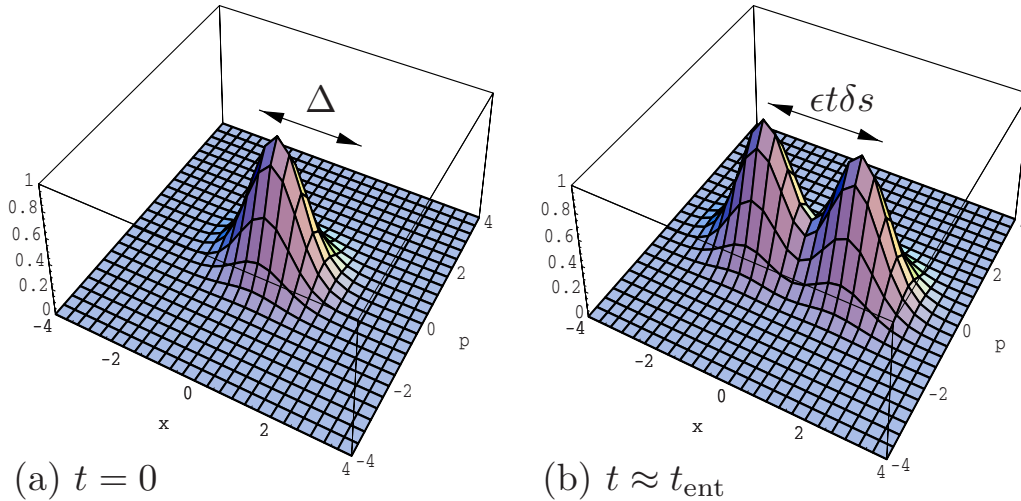


FIG. 1. (Color online) Wigner function  $W_P(x, p; t)$  of the pointer reduced state  $\rho_P(t)$ ;  $S$  is a spin one-half with two eigenvalues  $\pm \delta s/2$  and  $\langle \pm | \rho_S | \pm \rangle = 1/2$ . Note the absence of ripples: the reduced state is a mixture, not a superposition, of single-peak states. In the horizontal axis, position and momentum are measured in units of  $\Delta/2$  and  $\Delta p$ ; in the vertical axis, units are such that  $W_P(x, p; t)$  has maximum value 1.

The entanglement in the state (18) comes from the off-diagonal ( $s \neq s'$ ) contributions in  $\rho_S$ . Due to the peak structure of the pointer matrix elements (20), for fixed  $s \neq s'$ ,  $|\langle s, x | \rho_{PS}(t) | s', x' \rangle|$  reaches its maximal value when  $x = \epsilon st$  and  $x' = \epsilon st$ . For those values of  $x$  and  $x'$ ,

$$\langle s, x = \epsilon st | \rho_{PS}(t) | s', x' = \epsilon st \rangle = \langle s | \rho_S | s' \rangle \langle 0 | \rho_P | 0 \rangle \quad (23)$$

is time independent and proportional to  $\langle s | \rho_S | s' \rangle$ . Hence all coherences between different eigenstates of  $S$  present in the initial state of the object are still alive, no matter how large the time  $t$  is. At times  $t \gtrsim t_{\text{class}}$ ,  $\rho_{PS}(t)$  resembles a Schrödinger cat state—i.e., has nonzero matrix elements between macroscopically distinguishable pointer position eigenstates. For such an object-pointer state, no classical probabilistic interpretation is possible: one cannot assign a probability to the pointer being located, e.g., in the vicinity of  $x = \epsilon ts$ , henceforth implying that  $S$  has the value  $s$ . In a quantum measurement, the entanglement process must be completed by a decoherence process suppressing the coherences (23) for  $s \neq s'$ .

### B. Decoherence and “disentanglement” of object and pointer

We now turn to the decoherence brought about by the pointer-bath interaction  $H_{PB} = BX^\alpha$ , momentarily disregarding all other terms in the full Hamiltonian  $H$ . As shown in [21] for a similar model, a quantum superposition of coherent states of  $\mathcal{P}$  with well-separated peaks in position evolves under  $H_{PB}$  to a statistical mixture of these coherent states. In the situation under study here,  $S$  and  $\mathcal{P}$  are entangled, and then decoherence also modifies  $S$ . The present subsection highlights the fundamental role of this decoherence in a measurement (for more details, see [3,5,6]).

Assume object and pointer at time  $t_0$  entangled, with  $\rho_{PS} = \rho_{PS}^{\text{ent}}$  given by (18); the time  $t_0$  should be chosen larger than  $t_{\text{ent}}$ , possibly as large as the classical time scale introduced above,  $t_{\text{ent}} \ll t_0 \approx t_{\text{class}}$ . At time  $t_0$ , the state of  $S + \mathcal{P} + \mathcal{B}$  is  $\rho(t_0) = \rho_{PS}^{\text{ent}} \otimes \rho_B$  and the pointer-bath coupling  $H_{PB}$  is

switched on. To simplify the discussion, let us take for  $\rho_B$  a pure state  $\rho_B = |\Psi_0\rangle\langle\Psi_0|$ , where  $|\Psi_0\rangle = \otimes_{\nu} |\psi_{\nu}\rangle$  is a product of  $N$  single-degree-of-freedom wave functions. (All arguments below can be easily extended to a bath in an initial mixed state like  $\rho_B = \rho_B^{\text{(eq)}}$ .) Moreover, let us suppose that  $\langle \psi_{\nu} | B_{\nu} | \psi_{\nu} \rangle = 0$  and that higher moments  $\langle B_{\nu}^q \rangle = \langle \psi_{\nu} | B_{\nu}^q | \psi_{\nu} \rangle$  ( $q = 2, 3, \dots$ ) are bounded uniformly in  $\nu$ . The eigenstates  $|s, x\rangle$  are entangled at time  $t > t_0$  with the bath states  $|\Psi_x(t)\rangle = e^{-i(t-t_0)X^\alpha B/\hbar} |\Psi_0\rangle$ . The density operator of  $S + \mathcal{P} + \mathcal{B}$  reads

$$\begin{aligned} \rho(t) &= e^{-iH_{PB}(t-t_0)/\hbar} \rho_{PS}^{\text{ent}} \otimes \rho_B e^{iH_{PB}(t-t_0)/\hbar} \\ &= \sum_{s, s'} \int dx dx' \langle s, x | \rho_{PS}^{\text{ent}} | s', x' \rangle |s\rangle\langle s'| \otimes |x\rangle\langle x'| \\ &\quad \otimes |\Psi_x(t)\rangle\langle\Psi_{x'}(t)|. \end{aligned} \quad (24)$$

We now argue that for  $x \neq x'$ , the scalar product  $\langle \Psi_x(t) | \Psi_{x'}(t) \rangle$  is vanishingly small when the time span  $t - t_0$  is larger than a certain *decoherence time*  $t_{\text{dec}}(x, x')$ . Due to the additivity (3) of  $B$ ,

$$\begin{aligned} \langle \Psi_x(t) | \Psi_{x'}(t) \rangle &= \prod_{\nu=1}^N \langle \psi_{\nu} | e^{-i(t-t_0)(x'^{\alpha} - x^{\alpha})B_{\nu}/(\hbar\sqrt{N})} | \psi_{\nu} \rangle \\ &= \prod_{\nu=1}^N \left( 1 - \frac{(t-t_0)^2 (x'^{\alpha} - x^{\alpha})^2 \langle \psi_{\nu} | B_{\nu}^2 | \psi_{\nu} \rangle}{2N\hbar^2} \right. \\ &\quad \left. + O(N^{-3/2}) \right). \end{aligned} \quad (25)$$

Taking the limit  $N \rightarrow \infty$  for fixed values of  $t$ ,  $t_0$ ,  $x$ , and  $x'$  we obtain

$$\langle \Psi_x(t) | \Psi_{x'}(t) \rangle = e^{-D_t(x, x')} = \exp \left\{ -\frac{(t-t_0)^2}{t_{\text{dec}}(x, x')^2} + O(N^{-1/2}) \right\}, \quad (26)$$

$$t_{\text{dec}}(x, x') = \frac{\sqrt{2\hbar}}{|x'^{\alpha} - x^{\alpha}| \sqrt{\langle B^2 \rangle}}, \quad (27)$$

with  $\langle B^2 \rangle = \langle \Psi_0 | B^2 | \Psi_0 \rangle = N^{-1} \sum_{\nu} \langle \psi_{\nu} | B_{\nu}^2 | \psi_{\nu} \rangle$ . We have so far retraced the proof of the (classical) central-limit theorem.

Taking the partial trace of (24) over the bath Hilbert space yields

$$\begin{aligned} \rho_{\mathcal{P}\mathcal{S}}(t) = \sum_{s, s'} \int dx dx' \langle s | \rho_{\mathcal{S}} | s' \rangle \langle x_s(t_0) | \rho_{\mathcal{P}} | x'_s(t_0) \rangle e^{-D_t(x, x')} \\ \times |s\rangle \langle s'| \otimes |x\rangle \langle x'|. \end{aligned} \quad (28)$$

Due to the coupling with the bath, each matrix element of  $\rho_{\mathcal{P}\mathcal{S}}^{\text{ent}}$  is now multiplied by the scalar product (26). Let us consider a particular term  $s \neq s'$  in the sum in the right-hand side of (28). To simplify the forthcoming discussion, we assume that  $\alpha=1$ . As follows from the peak structure of the pointer coherences (20), only the terms satisfying  $x \simeq x_{s0} = t_0 \epsilon s$  and  $x' \simeq x_{s'0} = t_0 \epsilon s'$  with uncertainty  $\Delta$  contribute significantly to the integral over  $x$  and  $x'$ . For those terms,  $D_t(x, x') = D_t(x_{s0}, x_{s'0}) [1 + O(t_{\text{ent}}/t_0)]$ ; see (21), (26), and (27). Therefore, if  $t-t_0$  is large compared with  $t_{\text{dec}}(x_{s0}, x_{s'0})$  and  $t_0 \gg t_{\text{ent}}$ , the product  $\langle x_s(t_0) | \rho_{\mathcal{P}} | x'_s(t_0) \rangle e^{-D_t(x, x')}$  is vanishingly small for all values of  $(x, x')$ . The off-diagonal terms corresponding to  $s \neq s'$  then become negligible in the object-pointer state (28). It is worth emphasizing that  $t_{\text{dec}}(x_{s0}, x_{s'0})$  can be much smaller than the dissipation time scale on which the pointer-bath coupling irreversibly changes the pointer position.

We would like to point out that the aforementioned damping of the coherences is related to a lack of information about the bath in a more subtle way than what is suggested by the partial trace in (28). In fact, some partial knowledge of the bath state would not inhibit this decoherence. More precisely, in order to obtain some information on the coherences in the full density operator [terms proportional to  $|s\rangle \langle s'|$  with  $s \neq s'$  in (24)] at times  $t-t_0 \gg t_{\text{dec}}(x_{s0}, x_{s'0})$ , it is necessary to perform a measurement on some bath observable  $O_B$  satisfying  $\langle \Psi_{x_{s'0}}(t) | O_B | \Psi_{x_{s0}}(t) \rangle \neq 0$  at such time. It can be shown by repeating the arguments yielding to (26) that such an observable must be nonlocal; i.e., it must act nontrivially on *all* bath degrees of freedom except for a finite number of them. Considering that measuring such an observable is “unrealistic,” everything happens as if the  $s \neq s'$  terms have disappeared in (24). The crucial point is that the object-pointer state is entangled by  $H_{\mathcal{P}\mathcal{B}}$  with a *very large number*  $N$  of bath variables, so that information about the coherences is spread out between these many variables after some time. Macroscopically distinguishable object-pointer states are then entangled with bath states which are almost orthogonal in many subspaces of the bath Hilbert space. This makes the situation quite different from the entanglement discussed in Sec. IV A: there, the object state was entangled with a single pointer variable  $x$  and it was implicitly assumed that any pointer observable (in particular, its position  $X$ ) could be “observed” at some ultimate stage of the measurement. We refer the reader to [39] (Sec. 22.11), [20], [3] (Chap. 2), and [6] for related discussions on this very important conceptual point.

Let us define the decoherence time  $t_{\text{dec}}$  as the largest of the times  $t_{\text{dec}}(x_{s0}, x_{s'0})$  for all distinct eigenvalues  $s$  and  $s'$  such that  $\langle s | \rho_{\mathcal{S}} | s' \rangle \neq 0$ . For  $t-t_0 \gg t_{\text{dec}}$ , the object-pointer state has shed all terms  $s \neq s'$  in the double sum in the density operator (18),

$$\rho_{\mathcal{P}\mathcal{S}}(t) \simeq \sum_s \langle s | \rho_{\mathcal{S}} | s \rangle |s\rangle \langle s| \otimes \rho_{\mathcal{P}}^s(t), \quad (29)$$

wherein it has been assumed that  $t_{\text{dec}} \ll t-t_0 \ll t_B, T_{\mathcal{S}}, T_{\mathcal{P}}$  (so that  $H_B, H_{\mathcal{S}}$ , and  $H_{\mathcal{P}}$  can be neglected) and  $t_{\text{ent}} \ll t_0$  and we have set

$$\rho_{\mathcal{P}}^s(t) = \int dx dx' \langle x_s(t_0) | \rho_{\mathcal{P}} | x'_s(t_0) \rangle e^{-(t-t_0)^2/t_{\text{dec}}(x, x')^2} |x\rangle \langle x'|. \quad (30)$$

While  $\rho_{\mathcal{P}\mathcal{S}}(t)$  is not (and actually never can become) strictly diagonal in the position basis of the pointer, the matrix elements of the pointer state (30) almost vanish if  $|x-x'|$  is larger than either the uncertainty  $\lambda$  [see (20)] or the decoherence length  $\sqrt{2\hbar}/[\alpha\Delta^{\alpha-1}(t-t_0)\sqrt{\langle B^2 \rangle}]$  [see the second factor inside the integral in (30)].

It is worth noting that the object-pointer states appearing in the sum over  $s$  in (29) are product states;  $\rho_{\mathcal{P}\mathcal{S}}(t)$  is a statistical mixture of these states with probabilities  $p_s = \langle s | \rho_{\mathcal{S}} | s \rangle$ . Hence the decoherence *disentangles*  $\mathcal{S}$  and  $\mathcal{P}$ . This implies that in the time regime indicated after (29),  $\mathcal{S}$  and  $\mathcal{P}$  can be given independent states  $\rho_{\mathcal{S}}(t)$  and  $\rho_{\mathcal{P}}(t)$ ,

$$\rho_{\mathcal{S}}(t) = \sum_s p_s |s\rangle \langle s|, \quad \rho_{\mathcal{P}}(t) = \sum_s p_s \rho_{\mathcal{P}}^s(t). \quad (31)$$

The object  $\mathcal{S}$  is in one of the eigenstates  $|s\rangle$  with probability  $p_s$ , in agreement with von Neumann’s postulate. The pointer  $\mathcal{P}$  is in the quasiclassical state  $\rho_{\mathcal{P}}^s(t)$ , with the same probability.

### C. Summary

Let us sum up the discussion of the two previous subsections about the object-pointer entanglement produced by the interaction  $H_{\mathcal{P}\mathcal{S}}$  and the decoherence arising from the coupling with the bath  $H_{\mathcal{P}\mathcal{B}}$ . The dynamics implemented by  $H_{\mathcal{P}\mathcal{S}}$  uniquely ties up after the entanglement time  $t_{\text{ent}}$  each eigenvalue  $s$  of  $\mathcal{S}$  with a characteristic pointer position  $x_s(t)$ . Such neighboring pointer positions differ by more than the uncertainty  $\Delta$  then. Note that arbitrarily close eigenvalues cannot be resolved within a finite time: in fact,  $t_{\text{ent}}$  tends to be large for an object initially in a superposition of eigenstates  $|s\rangle$  with closely lying eigenvalues  $s$  (e.g., near an accumulation point of the spectrum)—i.e., for small values of  $\delta s$  in (21); this limits in practice the precision of the measurement of  $\mathcal{S}$ . In the absence of any other interaction, after a time  $t_{\text{class}} \gg t_{\text{ent}}$  the initial product state of the object and pointer has evolved into a Schrödinger cat state. Nothing irreversible is brought about by the dynamics: the entanglement can be as easily undone as done, by applying the Hamiltonian  $H_{\mathcal{P}\mathcal{S}}$  with the parameter reset  $\epsilon \rightarrow -\epsilon$ .

The dynamics generated by  $H_{\mathcal{P}\mathcal{B}}$  brings about decoherence. After the decoherence time  $t_{\text{dec}}$ , any pair of object-

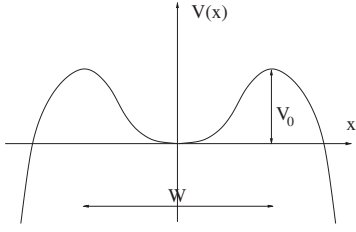


FIG. 2. Sketch of a candidate for the pointer potential. The height  $V_0$  of the potential barriers around  $x=0$  and the width  $W$  of the potential wall are much larger than  $k_B T$  and the thermal fluctuation  $\Delta_{\text{th}}$ .

pointer states corresponding to macroscopically distinguishable pointer positions are entangled with almost orthogonal bath states. After averaging the object-apparatus state over the bath variables, one obtains an object-pointer state  $\rho_{\mathcal{P}\mathcal{S}}(t)$  in which all information about the coherences between such states is missing; i.e., all coherences for pairs  $(s, s')$  of distinct eigenvalues are suppressed. The irretrievable loss of information about the bath goes hand in hand with the irreversibility of the object-pointer dynamics.

An object-pointer state  $\rho_{\mathcal{P}\mathcal{S}}(t)$  describes an accomplished measurement under two conditions.

(i) All coherences  $\langle s, x | \rho_{\mathcal{P}\mathcal{S}}(t) | s', x' \rangle$  corresponding to  $s \neq s'$  have disappeared, so that  $\mathcal{S} + \mathcal{P}$  is in a statistical mixture of separable states like in (29); this occurs at time  $t \gg t_{\text{dec}}$ .

(ii) The separation between the peaks of the distinguished pointer states  $\rho_{\mathcal{P}}^s(t)$  reaches a macroscopic value  $\Delta_{\text{class}}$ ; this occurs at time  $t_0 \gtrsim t_{\text{class}}$  [see (22)].

Only for  $t_0 \gtrsim t_{\text{class}}$  can a classical observer infer a measured value  $s$  by looking at the position of the pointer. Such a “reading” of the pointer, while still a physical process in principle perturbing  $\mathcal{P}$ , surely cannot blur the distinction of the peaks. Rather, the pointer will behave classically under a reading; i.e., it will not noticeably react.

#### D. Unstable pointer potentials and amplification

It is clear from (22) that condition (ii) can hardly arise unless the object-pointer coupling constant  $\epsilon$  is very large. This is related to the well-known amplification problem in quantum measurements [39]. In order to get rid of this unrealistic condition on  $\epsilon$ , one may consider a different situation than that described in Sec. IV A. Let us take a nonconfining pointer potential  $V(x)$  with two potential barriers separated by a distance  $W$ ; see Fig. 2. The height of these barriers is large compared with the thermal energy  $1/\beta$ . We now replace the initial states  $\rho_{\mathcal{P}} = \rho_{\mathcal{P}}^{\text{(eq)}}$  in (4a) and  $\rho_{\mathcal{P}\mathcal{B}}^{\text{(eq)}}$  in (4b) by *local equilibria* within the potential well. (This local equilibrium for the apparatus can be achieved by first preparing  $\mathcal{P}$  in some state localized near  $x=0$  at time  $t=-t_i$ , with  $t_i$  larger than the relaxation time, but small compared with the tunneling escape time, and then letting  $\mathcal{P}$  interact with  $\mathcal{B}$  until  $t=0$ .) Our previous statements about the distinct peaks in the pointer density produced by the object-pointer interaction remain valid for such initial states. The interaction  $H_{\mathcal{P}\mathcal{S}}$  is switched off at some time  $t_{\text{int}}$ . If  $t_{\text{int}}$  is larger than  $W/(\epsilon\delta s)$ , the separation between the peaks in the pointer density at

time  $t_{\text{int}}$  will be subsequently amplified by the pointer dynamics. Assuming also that  $\Delta \ll W \ll \epsilon\delta s T_{\mathcal{P}}, \Delta_{\text{class}}$ , one has  $t_{\text{int}} \approx W/(\epsilon\delta s) \ll T_{\mathcal{P}}, t_{\text{class}}$ . In this situation, the small quantum system  $\mathcal{S}$  must be able to perturb the pointer strongly enough in order to produce in it a “mesoscopic change” (i.e., a distance  $W$  between the peaks in its density), instead of a macroscopic change as required in Sec. IV A. In particular, if  $\rho_{\mathcal{P}}$  is a Gibbs state with position uncertainty  $\Delta = \Delta_{\text{th}} = [\beta V''(0)]^{-1/2}$ , this arises when the height  $V_0 \approx W^2 V''(0)$  of the two potential barriers satisfies  $\beta^{-1} \ll V_0 \ll M(\epsilon\delta s)^2$  and  $V_0 \ll \Delta_{\text{class}}^2 V''(0)$  [recall that  $T_{\mathcal{P}}^2 = M/V''(0)$ ]. Then condition (ii) of the preceding subsection will be fulfilled after the object-pointer interaction has been turned off, at time  $t_0 \approx T_{\mathcal{P}}$ . If moreover  $V_0 \ll V''(0)(\epsilon\delta s T_{\mathcal{S}})^2$ , the simplification of the dynamics discussed in Sec. III can be used since the object-pointer interaction time satisfies  $t_{\text{int}} \ll T_{\mathcal{S}}, T_{\mathcal{P}}$ .

### V. SIMULTANEOUS ENTANGLEMENT AND DECOHERENCE

We now present and discuss the main results of this work, before deriving them in Sec. VII. We are interested in the object-pointer dynamics when, unlike in the situation just described,  $\mathcal{S}$  and  $\mathcal{P}$  evolve under the *simultaneous action* of  $H_{\mathcal{P}\mathcal{S}}$  and  $H_{\mathcal{P}\mathcal{B}}$ . Furthermore, in contrast to Sec. IV B, we do not neglect the bath Hamiltonian  $H_{\mathcal{B}}$ .

#### A. Partial-equilibrium initial state

Let us first consider the evolution of the initial state (4a). Due to both the initial statistical independence and our special choice of the interactions, the density matrix of  $\mathcal{S} + \mathcal{P}$  retains at “short” times  $t \ll T_{\mathcal{S}}, T_{\mathcal{P}}$  a remarkably simple product structure (see the discussion in Sec. III)

$$\begin{aligned} \langle s, x | \rho_{\mathcal{P}\mathcal{S}}(t) | s', x' \rangle &= \langle s | \rho_{\mathcal{S}}^0(t) | s' \rangle \langle x_s(t) | \rho_{\mathcal{P}} | x'_s(t) \rangle \\ &\times \exp\{-D_t[x_s(t), x'_s(t); s, s'] - i\phi_t\}. \end{aligned} \quad (32)$$

Here  $\rho_{\mathcal{S}}^0(t)$  is given by (16),  $x_s(t) = x - t\epsilon s$ ,  $x'_s(t) = x' - t\epsilon s'$ , and  $\phi_t$  is a certain real phase (depending on  $t, x, x', s$ , and  $s'$ ) which we do not specify here since it is irrelevant for decoherence. We shall derive in Sec. VII the decoherence exponent  $D_t$

$$\begin{aligned} D_t(x, x'; s, s') &= \frac{1}{2\hbar^2} \int_0^t d\tau_1 \int_0^t d\tau_2 [(x' + \tau_1 \epsilon s')^\alpha - (x + \tau_1 \epsilon s)^\alpha] \\ &\times [(x' + \tau_2 \epsilon s')^\alpha - (x + \tau_2 \epsilon s)^\alpha] h(\tau_1 - \tau_2), \end{aligned} \quad (33)$$

where  $h(\tau_1 - \tau_2) = h_2(\tau_1, \tau_2)$  is the bath two-point function defined in (10). The first factor in (32) accounts for free evolution of the object initial state  $\rho_{\mathcal{S}}$ , as generated by  $H_{\mathcal{S}}$ , see (16). It is equal to  $p_s = \langle s | \rho_{\mathcal{S}} | s \rangle$  if  $s = s'$  and  $t \ll T_{\mathcal{S}}$ . The second factor is nothing but the matrix element (20) of the shifted pointer initial state. Here, the Hamiltonian  $H_{\mathcal{P}}$  does not show up because of our assumption  $t \ll T_{\mathcal{P}}$  and our choice of a quasiclassical initial state  $\rho_{\mathcal{P}}$ . Most important is now the third factor in (32); it accounts for decoherence—i.e., for the



suppression of coherences with respect to pointer displacements associated with different eigenvalues  $s \neq s'$ .

The exponent  $D_t$  has the following properties.

(a)  $D_t(x, x'; s, s') \geq 0$  for all values of  $x, x', s$ , and  $s'$ .

(b)  $D_t(x, x'; s, s') = 0$  initially (for  $t=0$ ) for all matrix elements and at all later times for the diagonal matrix elements ( $x=x'$  and  $s=s'$ ).

(c)  $D_t(x_s(t), x'_s(t); s, s') = D_{-t}(x, x'; s, s')$ .

The non-negativity (a) is a consequence of the fact that the correlation function  $h(t)$  and its real part  $\text{Re}h(t)$  are of positive type; i.e., they have non-negative Fourier transforms  $\widehat{h}(\omega)$  and  $\widehat{(\text{Re}h)}(\omega)$ . Actually, one may rewrite (33) as

$$D_t(x, x'; s, s') = \frac{1}{2\hbar^2} \int_0^\infty \frac{d\omega}{\pi} \widehat{(\text{Re}h)}(\omega) \left| \int_0^t d\tau [(x' + \tau\epsilon s)^\alpha - (x + \tau\epsilon s)^\alpha] e^{-i\omega\tau} \right|^2 \geq 0, \quad (34)$$

where we have used  $\widehat{(\text{Re}h)}(\omega) = \widehat{(\text{Re}h)}(-\omega)$  and  $\widehat{(\text{Im}h)}(\omega) = -\widehat{(\text{Im}h)}(-\omega)$ . Property (c) is easily checked by a change of the time integration variable in (34). Let us recall from Sec. IV A that the dynamics generated by  $H_{\mathcal{P}\mathcal{S}}$  maps the object-pointer coordinate  $(x, s)$  to  $(x_s(t), s)$  after time  $t$  and, similarly,  $(x', s')$  is mapped to  $(x'_s(t), s')$ . Hence one may interpret (c) as the invariance of  $D_t$  under time reversal—i.e., under  $t \rightarrow -t$  and the exchange of the initial and final coordinates.

### B. Equilibrium apparatus initial state

Our result for the initial state (4b) looks quite similar to that for the initial state (4a). Before stating it, let us introduce the effective pointer potential

$$V_{\text{eff}}(x) = V(x) - \hbar^{-1} \gamma_0 x^{2\alpha}, \quad (35)$$

wherein  $\gamma_0$  is given in terms of the imaginary part of the bath correlation function  $h(t)$  by

$$\gamma_0 = \int_{-\infty}^0 d\tau \text{Im}h(\tau). \quad (36)$$

We write  $Z_{\mathcal{P},\text{eff}} = \int dx e^{-\beta V_{\text{eff}}(x)}$  for the partition function associated with  $V_{\text{eff}}$ . It follows from the general properties of  $h(t)$  that  $0 \leq \gamma_0 \leq \hbar\beta h(0)/2$  (see Appendix C). Considering e.g., a linear pointer-bath coupling,  $2\gamma_0/\hbar$  is the mean force per unit length exerted by the bath on the pointer. Note that  $V_{\text{eff}}(x)$  is a nonconfining potential if  $V(x) = o(x^{2\alpha})$  at large distances. For instance, if  $\mathcal{P}$  is a harmonic oscillator [ $V(x) \propto x^2$  for all  $x$ ] and  $\alpha > 1$  then  $V_{\text{eff}}(x)$  looks like in Fig. 2. This means that an initial pointer density localized around  $x=0$  will tunnel away and eventually spread over the whole real line once the pointer-bath coupling is switched on. In such a case the apparatus equilibrium state  $\rho_{\mathcal{P}\mathcal{B}}^{(\text{eq})}$  must be replaced by a local thermal equilibrium (see Sec. IV D). This local equilibrium exists under certain conditions on the pointer-bath coupling to be discussed below.

As we shall show in Sec. VII C, the object-pointer density operator is given at times  $t \ll T_S, T_P$  by

$$\begin{aligned} \langle s, x | \rho_{\mathcal{P}\mathcal{S}}(t) | s', x' \rangle &= \langle s | \rho_{\mathcal{S}}^0(t) | s' \rangle R_t(x_s(t), x'_s(t); s, s') \\ &\times \exp\{-D_t(x_s(t), x'_s(t); s, s') - i\phi_t\}, \end{aligned} \quad (37)$$

with the same decoherence exponent  $D_t$  and phase  $\phi_t$  as above. The only difference between (37) and the formula (32) for the partial-equilibrium initial state lies in the replacement of the initial pointer density  $\langle x | \rho_{\mathcal{P}} | x' \rangle$  by the function  $R_t(x, x'; s, s')$ . For a time  $t$  short enough so that  $D_t(x, x'; s, s') \ll 1$ , this function is given by the Gibbs-type density

$$\begin{aligned} R_t(x, x'; s, s') &\simeq R_0(x, x') \\ &= Z_{\mathcal{P},\text{eff}}^{-1} e^{-\beta[V_{\text{eff}}(x) + V_{\text{eff}}(x')]/2} e^{-2\pi^2(x' - x)^2/\lambda_{\text{th}}^2}. \end{aligned} \quad (38)$$

For larger times  $t$  (with the proviso  $t \ll T_S, T_P$ ),  $R_t$  is given by the more complicated integral (85) or, in the special case  $\alpha = 2$ , by the formula (89) below. Let us only mention here that for  $\alpha = 1$ , Eq. (38) gives the correct answer up to a phase factor for all times  $t \ll T_S, T_P$ . Interestingly, (37) entails the following result on the reduced pointer initial state:

$$\langle x | \text{tr}_{\mathcal{B}}(\rho_{\mathcal{P}\mathcal{B}}^{(\text{eq})}) | x' \rangle = R_0(x, x'). \quad (39)$$

Comparing (38) with the expression of  $\langle x | \rho_{\mathcal{P}}^{(\text{eq})} | x' \rangle$  at high temperatures given in Sec. II, we see that the coupling between  $\mathcal{P}$  and  $\mathcal{B}$  can be fully accounted for by the effective potential (35). Furthermore, for a linear coupling  $\alpha = 1$  the matrix elements (38) can be approximated for small  $x$  and  $x'$  by the Gaussian (7) with an almost unchanged uncertainty in momentum,  $\Delta p \simeq 2\pi\hbar/\lambda_{\text{th}}$ , and a renormalized uncertainty in position  $\Delta_{\text{eff}} \geq \Delta_{\text{th}}$  given by  $\Delta_{\text{eff}}^{-2} = \beta V''_{\text{eff}}(0) = \beta[V''(0) - 2\gamma_0/\hbar]$  (see Sec. II).

Our results (37)–(39) rely, in addition to  $t \ll T_P, T_S$ , on two additional hypotheses: (a) the separation of time scales  $\hbar\beta \ll T_P$  or, equivalently, the separation of length scales  $\lambda_{\text{th}} \ll \Delta_{\text{th}}$  (see Sec. III B); (b) a weak enough pointer-bath coupling satisfying

$$\begin{cases} \eta_{\text{th}} < 1/\sqrt{2} & \text{if } \alpha = 1, \\ \eta_{\text{th}} \ll 1 & \text{if } \alpha > 1, \end{cases} \quad (40)$$

with

$$\eta_{\text{th}} = \langle B^2 \rangle^{1/2} \Delta_{\text{th}}^\alpha \beta.$$

Here  $\langle B^2 \rangle = \text{tr}_{\mathcal{B}}(B^2 \rho_{\mathcal{B}}^{(\text{eq})}) = h(0)$  is the thermal variance of the bath coupling agent. Condition (40) is motivated by the following requirement: The effective potential (35) must have a local minimum at  $x=0$  and the height of the potential barriers surrounding the origin must be large compared with the thermal energy  $1/\beta$ . Only under that condition can pointer and bath be prepared in a local thermal state in which the pointer reduced state has a single peak at the origin like in Fig. 1(a). If the coupling  $H_{\mathcal{P}\mathcal{B}}$  induces an instability in the pointer-bath dynamics, we must replace the Gibbs state  $\rho_{\mathcal{P}\mathcal{B}}^{(\text{eq})}$  in (4b) by that local thermal state, as explained in Sec. IV D. Note that we exclude here pointers being at a critical point of a phase transition considered in [18, 19].

We first consider the case  $\alpha=1$ . If  $V(x)=V''(0)x^2/2$ , the aforementioned requirement is met whenever  $V''_{\text{eff}}(0) > 0$ —i.e.,  $\gamma_0/\hbar < V''(0)/2$ . This stability condition is well known for a harmonic oscillator interacting linearly with a bath of harmonic oscillators [40]. For a potential  $V(x)$  which is nonquadratic at large distances  $|x| \geq W$ , we must stipulate a bit more, e.g.,  $\gamma_0/\hbar < V''(0)/4$ , in order that the height of the two potential barriers be large compared with  $1/\beta$ . Bearing in mind that  $\gamma_0 \leq \hbar\beta\langle B^2 \rangle/2$ , the latter condition is satisfied under our hypothesis (40). Most importantly, it implies  $\Delta_{\text{th}} \leq \Delta_{\text{eff}} \leq \sqrt{2}\Delta_{\text{th}}$ , so that the various length scales are ordered as  $\lambda_{\text{th}} \ll \Delta_{\text{th}} \approx \Delta_{\text{eff}} \ll W$ .

Now turning to the case  $\alpha > 1$  we insert  $V(x) \approx V''(0)x^2/2$  into (35) and find a distance between the left and right maxima of  $V_{\text{eff}}(x)$  equal to  $W_{\text{eff}} = 2[\hbar V''(0)/(2\alpha\gamma_0)]^{1/(2\alpha-2)}$ , these maxima equaling  $V''(0)[\hbar V''(0)/\gamma_0]^{1/(\alpha-1)}$  up to a factor of the order of unity. As a result, (40) implies the required stability of  $V_{\text{eff}}(x)$ . According to the discussion of Sec. IV D, the object-pointer coupling can be switched off at time  $t_{\text{int}} \approx W_{\text{eff}}/(\epsilon\delta s) \approx [\hbar V''(0)/\gamma_0]^{1/(2\alpha-2)}(\epsilon\delta s)^{-1}$ . This time must be chosen small compared with  $T_S$  and  $T_P$  and large compared with the entanglement time  $t_{\text{ent}} = \Delta_{\text{th}}/(\epsilon\delta s)$ , so as to fulfill (2) and (40).

By comparing (32) and (37) we may conclude that the coherences of  $\rho_{PS}(t)$  for  $s \neq s'$  decay to zero in same way for the two initial states (4), at least in the early time regime when these coherences are not yet very small. Furthermore, in view of (38) the whole discussion of Sec. IV about the emergence of classically discernible peaks remains qualitatively valid.

## VI. DECOHERENCE TIMES

Before presenting a derivation of our main results (32) and (37) in the next section, we focus our attention to the decoherence factor  $e^{-D_t}$ . It has been stressed in Sec. IV B that the object-pointer matrix elements

$$\begin{aligned} \rho_{PS}^{\text{peak}}(t; s, s') &= \langle s, x = \epsilon t s | \rho_{PS}(t) | s', x' = \epsilon t s' \rangle \\ &= \langle s | \rho_S^0(t) | s' \rangle R_t(0, 0; s, s') e^{-D_t^{\text{peak}}(s, s') - i\phi_t^{\text{peak}}} \end{aligned} \quad (41)$$

are of particular importance for decoherence in a quantum measurement. Here  $R_t(0, 0; s, s')$  is equal to  $\langle 0 | \rho_P | 0 \rangle$  for the initial state (4a), to  $Z_{P, \text{eff}}^1$  for the initial state (4b) if  $D_t^{\text{peak}}(s, s') \leq 1$ , and to a more complicated function of  $s$  and  $s'$  for the initial state (4b) if  $D_t^{\text{peak}}(s, s') \geq 1$ . The main difference between (41) and (23) lies in the presence of the damping factor  $\exp\{-D_t^{\text{peak}}(s, s')\}$  given by

$$\begin{aligned} D_t^{\text{peak}}(s, s') &= D_t(0, 0; s, s') \\ &= \frac{\epsilon^2 \alpha}{2\hbar^2} (s'^\alpha - s^\alpha)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \tau_1^\alpha \tau_2^\alpha h(\tau_1 - \tau_2). \end{aligned} \quad (42)$$

### A. How does $D_t^{\text{peak}}$ grow with time?

The decoherence factor (42) is positive, vanishes for  $s = s'$  [see (a) and (b) in Sec. V A], and satisfies the following:

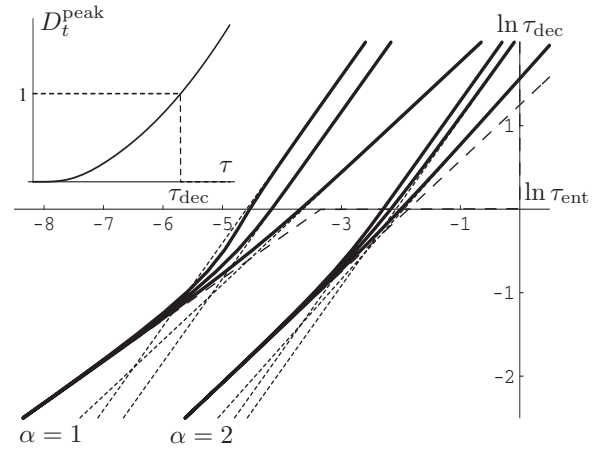


FIG. 3. Decoherence against entanglement times in units of  $T_B = \hbar\beta$  for the harmonic oscillator bath considered in Sec. VI D with  $\eta = 10^{-1}$ ,  $c = 1$ , and  $w_D = 5$  (log-log scale). Solid curves, exact results for  $(\alpha, m) = (1, 5)$ ,  $(1, 3)$ ,  $(1, 1)$ ,  $(2, 5)$ ,  $(2, 3)$ , and  $(2, 1)$  (from left to right). Dashed lines, approximate expressions for  $\tau_{\text{dec}} \leq w_D^{-1}$  (dashed curves) and  $\tau_{\text{dec}} \geq 1$  (dotted curves); see text. Inset: decoherence exponent  $D_t^{\text{peak}}$  as a function of  $\tau = t/T_B$  for  $(\alpha, m) = (1, 3)$ .

(d)  $D_t^{\text{peak}}(s, s')$  is an *increasing convex function* of time if  $s^\alpha \neq s'^\alpha$ .

(e)  $D_t^{\text{peak}}(s, -s) = 0$  if  $\alpha$  is even.

(f)  $D_t(x, x'; s, s') = D_t^{\text{peak}}(s, s') [1 + O((|x| + |x'|)(\epsilon t |s - s'|)^{-1})]$  for  $|x|, |x'| \ll \epsilon t |s - s'|$ .

Property (d) means that, quite generally, the graph of  $D_t^{\text{peak}}$  looks qualitatively like in the inset in Fig. 3. To establish this result, we take  $x = x' = 0$  in (34), differentiate both sides with respect to  $t$ , and do the time integration by parts to get

$$\begin{aligned} \frac{\partial}{\partial t} D_t^{\text{peak}}(s, s') &= \frac{\epsilon^2 \alpha}{\hbar^2} (s'^\alpha - s^\alpha)^2 \alpha t^{2\alpha} \int_0^\infty \frac{d\omega}{\pi} \frac{\widehat{\text{Re}h}(\omega)}{\omega} \\ &\quad \times \int_0^1 du (1-u)^{\alpha-1} \sin(\omega t u). \end{aligned} \quad (43)$$

Using the fact that the function  $(1-u)^{\alpha-1}$  is positive and decreasing between 0 and 1, it is easy to show that the integral over  $u$  in (43) is positive for almost all  $\omega \geq 0$ . Bearing in mind that  $\widehat{\text{Re}h}(\omega) \geq 0$ , this establishes that  $\partial D_t^{\text{peak}} / \partial t > 0$  for  $t > 0$ . Hence  $D_t^{\text{peak}}$  is an increasing function of  $t$ . By a similar argument,  $\partial^2 D_t^{\text{peak}} / \partial t^2 > 0$  and thus  $D_t^{\text{peak}}$  is convex.

According to property (e), if  $\alpha$  is even and the spectrum of  $S$  is symmetric with respect to  $s = 0$ , the coherences (41) for  $s' = -s$  are not damped. This comes from the symmetry  $x \leftrightarrow -x$  of the Hamiltonian  $H_{PB}$  in (3), which allows for the existence of decoherence-free subspaces [41]. Due to these long-living coherences,  $\mathcal{P} + \mathcal{S}$  fails to reach (at least within a time span  $t \leq T_P, T_S$ ) the statistical mixture required to be able to give a classical result to the measurement. We exclude that case from now on. More precisely, we assume that if  $\alpha$  is even, then  $s/s'$  is not close to  $-1$  for all pairs  $(s, s')$  of eigenvalues such that  $\langle s | \rho_S | s' \rangle \neq 0$ ; i.e.,  $|s'^\alpha - s^\alpha| / |s' - s|^\alpha$  is bounded below by a constant  $c_\alpha^{\text{min}} > 0$  of the order of unity.

With this restriction, for  $(|x|+|x'|)/(\epsilon t|s-s'|)$  sufficiently small the error term in property (f) is bounded by  $(|x|+|x'|)(\epsilon t|s-s'|)^{-1}$  times a constant independent of  $x, x', s, s',$  and  $t$  [42].

We are concerned in this section with determining the time scale  $t_{\text{dec}}(s, s')$  characterizing the growth of  $D_t^{\text{peak}}(s, s')$  and the corresponding decay of the ( $s \neq s'$ ) coherences (41). This time, to be called the decoherence time, is defined implicitly as  $D_{t=t_{\text{dec}}}^{\text{peak}}(s, s')=1$ —i.e.,

$$\left(\frac{t_{\text{ent}}(s, s')}{\eta^{1/\alpha}}\right)^{2\alpha} = \frac{c_\alpha(s, s')^2}{(\hbar\beta)^2} \int_0^{t_{\text{dec}}(s, s')} d\tau_1 \int_0^{\tau_1} d\tau_2 \tau_1^\alpha \tau_2^\alpha \times \frac{\text{Reh}(\tau_1 - \tau_2)}{\langle B^2 \rangle}, \quad (44)$$

where

$$t_{\text{ent}}(s, s') = \frac{\Delta}{\epsilon|s' - s|} \quad (45)$$

is the entanglement time (whose physical interpretation has been illustrated in Sec. IV A),  $\eta$  is the (fluctuation of the) initial pointer-bath coupling energy in units of  $k_B T$ ,

$$\eta = \langle B^2 \rangle^{1/2} \Delta \alpha \beta \approx \beta [\text{tr}(H_{\mathcal{P}\mathcal{B}}^2 \rho_{\mathcal{P}} \otimes \rho_{\mathcal{B}}^{(\text{eq})})]^{1/2}, \quad (46)$$

$c_\alpha(s, s')=1$  if  $\alpha=1$ , and

$$c_\alpha(s, s') = \frac{|s'^\alpha - s^\alpha|}{|s' - s|^\alpha} \quad \text{if } \alpha \geq 1. \quad (47)$$

For the initial state (4b), one must set  $\Delta = \Delta_{\text{th}}$  in (45) and (46) and  $\eta = \eta_{\text{th}}$  must be small enough; see (40). By inspection of (44),  $t_{\text{dec}}(s, s')$  depends on the object-pointer and pointer-bath coupling constants  $\epsilon$  and  $\eta$  through a single parameter  $\epsilon\eta^{1/\alpha}$ . Recalling that (32)–(42) are valid with the proviso  $t \ll T_S, T_P$ , the “free” evolutions of  $S$  and  $X$  must be slow compared to  $t_{\text{dec}}(s, s')$ —i.e.,

$$t_{\text{dec}}(s, s') \ll T_S, T_P, \quad s \neq s'. \quad (48)$$

For given  $s \neq s'$ , if  $t_{\text{dec}}(s, s') \geq t_{\text{ent}}(s, s')$ , then at time  $t \gg t_{\text{dec}}(s, s')$  the peaks at  $(x, x') = (\epsilon t s, \epsilon t s')$  of the pointer coherences in (32) and (37) (second factors on the right-hand side) are flattened down by decoherence (third factors), so that  $\langle s, x | \rho_{\mathcal{P}\mathcal{S}}(t) | s', x' \rangle \approx 0$  for all values of  $(x, x')$ . This statement follows from a similar argument as in Sec. IV B and from property (f) (see the beginning of this section). It is worth emphasizing that if, unlike in the situation just described,  $t_{\text{dec}}(s, s')$  is smaller than  $t_{\text{ent}}(s, s')$ , then the coherence  $\langle s, x | \rho_{\mathcal{P}\mathcal{S}}(t) | s', x' \rangle$  may still be large at time  $t_{\text{dec}}(s, s')$  for some  $(x, x') = (\epsilon t s, \epsilon t s')$  with uncertainty  $\Delta$ . In such a case the decoherence time must be defined as the time  $t$  at which the minimum of  $D_t(x, x'; s, s')$  over all values of  $(x, x')$  is equal to 1. We postpone to a separate work the determination of that decoherence time.

The decoherence time  $t_{\text{dec}}$  of the measurement is the largest of the times  $t_{\text{dec}}(s, s')$  for all pairs of distinct eigenvalues  $(s, s')$  such that  $\langle s | \rho_S | s' \rangle \neq 0$  [with the proviso  $t_{\text{dec}} \geq t_{\text{ent}} = \Delta/(\epsilon\delta s)$  in light of the discussion in the preceding paragraph]. This amounts to replacing  $|s'^\alpha - s^\alpha|$  in (42) by its minimum value over all such pairs  $(s, s')$  (recall that  $D_t^{\text{peak}}$  is

an increasing function of time). For  $\alpha=1$  this minimum value is by definition equal to  $\delta s$  (Sec. IV A); for  $\alpha \geq 2$ , it depends on the spectrum of  $S$  in a more subtle way [43]. At times  $t \gg t_{\text{dec}}$ , the object-pointer state  $\rho_{\mathcal{P}\mathcal{S}}(t)$  is very close to the separable state (29). In other words,  $S$  and  $\mathcal{P}$  are in the statistical mixture (31) with the probabilities  $p_s = \langle s | \rho_S | s \rangle$  and with pointer states  $\rho_{\mathcal{P}}^s(t)$  given by

$$\langle x | \rho_{\mathcal{P}}^s(t) | x' \rangle = R_t(x_s(t), x_s'(t); s, s) \exp\{-D_{-}(x, x'; s, s) - i\phi_t\}, \quad (49)$$

with  $R_t(x, x'; s, s)$  equal to  $\langle x | \rho_{\mathcal{P}} | x' \rangle$  for the initial state (4a) and to the Gibbs-type density (38) for the initial state (4b) when  $D_{-}(x, x'; s, s) \leq 1$ . We have used in (49) the time-invariance property (c); see Sec. V A. The initial superpositions of eigenstates  $|s\rangle$  have disappeared by *indirect decoherence via the pointer*. The pointer is in a statistical mixture of quasiclassical states having densities localized around  $x = t\epsilon s$  with uncertainty  $\Delta$ . The essence of quantum measurements lies in this loss of coherences: for indeed, as already pointed out in Sec. IV C it is only when all object-pointer coherences for  $s \neq s'$  are vanishingly small that a classical probability can be given for the result of the measurement.

The pointer matrix elements (49) are also damped by decoherence via the last exponential factor in (49). One can show, however, that for relevant values of  $x$  and  $x'$  satisfying  $|x - \epsilon t s| \leq \Delta$  and  $|x' - \epsilon t s| \leq \Delta$ , the corresponding damping time is much larger than  $t_{\text{dec}}$ , at least in the two limiting regimes  $t_{\text{dec}} \ll t_B$  and  $t_{\text{dec}} \gg t_B$  studied below. The special case  $\alpha=1$  will be discussed in the next subsection.

It is worthwhile mentioning here that one should expect that  $t_{\text{dec}} \ll t_{\text{class}}$ , save for extremely large object-pointer coupling constants  $\epsilon$ . Object and pointer are then never in a Schrödinger cat state as in (18), because decoherence subdues linear superpositions [via the third factors in (32) and (37)] faster than entanglement between  $\mathcal{P}$  and  $\mathcal{S}$  can produce them [second factors in (32) and (37)]. Due to the simultaneous action of  $H_{\mathcal{P}\mathcal{S}}$  and  $H_{\mathcal{P}\mathcal{B}}$ , the whole measurement process directly produces the mixture of macroscopically distinct pointer states  $\rho_{\mathcal{P}}^s(t)$ , without allowing for the intermediate appearance of macroscopic superpositions. This is one of the central results of the present paper. In the situation described in Secs. IV D and V B; i.e., if the (effective) pointer potential is unstable and the object-pointer interaction is switched off at time  $t_{\text{int}} \approx W/(\epsilon\delta s) \ll T_P, t_{\text{class}}$ , even mesoscopic superpositions do not appear at any stage of the measurement when  $t_{\text{dec}} \ll t_{\text{int}}$ .

No assumption whatsoever was made on the bath correlation time  $T_B$  to establish (32)–(42). Our results therefore go beyond the so-called Markovian limit which would require  $T_B \ll t_{\text{dec}}$ . This is an important point, since for sufficiently large  $\epsilon$  decoherence will take place within the “non-Markovian” regime  $t \leq T_B$ . Explicit asymptotical results for  $t_{\text{dec}}$  can now be drawn from the foregoing expressions for both  $t_{\text{dec}} \ll t_B$  and  $t_{\text{dec}} \gg T_B$ .

## B. Interaction-dominated regime $t_{\text{dec}} \ll t_B$

In the (non-Markovian) regime  $t_{\text{dec}} \ll t_B$ , the dynamics is dominated by the interactions  $H_{\mathcal{P}\mathcal{S}}$  and  $H_{\mathcal{P}\mathcal{B}}$ . For  $t \ll t_B$ , one

may approximate  $h(\tau)$  in (42) by the thermal variance  $h(0) = \langle B^2 \rangle$  of the bath coupling agent. This yields

$$D_t^{\text{peak}}(s, s') = \left( \frac{t}{t_{\text{dec}}(s, s')} \right)^{2\alpha+2} \quad (\text{interaction-dominated regime}), \quad (50)$$

$$t_{\text{dec}}(s, s') = \left( \frac{\sqrt{2}(\alpha+1)}{c_\alpha(s, s')} \right)^{1/(\alpha+1)} \left( \frac{t_{\text{ent}}(s, s')}{\hbar\beta\eta^{1/\alpha}} \right)^{\alpha/(\alpha+1)} \hbar\beta, \quad (51)$$

where we have used the entanglement time and dimensionless parameters (45)–(47). This result makes sense with the proviso  $t_{\text{dec}}(s, s') \ll t_B, T_S, T_P$ . The fact that  $D_t^{\text{peak}} \propto t^{2\alpha+2}$  could be expected from (26): indeed, a contribution of  $t^2(x'^\alpha - x^\alpha)^2$  to the decoherence exponent was found for  $H_{\mathcal{P}\mathcal{S}}=0$  and fixed  $x$  and  $x'$ ; recalling that for  $H_{\mathcal{P}\mathcal{S}}$  given by (3) the positions of the peaks grow proportionally with time, we recover the above-mentioned power law. It should also be noted that  $t_{\text{dec}}$  depends on the bath through the single parameter  $\eta$ . This comes from the fact that the bath dynamics can be ignored when  $t_{\text{dec}} \ll t_B$  [44].

Invoking (42) and  $(\text{Re}h)(\omega) \geq 0$ , it is easy to show that (50) gives an upper bound on  $D_t^{\text{peak}}(s, s')$  for all times  $t \geq 0$ . By property (d) in Sec. VI A, in the regime  $t_{\text{dec}} \geq t_B$  the decoherence time must be larger than the right-hand side of the asymptotic formula (51).

It has been stressed above that the interpretation of  $t_{\text{dec}}$  as the decoherence time of the measurement relies on the assumption  $t_{\text{ent}} \leq t_{\text{dec}}$ . We now argue that this condition is fulfilled if the pointer-bath coupling energy is of the order or smaller than  $k_B T$  (i.e.,  $\eta \lesssim 1$ ) and  $s'/s$  is not very close to unity. In fact, under these assumptions one has even  $t_{\text{ent}}(s, s') \ll t_{\text{dec}}(s, s')$ . This follows from (51), the consistency condition  $t_{\text{dec}}(s, s') \ll t_B$ , and the inequality  $t_B \leq \hbar\beta$ , which imply  $t_{\text{dec}}(s, s') \ll \hbar\beta$  and thus  $t_{\text{ent}}(s, s') \ll \hbar\beta$ . In contrast, if  $\alpha \geq 2$  and  $|s'/s - 1| \ll 1$ , one sees by inspection of (47) that  $c_\alpha(s, s') \approx \alpha|1 - s'/s|^{1-\alpha} \gg 1$ . Hence the first factor on right-hand side of (51) is small and one may have  $t_{\text{ent}}(s, s') \geq t_{\text{dec}}(s, s')$ . This corresponds to an initial superposition of eigenstates  $|s\rangle$  with closely lying eigenvalues, as discussed in Sec. IV C. Similarly, one may have  $t_{\text{ent}}(s, s') \geq t_{\text{dec}}(s, s')$  for a strong pointer-bath coupling—i.e., for  $\eta \gg 1$ .

It is worthwhile comparing the strength of decoherence for different values of the exponent  $\alpha$  in the coupling Hamiltonian  $H_{\mathcal{P}\mathcal{B}}$ , keeping its magnitude  $\eta/\beta$  constant. We find that  $t_{\text{dec}}$  is smaller in the nonlinear case  $\alpha > 1$  in comparison with the linear case  $\alpha = 1$  by a factor of  $t_{\text{dec}}^{(\alpha>1)}/t_{\text{dec}}^{(\alpha=1)}$  of the order of  $c_\alpha^{-1/(\alpha+1)}(t_{\text{ent}}/t_{\text{dec}}^{(\alpha=1)})^{(\alpha-1)/(\alpha+1)} \ll 1$ . Interestingly, a linear pointer-bath coupling is much less efficient than a nonlinear one in suppressing the coherences (41) for  $s \neq s'$ . This has the following important consequence: for a pointer-bath coupling of the form  $H_{\mathcal{P}\mathcal{B}} = Bf(X)$  with  $f(x)$  a smooth real function, a dipolelike approximation consisting in linearizing  $f(x)$  may lead to an overestimation of the decoherence time  $t_{\text{dec}}$  even if  $f''(0)\Delta/f'(0)$  is small. Actually, the quadratic cou-

pling  $Bf''(0)X^2/2$  gives a smaller decoherence time than the linear coupling  $Bf'(0)X$  when  $t_{\text{ent}}/\hbar\beta \lesssim \eta^{-1}[f''(0)\Delta/f'(0)]^2$  with  $\eta = f'(0)\langle B^2 \rangle^{1/2}\Delta\beta$ .

We can now give an explicit condition ensuring that  $t_{\text{dec}}$  is smaller than the time  $t_{\text{int}} \approx W/(\epsilon\delta s)$  needed by object-pointer entanglement to produce superpositions of pointer positions separated by the mesoscopic length  $W$ ,  $\Delta \ll W \ll \Delta_{\text{class}}$ :  $t_{\text{dec}}$  must be large compared with  $(\Delta/W)^{\alpha+1}\hbar\beta/(\eta c_\alpha)$ . In this limit decoherence is so fast that these mesoscopic superpositions do not appear at any moment during the measurement.

As pointed out in Sec. VI A, it is appropriate to demonstrate that the decay of the pointer matrix elements (49) remains negligible for times  $t$  until well after the disappearance of the off-diagonal ( $s \neq s'$ ) terms in  $\rho_{\mathcal{P}\mathcal{S}}(t)$ . Due to the peak structure of the pointer density [first factor in the right-hand side of (49)], the relevant values of  $x, x'$  are such that  $|x_s(t)|, |x'_s(t)| \leq \Delta$ . Given  $t_{\text{ent}} \ll t_{\text{dec}}$ , such  $x$  and  $x'$  are separated by a distance  $|x - x'| \leq 2\Delta$  much smaller than the inter-peak distance  $t_{\text{dec}}\epsilon\delta s$  relevant for the decay of the ( $s, s' = s + \delta s$ ) matrix elements of  $\rho_{\mathcal{P}\mathcal{S}}(t)$ . We restrict ourselves to the case  $\alpha = 1$  and consider the limit  $t \ll t_B \leq \hbar\beta$ . Setting  $s = s'$ , inverting the sign of the time  $t$ , and replacing  $h(\tau_1 - \tau_2)$  by  $\langle B^2 \rangle$  in (33), one finds that  $D_{-t}(x, x'; s, s) = \langle B^2 \rangle t^2 (x - x')^2 / (2\hbar^2)$ . Note that this decoherence exponent is the same as in (30). If  $\eta \lesssim 1$  and  $x, x'$ , and  $t$  are in the range mentioned above, then  $D_{-t}(x, x'; s, s) \leq 2\eta^2 t^2 (\hbar\beta)^{-2} \ll 1$ . Hence the decoherence caused by the pointer-bath coupling has a small effect on the pointer states  $\rho_{\mathcal{P}}^s(t)$  up to times  $t \lesssim t_{\text{dec}}(s)$ , the decoherence factor in (49) being still close to unity. So, indeed, the bath does away with the “off-diagonal” ( $s \neq s'$ ) object-pointer matrix elements before the “diagonal” ones change noticeably.

### C. Markov regime $t_{\text{dec}} \gg T_B$

When  $t_{\text{dec}} \gg t_B$  the off-diagonal matrix elements (41) have no time to decay between  $t=0$  and  $t_B$ . Decoherence may then take place within the so-called Markov regime  $t \gg T_B$ , also known in the mathematical literature as the singular-coupling limit [45,46]. Note that under our condition  $t_{\text{dec}} \ll T_S, T_P$  it is not appropriate to use a rotating-wave approximation. Decoherence is governed in the Markov regime by the small-frequency behaviors of the Fourier transforms  $(\text{Re}h)(\omega)$  and  $(\text{Im}h)(\omega)$  of the real and imaginary parts of the bath correlator  $h(t)$ . We shall make use of a few properties of these Fourier transforms, which are explained in more detail in Appendix C. We assume that  $(\text{Im}h)(\omega) \sim -i\hat{\gamma}\omega^m$  for  $\omega \ll T_B^{-1}$ ,  $\hat{\gamma}$  being a positive constant. Bearing in mind that  $(\text{Im}h)(\omega)$  is an odd function of  $\omega$  and must be regular enough (i.e., admit differentials of sufficiently high orders) in such a way that  $\text{Im}h(t)$  decays rapidly to zero as  $t \rightarrow \pm\infty$ , we take  $m$  to be a positive odd integer. By analogy with the case of a bath of harmonic oscillators linearly coupled to  $\mathcal{P}$ , we speak of *Ohmic damping* when  $m=1$  and of *super-Ohmic damping* when  $m > 1$  [24,25]. The behavior of  $(\text{Re}h)(\omega)$  at small frequencies can be deduced from that of  $(\text{Im}h)(\omega)$  thanks to the Kubo-Martin-Schwinger (KMS) relation (C6). Such a rela-

tion holds because the average in the correlation function  $\widehat{h(t)}$  is taken with respect to a bath Gibbs state [47]. It implies  $\widehat{(\text{Re}h)}(\omega) \sim 2\hat{\gamma}\omega^{m-1}/(\hbar\beta)$ .

Let us first discuss the super-Ohmic case  $m \geq 3$ . The frequency integral in (43) can be rewritten after an integration by parts as

$$\begin{aligned} & \int_0^\infty \frac{d\omega}{\pi} \frac{\widehat{(\text{Re}h)}(\omega)}{\omega} \int_0^1 du (1-u)^{\alpha-1} \sin(\omega tu) \\ &= t^{-1} \int_0^\infty \frac{d\omega}{\pi} \frac{\widehat{(\text{Re}h)}(\omega)}{\omega^2} [1 - \delta_{\alpha 1} \cos(\omega t) \pm (\alpha - 1) \\ & \quad \times \int_0^1 du (1-u)^{\alpha-2} \cos(\omega tu)] \simeq t^{-1} \int_0^\infty \frac{d\omega}{\pi} \frac{\widehat{(\text{Re}h)}(\omega)}{\omega^2}, \end{aligned} \quad (52)$$

where we have neglected in the last expression the oscillatory integrals by invoking  $t \gg T_B$ . By inspection of (C8) we conclude that for  $m \geq 3$  the frequency integral in (43) can be approximated by  $t^{-1} |\int_0^\infty d\tau \tau \text{Re}h(\tau)|$ .

For an Ohmic bath  $m=1$ , the last integral in (52) diverges. We now argue that one can replace  $\widehat{(\text{Re}h)}(\omega)$  by  $\widehat{(\text{Re}h)}(0) = 2\hat{\gamma}(\hbar\beta)^{-1}$  on the left-hand side of (52), which becomes

$$\int_0^\infty \frac{d\omega}{\pi} \frac{\widehat{(\text{Re}h)}(0)}{\omega} \int_0^1 du (1-u)^{\alpha-1} \sin(\omega tu) = \frac{\widehat{(\text{Re}h)}(0)}{2\alpha} \quad (53)$$

in the limit  $t \gg T_B$ . [We have used  $\int d\omega \sin(\omega tu)/\omega = \pi$  for  $tu > 0$ .] Note that this amounts to replacing  $\text{Re}h(t)$  by a white-noise correlator  $2\hat{\gamma}(\hbar\beta)^{-1} \delta(t)$  in (42). Let us estimate the error introduced in the frequency integral in (43) by this substitution. This error is given by the left-hand side of (52) modulo the replacement of  $\widehat{(\text{Re}h)}(\omega)$  by  $\widehat{(\text{Re}h)}(\omega) - \widehat{(\text{Re}h)}(0)$ . Disregarding oscillatory integrals as in the case  $m \geq 3$ , the error is equal in the limit  $t \gg T_B$  to  $t^{-1} \int_0^\infty d\omega [\widehat{(\text{Re}h)}(\omega) - \widehat{(\text{Re}h)}(0)] \omega^{-2} / \pi$ . The latter integral converges since  $\widehat{(\text{Re}h)}(\omega) - \widehat{(\text{Re}h)}(0)$  behaves like  $\omega^2$  for small  $\omega$ . Comparing with (53) [see also (C9)], one concludes that the relative error introduced in (43) by the substitution of  $\widehat{(\text{Re}h)}(\omega)$  by its value for  $\omega=0$  is small, of the order of  $T_B/t$ . Hence, for  $m=1$  the frequency integral in (43) can be approximated by  $\widehat{(\text{Re}h)}(0)/(2\alpha) = \alpha^{-1} \int_0^\infty d\tau \text{Re}h(\tau)$ .

Collecting the above results and integrating (43) with respect to time, we find in the Ohmic case  $m=1$

$$D_t^{\text{peak}}(s, s') = \left( \frac{t}{t_{\text{dec}}(s, s')} \right)^{2\alpha+1} \quad (\text{Ohmic}), \quad (54)$$

$$\begin{aligned} t_{\text{dec}}(s, s') &= \left( \frac{(2\alpha+1)\langle B^2 \rangle \hbar \beta}{c_\alpha(s, s')^2 \int_0^\infty d\tau \text{Re}h(\tau)} \right)^{1/(2\alpha+1)} \\ & \quad \times \left( \frac{t_{\text{ent}}(s, s')}{\hbar \beta \eta^{1/\alpha}} \right)^{2\alpha/(2\alpha+1)} \hbar \beta \end{aligned} \quad (55)$$

and in the super-Ohmic case  $m \geq 3$

$$D_t^{\text{peak}}(s, s') = \left( \frac{t}{t_{\text{dec}}(s, s')} \right)^{2\alpha} \quad (\text{super-Ohmic}), \quad (56)$$

$$t_{\text{dec}}(s, s') = \left( \frac{2\langle B^2 \rangle \hbar^2 \beta^2}{c_\alpha(s, s')^2 \left| \int_0^\infty d\tau \tau \text{Re}h(\tau) \right|} \right)^{1/(2\alpha)} \frac{t_{\text{ent}}(s, s')}{\eta^{1/\alpha}}, \quad (57)$$

with the proviso  $T_B \ll t_{\text{dec}}(s, s') \ll T_S, T_P$ . We can interpret the growth of  $D_t^{\text{peak}}$  like  $t^{2\alpha+1}$  in the Ohmic case by saying that for fixed  $x$  and  $x'$ , in the Markov regime  $D_t$  must be proportional to  $t(x'^\alpha - x^\alpha)^2$  (the fact that  $D_t \propto t$  is well known [3]); the indicated time behavior of  $D_t^{\text{peak}}$  then follows by replacing  $(x, x')$  by  $(\epsilon t s, \epsilon t s')$ .

By using  $|h(\tau)| \leq \langle B^2 \rangle$  and  $\text{Re}h(\tau) = 0$  for  $\tau \gg T_B$  (Sec. III A), one finds that the integrals  $\int_0^\infty d\tau \text{Re}h(\tau)$  and  $|\int_0^\infty d\tau \tau \text{Re}h(\tau)|$  are at most of the order of  $\langle B^2 \rangle T_B$  and  $\langle B^2 \rangle T_B^2$ , respectively. If  $|s'/s-1|$  is not close to unity [so that  $c_\alpha(s, s')$  in (47) is not very large], the factor inside the parentheses in (57) is of the order of  $(\hbar\beta/T_B)^2$  or larger. Thus, for coupling strength  $\eta \lesssim \hbar\beta/T_B$  the condition  $t_{\text{ent}}(s, s') \leq t_{\text{dec}}(s, s')$  holds in the Markov regime for super-Ohmic baths. The situation is different for Ohmic baths: then, by (55), the condition in question is violated even for small  $\eta$  if the entanglement time  $t_{\text{ent}}(s, s')$  is large enough compared with  $\hbar\beta$ . More precisely, still assuming that  $c_\alpha(s, s')$  is of the order of unity,  $t_{\text{dec}}(s, s')$  becomes smaller than  $t_{\text{ent}}(s, s')$  when

$$\frac{t_{\text{ent}}(s, s')}{\hbar \beta} \gtrsim \frac{\langle B^2 \rangle \hbar \beta}{\eta^2 \int_0^\infty d\tau \text{Re}h(\tau)} \gtrsim \frac{\hbar \beta}{\eta^2 T_B}. \quad (58)$$

For super-Ohmic baths, the decoherence time (57) decreases by increasing  $\alpha$  for  $\eta \lesssim \hbar\beta/T_B$  and  $|s'/s-1|$  not close to unity, i.e., provided that  $t_{\text{dec}}(s, s') \geq t_{\text{ent}}(s, s')$ . Then  $t_{\text{dec}}^{(\alpha>1)}/t_{\text{dec}}^{(\alpha=1)} \leq [\eta T_B/(\hbar\beta)]^{1-1/\alpha} \leq 1$ . Thus, for fixed weak enough coupling strength  $\eta$ , nonlinear pointer-bath couplings always win over a linear coupling in efficiency for decoherence. This is in striking contrast with what happens in the Ohmic case. Actually, for a Ohmic bath nonlinear couplings become less efficient than a linear coupling when  $t_{\text{ent}}(s, s')$  is large enough so as to fulfill (58). More precisely, we find by using  $c_\alpha(s, s') \approx 1$  and (58) that the decoherence time (55) is larger in the nonlinear case  $\alpha > 1$  than in the linear case  $\alpha = 1$  by a factor of  $t_{\text{dec}}^{(\alpha>1)}/t_{\text{dec}}^{(\alpha=1)}$  of the order of  $(t_{\text{ent}}/t_{\text{dec}}^{(\alpha=1)})^{(2\alpha-2)/(2\alpha+1)} \geq 1$ .

Finally, it is worth mentioning that Ohmic baths win in efficiency over super-Ohmic baths. This can be shown by noting that  $(t_{\text{dec}}^{\text{Ohm}}/t_{\text{dec}}^{\text{sup-Ohm}})^{2\alpha+1}$  is equal (up to a numerical factor of the order of unity) to the product of  $|\int_0^\infty d\tau \tau \text{Re}h(\tau)|[\hbar\beta\int_0^\infty d\tau \text{Re}h(\tau)]^{-1}$  by  $\hbar\beta/t_{\text{dec}}^{\text{sup-Ohm}}$ . Since the last factor must be small compared with 1 for consistency (recall that  $\hbar\beta \leq T_B$ ), it follows that  $t_{\text{dec}}^{\text{Ohm}}(s, s')$  is smaller than  $t_{\text{dec}}^{\text{sup-Ohm}}(s, s')$ .

One may wonder if the results of this section could be strongly modified if a direct coupling between the object  $\mathcal{S}$  and bath  $\mathcal{B}$  (which we do not admit in the present model) was allowed for. It is clear that one can answer this question by the negative when the object-pointer coupling constant  $\epsilon$  is large enough—i.e., for small enough  $t_{\text{ent}}$ . In order to estimate how small must be  $t_{\text{ent}}$ , let us couple  $\mathcal{S}$  and  $\mathcal{B}$  via the Hamiltonian  $H_{SB} = \Delta^\alpha (S/\delta_S)^\alpha B$ . This Hamiltonian has a magnitude comparable with the pointer-bath coupling (3) in the initial state (4a). We first consider Ohmic baths. It is known that the decay of the off-diagonal matrix elements  $\langle s|\rho_S(t)|s'\rangle$  resulting from the coupling  $H_{SB}$  then goes like  $\exp[-t/T_{\text{dec}}(s, s')]$  in the Markov regime (we ignore here the object-pointer coupling) [3]. If  $T_{\text{dec}}(s, s') \ll T_S$ , a condition fulfilled if  $[S, H_S] = 0$  (pure dephasing regime  $T_S = \infty$ ), the corresponding decoherence time is given by  $T_{\text{dec}}(s, s') = \hbar^2(\delta_S/\Delta)^{2\alpha}|s'^\alpha - s^\alpha|^{-2}/\int_0^\infty d\tau \text{Re}h(\tau)$  [21]. The ratio between  $T_{\text{dec}}(s, s')$  and the decoherence time (55) for  $s' = s + \delta s$  is  $[(c_\alpha \eta)^{-2} t_{\text{ent}}^{-1} (\hbar\beta)^2 (B^2)/\int_0^\infty d\tau \text{Re}h(\tau)]^{2\alpha/(2\alpha+1)}$  up to an irrelevant factor. Taking into account that  $\int_0^\infty d\tau \text{Re}h(\tau) \leq \langle B^2 \rangle T_B$ , we see that it is well justified to neglect the coupling of the object with all degrees of freedom of the apparatus but the pointer provided that  $t_{\text{ent}}/(\hbar\beta) \ll (\hbar\beta/T_B)(c_\alpha \eta)^{-2}$ . For a super-Ohmic bath, if  $[S, H_S] = 0$ , then the modulus of the off-diagonal matrix element  $\langle s|\rho_S(t)|s'\rangle$  decays to a nonzero value under the coupling  $H_{SB}$  (for a discussion on this saturation of decoherence see, e.g., [48]), whereas indirect decoherence via the pointer leads to a complete decay of the object-pointer coherences [this decay being given by the decoherence exponent (56)]. It is also easy to show that  $T_{\text{dec}}(s, s')$  is much larger than the decoherence time (51) provided that  $t_{\text{ent}}/(\hbar\beta) \ll (\hbar\beta/T_B)^{1+1/\alpha}(c_\alpha \eta)^{-2-1/\alpha}$ .

### D. Bath of harmonic oscillators linearly coupled to $\mathcal{P}$

To study the transition between the limiting time regimes discussed in the two preceding subsections, let us consider a bath of  $N \gg 1$  harmonic oscillators,  $H_B = \sum_\nu \hbar \omega_\nu (b_\nu^\dagger b_\nu + 1/2)$ , coupled to the pointer via a coupling agent  $B$  linear in each of its creation and annihilation operators  $b_\nu^\dagger$  and  $b_\nu$ ,  $B = \sum_\nu (\kappa_\nu b_\nu^\dagger + \kappa_\nu^* b_\nu)/\sqrt{N}$  [24]. Here  $\omega_\nu$  is the frequency and  $\kappa_\nu$  the coupling constant of the  $\nu$ th oscillator. We shall take the following specific choice for the power spectrum function:

$$J(\omega) = \frac{\pi}{N} \sum_{\nu=1}^N |\kappa_\nu|^2 \delta(\omega - \omega_\nu) = \hat{\gamma} \omega^m e^{-\omega^2/\omega_D^2}, \quad (59)$$

wherein  $m$  is an odd positive integer,  $\hat{\gamma} > 0$ , and  $\omega_D$  is a cutoff frequency. We recall that the case  $m=1$  corresponds to an Ohmic damping, whereas one speaks of super-Ohmic damping for  $m > 1$ . For instance,  $m=d$  or  $d+2$  for a phonon

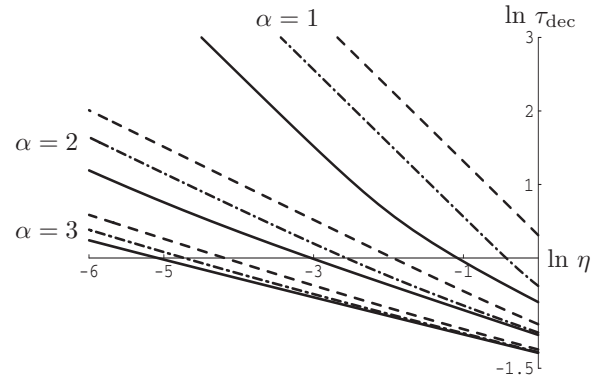


FIG. 4. Decoherence time  $\tau_{\text{dec}}$  in units of  $T_B = \hbar\beta$  as a function of the pointer-bath coupling strength  $\eta$  for the same bath as in Fig. 3 with  $\tau_{\text{ent}} = 0.1$ ,  $m=3$ , and  $c=1$ . Three distinct values of  $w_D$  are shown:  $w_D=2$  (solid curves),  $w_D=5$  (dotted-dashed curves), and  $w_D=10$  (dashed curves). For each of these values,  $\tau_{\text{dec}}$  is shown for  $\alpha=1, 2$ , and  $3$  (from top to bottom).

bath in  $d$  dimensions, depending on the underlying symmetries [25]. As is well known [24,25], the imaginary part of the bath correlation function  $h(t)$  is temperature independent, its Fourier transform being given by  $i(\text{Im}h)(\omega) = J(\omega)$  for  $\omega \geq 0$ . By the KMS property (C6) this implies  $(\text{Re}h)(\omega) = \coth(\hbar\beta\omega/2)J(|\omega|)$ . If  $w_D = \hbar\omega_D\beta > 1$ , the thermal time  $T_B = \hbar\beta$  is the largest decay time of  $\text{Re}h(t)$ . The other time scale characterizing the variations of  $\text{Re}h(t)$  is the inverse cutoff frequency  $t_B = \omega_D^{-1} < T_B$ . By (44), the decoherence and entanglement times in units of  $T_B$ ,  $\tau_{\text{dec}} = t_{\text{dec}}/T_B$  and  $\tau_{\text{ent}} = t_{\text{ent}}/T_B$ , are given by

$$\frac{\tau_{\text{ent}}^{2\alpha}}{c_\alpha^2 \eta^2} = \frac{\int_0^\infty dw \coth(w/2) w^m e^{-w^2/w_D^2} \left| \int_0^{\tau_{\text{dec}}} d\tau \tau^\alpha e^{-i w \tau} \right|^2}{2 \int_0^\infty dw \coth(w/2) w^m e^{-w^2/w_D^2}}, \quad (60)$$

where we have expressed  $\text{Re}h(t)$  in terms of its Fourier transform and relied on (59). We did not write explicitly in (60) the dependence of  $\tau_{\text{ent}}$ ,  $\tau_{\text{dec}}$ , and  $c_\alpha$  on  $(s, s')$ . The right-hand side of (60) is shown in the inset in Fig. 3. We have computed numerically the integrals appearing in this right-hand side for various values of  $\alpha$ ,  $m$ , and  $w_D$ , so as to obtain  $\tau_{\text{dec}}$  as a function of  $\tau_{\text{ent}}$  and  $\eta$ . The main results are shown in Figs. 3 and 4. For fixed  $\alpha$  and  $\eta$ , the plain curves representing  $\tau_{\text{dec}}$  in Fig. 3 split by increasing  $\tau_{\text{ent}}$  into distinct branches corresponding to distinct  $m$ 's, as predicted by (54) and (56). This splitting occurs when  $\tau_{\text{dec}}$  is in the transition region  $w_D^{-1} \leq \tau_{\text{dec}} \leq 1$ . After this splitting  $\tau_{\text{dec}}$  is larger for larger  $m$ . In particular, a Ohmic bath ( $m=1$ ) has a smaller decoherence time than a super-Ohmic bath ( $m=3, 5, \dots$ ) as stated above. For comparison, the power-law behaviors found in Secs. VIB and VIC in the small-time ( $\tau_{\text{dec}} \ll w_D^{-1}$ ) and Markov ( $\tau_{\text{dec}} \gg 1$ ) regimes are also shown in Fig. 3 (dashed lines). A remarkably good agreement between the exact and asymptotic behaviors of  $\tau_{\text{dec}}$  is obtained: the exact results are well approximated by their small-time behaviors (51) up to

$\tau_{\text{dec}} \leq \omega_D^{-1}$  and they are hardly distinguishable from the Markov approximation as soon as  $\tau_{\text{dec}} \geq 1$ . Our aforementioned statement that a nonlinear pointer-bath coupling is more efficient for decoherence than a linear one when  $\tau_{\text{ent}}$  is not too large (and even for arbitrarily large  $\tau_{\text{ent}}$  if  $m \geq 3$  and  $\eta$  is small enough) is well confirmed. Indeed, it is seen in Fig. 4 that for a pointer-bath coupling strength  $\eta \ll 1$ ,  $\tau_{\text{dec}}$  becomes significantly smaller when the value of  $\alpha$  is increased from  $\alpha=1$  to  $\alpha=3$ . If the dotted lines (Markovian results) in Fig. 3 were drawn farther to the right, the two lines corresponding to  $(\alpha, m)=(1, 1)$  and  $(\alpha, m)=(2, 1)$  would intersect; after this intersection (not shown in the figure), the reverse situation of higher values of  $\alpha$  leading to higher values of  $\tau_{\text{dec}}$  occurs. In contrast, for  $m=3, 5, \dots$ , the dotted lines associated with  $\alpha=1$  and  $\alpha=2$  never intersect (they are parallel); hence,  $\tau_{\text{dec}}$  decreases with  $\alpha$  and  $\tau_{\text{ent}} \leq \tau_{\text{dec}}$  for all values of  $\tau_{\text{ent}}$  (more precisely,  $\tau_{\text{ent}} \leq 10^{-4} \tau_{\text{dec}}$  for  $\alpha=1$  and  $\tau_{\text{ent}} \leq 10^{-2} \tau_{\text{dec}}$  for  $\alpha=2$ ). We also emphasize that  $\tau_{\text{dec}}$  increases in Fig. 4 with the cutoff frequency  $\omega_D$ . Even though the results in Figs. 3 and 4 correspond to the simplifying choice of a bath of harmonic oscillators with power spectrum function (59), for more general baths they should still give the correct qualitative picture.

### E. Bath at very low temperature

We have so far considered baths at finite temperature. Motivated by experiments in solid-state physics, we shall now discuss the case of a bath initially in thermal equilibrium at very low temperature. Strictly speaking, for the equilibrium apparatus initial state (4b) extremely low temperatures have to be proscribed because of our hypothesis  $\hbar\beta \ll T_p$ . However, taking, e.g.,  $T_p=1$  s, this separation of time scales holds even for the smallest temperatures that can be achieved in experiments. Furthermore, the stability conditions (40) have a better chance to be met at low temperature  $T$  since  $\Delta_{\text{th}}$  decreases with  $T$ . To be specific, we consider the same bath of harmonic oscillators as in the previous subsection, but now in the limit  $\omega_D = \hbar\omega_D\beta \gg 1$ . Then only spontaneous emission plays a role in the pointer-bath interaction. In other words,  $\widehat{\text{Reh}}(\omega) = \coth(\hbar\beta\omega/2)J(|\omega|)$  can be approximated by  $J(|\omega|)$ . The zero-temperature variance of  $B$  equals  $\langle B^2 \rangle = \int d\omega J(|\omega|)/2\pi$ . For our choice (59) of the power spectrum function, this gives  $2\pi\langle B^2 \rangle = \hat{\gamma}\omega_D^{m+1}[(m-1)/2]!$ . The analog of (60) reads

$$\frac{(\omega_D t_{\text{ent}})^{2\alpha}}{c_\alpha^2 \eta_D^2} = \frac{1}{[(m-1)/2]!} \int_0^\infty dv v^m e^{-v^2} \left| \int_0^{\omega_D t_{\text{dec}}} du u^\alpha e^{-ivu} \right|^2, \quad (61)$$

where  $\eta_D$  is now the pointer-bath coupling strength in units of  $\hbar\omega_D$ ,  $\eta_D = \langle B \rangle^{1/2} \Delta^\alpha / (\hbar\omega_D)$ . Equation (61) holds provided that  $\omega_D t_{\text{dec}} \ll \omega_D$ —i.e.,  $t_{\text{dec}} \ll \hbar\beta$ . This equation is the same as (60) apart from the substitutions  $\tau_{\text{ent}} \rightarrow \omega_D t_{\text{ent}}$ ,  $\tau_{\text{dec}} \rightarrow \omega_D t_{\text{dec}}$ ,  $\eta \rightarrow \eta_D$ , and  $\coth(\omega/2)e^{-(\omega/\omega_D)^2} \rightarrow e^{-v^2}$ . Explicit formulas for  $t_{\text{dec}}$  can be given as before when  $t_{\text{dec}}$  is small or large compared with  $\omega_D^{-1}$ . One reads the small-time result directly on (51) by transforming this expression according to the recipe mentioned above. This gives  $\omega_D t_{\text{dec}}$

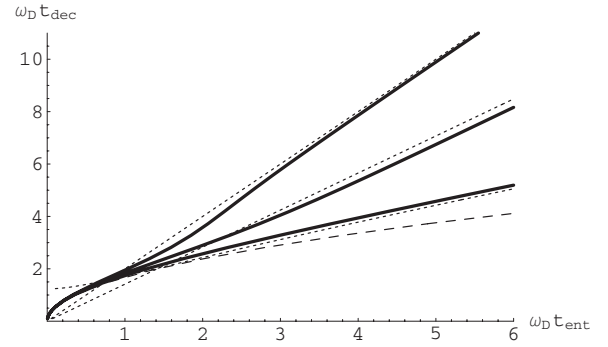


FIG. 5. Decoherence against entanglement times in units of  $\omega_D^{-1}$  for the bath at zero temperature of Sec. VI E with  $\alpha=1$ ,  $\eta_D=1$ , and  $m=5, 3, 1$  (solid curves, from top to bottom). The approximations (51) and (63) for  $\omega_D t_{\text{dec}} \ll 1$  and  $\omega_D t_{\text{dec}} \gg 1$  are shown as dashed lines.

$\propto (c_\alpha^{-1/\alpha} \eta_D^{-1/\alpha} \omega_D t_{\text{ent}})^{\alpha/(\alpha+1)}$  for  $t_{\text{dec}} \ll \omega_D^{-1}$ . Similarly, for a super-Ohmic bath  $t_{\text{dec}} \propto c_\alpha^{-1/\alpha} \eta_D^{-1/\alpha} t_{\text{ent}}$  when  $\omega_D^{-1} \ll t_{\text{dec}} \ll \hbar\beta$ . To find the proportionality factor, it is enough to realize that  $|\int_0^\infty d\tau \tau \widehat{\text{Reh}}(\tau)| = \int_0^\infty d\omega (\widehat{\text{Reh}}(\omega)) \omega^{-2} / \pi$  has to be interpreted in (57) as  $\hat{\gamma}\omega_D^{m-1} \int_0^\infty dv v^{m-2} e^{-v^2} / \pi = 2\omega_D^{-2} \langle B^2 \rangle / (m-1)$ . The Ohmic case requires some extra work. For indeed, replacing  $\widehat{\text{Reh}}(\omega)$  by  $J(|\omega|)$  leads to a vanishing second member in (53), even though the frequency integrals in the second and third members in Eq. (52) are still divergent. One actually finds

$$\int_0^\infty dv e^{-v^2} \int_0^1 du (1-u)^{\alpha-1} \sin(\omega_D t u v) \sim \frac{\ln(\omega_D t) + k_\alpha}{\omega_D t} \quad \text{as } \omega_D t \rightarrow \infty, \quad (62)$$

with  $k_1 \approx 0.2886$  and  $k_\alpha = k_1 - 1 - 1/2 - \dots - 1/(\alpha-1)$  if  $\alpha \geq 2$ . Substituting the frequency integral in (43) by the right-hand side of (62), one gets

$$t_{\text{ent}}(s, s') = [c_\alpha(s, s') \eta_D]^{1/\alpha} t_{\text{dec}}(s, s') \{ \ln[\omega_D t_{\text{dec}}(s, s')] + k_\alpha - (2\alpha)^{-1} \}^{1/(2\alpha)} \quad (\text{Ohmic}),$$

$$t_{\text{ent}}(s, s') = [c_\alpha(s, s') \eta_D / \sqrt{m-1}]^{1/\alpha} t_{\text{dec}}(s, s') \quad (\text{super-Ohmic}). \quad (63)$$

Instead of going through a proof of (62), which would lead us too far into technical details, let us compare the formulas (63) to the exact results obtained by numerical evaluations of the integrals in (61). It is seen in Fig. 5 that the approximate values (63) closely follow the exact curves when  $\omega_D t_{\text{dec}}$  becomes large (in fact, even for  $\omega_D t_{\text{dec}} \approx 2$  in the Ohmic case  $m=1$ ). Similar pictures are found for higher  $\alpha$ 's. Let us remark on (63) that for a given  $t_{\text{ent}}$ , the ratio between the decoherence times for Ohmic and super-Ohmic baths is logarithmically small in the dimensionless time  $\omega_D t_{\text{dec}}$ . Hence a Ohmic bath is not dramatically more efficient than a super-Ohmic bath at very low temperature, in contrast with our previous findings at “high” temperatures.

## VII. DERIVATION OF THE RESULT

We here fill in the derivation of the results presented in Sec. V. An alternative derivation of (32) based on the time-dependent Redfield equation can be found in [49]. Our approach below is nonperturbative in the pointer-bath coupling, but makes use of the QCLT (Sec. III A) which holds due to the additivity (3) of the bath coupling agent.

### A. Object-pointer dynamics

According to the results of Sec. III we may drop the Hamiltonians  $H_S$  and  $H_P$  in the full Hamiltonian  $H$ , with the proviso that the object initial state  $\rho_S$  is replaced by  $\rho_S^0(t)$  given by (16). This means that at times  $t \ll T_S, T_P$  the exact evolution operator of  $S+P+B$  in (8) can be approximated as  $e^{-itH/\hbar} \simeq e^{-itH_B/\hbar} \tilde{W}(t,0) e^{-it(H_S+H_P)/\hbar}$ , where

$$\begin{aligned} \tilde{W}(t,0) &= e^{itH_B/\hbar} e^{-it(H_B+H_{PS}+H_{PB})/\hbar} \\ &= \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau [\epsilon SP + X^\alpha \tilde{B}(\tau)] \right\} \end{aligned} \quad (64)$$

is the approximate evolution operator in the interaction picture. Note that  $\tilde{B}(\tau)$  is different from  $B$  as soon as  $\tau \gtrsim t_B$ . Since we do not assume here  $t$  to be small compared with  $t_B$ , we keep the time dependence of the bath coupling agent in (64). In view of the product structure  $\rho(0) = \rho_S \otimes \rho_{PB}$  of the initial state and by cyclic invariance of the trace, the object-pointer state (8) becomes

$$\rho_{PS}(t) \simeq \text{tr}_B[\tilde{W}(t,0) \rho_S^0(t) \otimes \rho_{PB} \tilde{W}(t,0)^\dagger], \quad t \ll T_S, T_P. \quad (65)$$

The pointer Hamiltonian  $H_P$  is absent in (65) since  $e^{-itH_P/\hbar} \rho_{PB} e^{itH_P/\hbar} \simeq \rho_{PB}$  at times  $t \ll T_P$  for the initial states under study; see Sec. III.

The approximate evolution operator (64) can be simplified by using the exact identity

$$\tilde{W}(t,0) = e^{-it\epsilon SP/\hbar} \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_0^t d\tau (X + \tau \epsilon S)^\alpha \tilde{B}(\tau) \right]. \quad (66)$$

We forego the proof of this (generalized Baker-Campbell-Hausdorff) identity, which uses the role of the momentum as generator of displacements,  $e^{-it\epsilon SP/\hbar} (X + t\epsilon S)^\alpha e^{it\epsilon SP/\hbar} = X^\alpha$ . Employing (66) in (65) and setting  $x_s(t) = x - t\epsilon S$  and  $x_{s'}(t) = x' - t\epsilon S'$  as before we get

$$\begin{aligned} \langle s, x | \rho_{PS}(t) | s', x' \rangle &= \langle s | \rho_S^0(t) | s' \rangle \langle x_s(t) | \text{tr}_B \left( \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau x_s(t-\tau)^\alpha \right. \right. \\ &\quad \left. \left. \times \tilde{B}(\tau) \right\} \rho_{PB} \left[ \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau x_{s'}^\alpha(t-\tau) \tilde{B}(\tau) \right\} \right]^\dagger \right) \\ &\quad \left. | x_{s'}^\alpha(t) \right). \end{aligned} \quad (67)$$

The next step consists in evaluating the trace over the bath in

(67) by taking advantage of Wick's theorem (12). We discuss the two initial states (4a) and (4b) separately.

### B. Partial-equilibrium initial state

For the partial-equilibrium initial state (4a) one has  $\rho_{PB} = \rho_P \otimes \rho_B^{(eq)}$  and the last matrix element in (67) is the product of a pointer and a bath expectation values,

$$\begin{aligned} \langle s, x | \rho_{PS}(t) | s', x' \rangle &= \langle s | \rho_S^0(t) | s' \rangle \\ &\quad \times \langle x_s(t) | \rho_P | x_{s'}^\alpha(t) \rangle K_t(x_s(t), x_{s'}^\alpha(t); s, s') \end{aligned} \quad (68)$$

with

$$\begin{aligned} K_t(x, x'; s, s') &= \left\langle \left[ \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau x_{s'}^\alpha(-\tau) \tilde{B}(\tau) \right\} \right]^\dagger \right. \\ &\quad \left. \times \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau x_s(-\tau) \tilde{B}(\tau) \right\} \right\rangle. \end{aligned} \quad (69)$$

Here  $\langle \cdots \rangle = Z_B^{-1} \text{tr}_B(\cdots e^{-\beta H_B})$  denotes the average with respect to the free bath thermal state. The QCLT and the additivity (3) of the bath coupling agent imply

$$\begin{aligned} F_{t,0}[k, l] &= \left\langle \left[ \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau k(\tau) \tilde{B}(\tau) \right\} \right]^\dagger \right. \\ &\quad \left. \times \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau l(\tau) \tilde{B}(\tau) \right\} \right\rangle \\ &= \exp \left\{ -\frac{1}{\hbar^2} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 [k(\tau_1) - l(\tau_1)] \right. \\ &\quad \left. \times [k(\tau_2) h(\tau_2, \tau_1) - l(\tau_2) h(\tau_1, \tau_2)] \right\}, \end{aligned} \quad (70)$$

where  $k(\tau)$  and  $l(\tau)$  are two arbitrary real-valued functions and  $h(\tau_1, \tau_2) = h(\tau_1 - \tau_2)$  is the two-point bath correlator; see (10). The identity (70) is equivalent to Wick's theorem (12). For ordered times  $t > t_1 > t_2 > \cdots > t_n > 0$  one actually gets (12) from (70) by setting  $k=0$  in (70) and taking the functional derivative of both members with respect to  $l(t_1), \dots, l(t_n)$  at  $l=0$ . The proof of the converse statement is deferred to Appendix D. By using the parity properties  $\text{Re}h(\tau) = \text{Re}h(-\tau)$  and  $\text{Im}h(\tau) = -\text{Im}h(-\tau)$  of the real and imaginary parts of  $h$  and employing (70) in (69) we get

$$K_t(x, x'; s, s') = e^{-D_t(x, x'; s, s') - i\phi_t(x, x'; s, s')}, \quad (71)$$

with a decoherence exponent  $D_t$  and a phase  $\phi_t$  given by

$$\begin{aligned} (D_t + i\phi_t)(x, x'; s, s') &= \frac{1}{\hbar^2} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 [x_{s'}^\alpha(-\tau_1)^\alpha - x_s(-\tau_1)^\alpha] \\ &\quad \times \{ [x_{s'}^\alpha(-\tau_2)^\alpha - x_s(-\tau_2)^\alpha] \text{Re}h(\tau_1 - \tau_2) \\ &\quad - i [x_{s'}^\alpha(-\tau_2)^\alpha + x_s(-\tau_2)^\alpha] \text{Im}h(\tau_1 - \tau_2) \}. \end{aligned}$$



Thus (68) reduces to the result (32) announced in Sec. V A.

### C. Equilibrium apparatus initial state

Before deriving the expression corresponding to (68) in the case  $\rho_{PB} = \rho_{PB}^{(\text{eq})}$ , we determine the pertinent initial density matrix of the pointer,

$$R_0(x, x') = \langle x | \text{tr}_B(\rho_{PB}^{(\text{eq})}) | x' \rangle. \quad (72)$$

It is convenient to introduce the  $x$ -dependent bath average

$$\langle O_B \rangle_x = Z_x^{-1} \text{tr}_B(O_B e^{-\beta(H_B + x^\alpha B)}), \quad Z_x = \text{tr}_B(e^{-\beta(H_B + x^\alpha B)}), \quad (73)$$

with  $x$  a real number (the pointer position, for us). Note that

$$e^{-\beta(H_B + x^\alpha B)} = e^{-\beta H_B} \mathcal{T} \exp \left\{ -\frac{x^\alpha}{\hbar} \int_0^{\hbar\beta} dz \tilde{B}(-iz) \right\}, \quad (74)$$

with  $\tilde{B}(-iz) = e^{zH_B/\hbar} B e^{-zH_B/\hbar}$ . The normalization factor  $Z_x$  can be determined by applying Wick's identity (70) with  $t = -i\hbar\beta$ ,  $k(\tau) = 0$ , and  $l(\tau) = x^\alpha$ . This gives

$$Z_x = Z_0 \exp \left\{ \frac{x^{2\alpha} \beta \gamma_0}{\hbar} \right\}, \quad (75)$$

with

$$\gamma_0 = \frac{1}{\hbar\beta} \int_0^{\hbar\beta} dz_1 \int_0^{z_1} dz_2 h(-iz_2). \quad (76)$$

By using the analyticity and KMS properties of the bath correlator  $h(\tau)$ , one can show that  $0 \leq \gamma_0 \leq \hbar\beta(B^2)/2$  and that  $\gamma_0 = \gamma(0)$  coincides with the following integral evaluated at  $t=0$ :

$$\gamma(t) = \int_{-\infty}^t d\tau \text{Im} h(\tau). \quad (77)$$

Details of this derivation are deferred to Appendix C.

Employing (75) in the approximation (17) for the apparatus initial state and inserting the high-temperature expression of  $\langle x | e^{-\beta H_{PB}'} | y \rangle$  (see Sec. II) yields

$$\begin{aligned} R_0(x, x') &= Z_{PB}^{-1} \int dy Z_y \langle x | e^{-\beta H_{PB}'} | y \rangle \langle y | e^{-\beta H_{PB}'} | x' \rangle \\ &= Z_0 Z_{PB}^{-1} \int dy e^{\beta \gamma_0 y^{2\alpha}/\hbar} e^{-\beta[V(x)+V(x')+2V(y)]/4} \\ &\quad \times e^{-4\pi^2[(x-y)^2+(x'-y)^2]/\lambda_{\text{th}}^2}, \end{aligned} \quad (78)$$

with  $\lambda_{\text{th}} = 2\pi\hbar(\beta/M)^{1/2}$ . The stability condition (40) does not guarantee that  $V(x)$  compensates  $-\gamma_0 x^{2\alpha}/\hbar$  when  $x \rightarrow \pm\infty$ . If this is not the case—i.e., if the effective potential  $V_{\text{eff}}(x) = V(x) - \gamma_0 x^{2\alpha}/\hbar$  has the shape shown in Fig. 2—the integrals in (78) diverge. This reflects the fact that the pointer interacting with the bath will tunnel to infinity after a certain time. Since we restrict our attention to initial states describing a pointer initially localized inside the potential well of  $V_{\text{eff}}(x)$ , we shall disregard this convergence problem by adding to  $V(x)$  a positive potential vanishing for  $|x| \leq W_{\text{eff}}$  and

diverging exponentially for  $x \rightarrow \pm\infty$ . This regularization trick amounts to replace  $\rho_{PB}^{(\text{eq})}$  in (72) by the local thermal state of the apparatus discussed in Sec. V B. After this regularization, the main contribution in the  $y$  integral in (78) comes from small values of  $y$ ,  $|y| \leq \Delta_{\text{th}}$ . In fact, the first exponential in this integral is a slowly varying function on the scale  $\lambda_{\text{th}}$  since  $(\beta\gamma_0/\hbar)^{-1/(2\alpha)} \geq 2^{1/\alpha} \Delta_{\text{th}} \gg \lambda_{\text{th}}$ , as follows from (40) and  $\gamma_0 \leq \hbar\beta(B^2)/2$ . Due to the presence of the last exponential in (78), this first exponential can be approximated by  $e^{\beta\gamma_0(x+x')^{2\alpha}/(2^{2\alpha}\hbar)}$  and taken out of the integral. Similarly, the second exponential varies noticeably on the scale  $\Delta_{\text{th}} \gg \lambda_{\text{th}}$  and can be approximated by  $e^{-\beta[V(x)+V(x')+2V(x/2+x'/2)]/4}$  and taken out of integral in (78). Thus (40) and  $\Delta_{\text{th}} \gg \lambda_{\text{th}}$  entail

$$R_0(x, x') \simeq Z_{P,\text{eff}}^{-1} e^{-\beta[V_{\text{eff}}(x)+V_{\text{eff}}(x')]/2} e^{-2\pi^2(x-x')^2/\lambda_{\text{th}}^2} \quad (79)$$

with  $V_{\text{eff}}(x)$  given by (35) and  $Z_{P,\text{eff}} = \int dx e^{-\beta V_{\text{eff}}(x)}$ . We have used in (79) the approximations  $\beta V(x/2+x'/2) \simeq \beta[V(x)+V(x')]/2$  and  $\beta\gamma_0(x+x')^{2\alpha}/(2^{2\alpha}\hbar) \simeq \beta\gamma_0(x^{2\alpha}+x'^{2\alpha})/(2\hbar)$ . This introduces an error which is negligible against  $(x-x')^2/\lambda_{\text{th}}^2$  for  $|x|, |x'| \leq \Delta_{\text{th}}$  and  $\Delta_{\text{th}} \gg \lambda_{\text{th}}$ . Hence the pointer is in a Gibbs-type state with an effective potential  $V_{\text{eff}}(x)$ , as announced in (38).

We can now proceed to evaluating (67). Repeating the steps yielding to (78) and using the notation (73),

$$\begin{aligned} \langle s, x | \rho_{PB}(t) | s', x' \rangle &= Z_{PB}^{-1} \langle s | \rho_S^0(t) | s' \rangle \int dy Z_y \langle x_s(t) | e^{-\beta H_{PB}'} | y \rangle \\ &\quad \times \langle y | e^{-\beta H_{PB}'} | x'_s(t) \rangle \left\langle \left[ \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau \right. \right. \right. \\ &\quad \times x'_s(t-\tau) \alpha \tilde{B}(\tau) \left. \left. \left. \right]^\dagger \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau \right. \right. \right. \\ &\quad \times x_s(t-\tau) \alpha \tilde{B}(\tau) \left. \left. \left. \right] \right\rangle_y. \end{aligned} \quad (80)$$

We set  $\delta\tilde{B}(\tau, y) = \tilde{B}(\tau) - \langle \tilde{B}(\tau) \rangle_y$  and consider the (quantum) characteristic functional

$$\begin{aligned} F_{t,y}[k, l] &= \left\langle \left[ \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau k(\tau) \delta\tilde{B}(\tau, y) \right\} \right]^\dagger \right. \\ &\quad \times \left. \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau l(\tau) \delta\tilde{B}(\tau, y) \right\} \right\rangle_y. \end{aligned} \quad (81)$$

It is shown in Appendix D that all the correlation functions  $\langle \delta\tilde{B}(\tau_1, y) \cdots \delta\tilde{B}(\tau_n, y) \rangle_{B,y}$  are independent of  $y$ —i.e.,

$$F_{t,y}[k, l] = F_{t,0}[k, l] \quad (82)$$

for any  $y$ ,  $t$ ,  $k(\tau)$ , and  $l(\tau)$ . Wick's theorem (12) also entails (see Appendix D)

$$\langle \tilde{B}(\tau) \rangle_x = -\frac{2x^\alpha}{\hbar} \gamma(\tau), \quad (83)$$

with  $\gamma(\tau)$  given by (77). Formula (83) is reminiscent of linear response theory since  $\gamma(t) = \gamma_0 - \hbar \int d\tau \chi(\tau) \theta(t-\tau)/2$ , where  $\theta(\tau)$  is the Heaviside function and  $\chi(\tau)$

$= -(2/\hbar)\theta(\tau)\text{Im}h(\tau)$  the linear susceptibility. Let us point out, however, that (83) is exact to all orders in  $x^\alpha$ . Collecting the above results one finds

$$\langle s, x | \rho_{\mathcal{P}S}(t) | s', x' \rangle = \langle s | \rho_S^0(t) | s' \rangle R_t(x_s(t), x'_s(t); s, s') \times K_t(x_s(t), x'_s(t); s, s'), \quad (84)$$

where  $K_t(x, x'; s, s')$  is given by (69)

$$R_t(x, x'; s, s') = C e^{-\beta[V_{\text{eff}}(x) + V_{\text{eff}}(x')]/2} e^{-4\pi^2(x^2 + x'^2)/\lambda_{\text{th}}^2} \times \int d\xi \exp \left\{ -\xi^2 + \sqrt{8\pi^2} \xi \frac{x+x'}{\lambda_{\text{th}}} - 2i\xi^\alpha g_t(x, x'; s, s') \right\}, \quad (85)$$

$C$  is a time-independent normalization constant, and

$$g_t(x, x'; s, s') = (8\pi^2)^{-\alpha/2} \frac{\lambda_{\text{th}}^\alpha}{\hbar^2} \int_0^t d\tau \gamma(\tau) [x'_s(-\tau)^\alpha - x_s(-\tau)^\alpha]. \quad (86)$$

Therefore, (71) and (84) account for (37). Moreover, by using (4b), (16), (72), and (84), and  $K_0=1$  one easily establishes that  $R_0(x, x'; s, s') = R_0(x, x')$ . Taking  $t=0$  in (85), evaluating the Gaussian integral, and comparing with (79), one gets  $C = \pi^{-1/2} Z_{\mathcal{P}, \text{eff}}^{-1}$ .

#### D. Justification of the approximation (38) for $R_0$

We here want to derive the inequality

$$g_t(x, x'; s, s')^2 \leq (8\pi^2)^{-\alpha} \frac{\lambda_{\text{th}}^{2\alpha} \beta \gamma_0}{\hbar} D_t(x, x'; s, s'). \quad (87)$$

Let us first point out that, putting together (87), the stability condition (40), the separation of length scales  $\lambda_{\text{th}} \ll \Delta_{\text{th}}$ , and the bound  $2\beta\gamma_0/\hbar \leq \eta_{\text{th}}^2/\Delta_{\text{th}}^{2\alpha}$ , it follows that  $g_t(x, x'; s, s')^2 \ll D_t(x, x'; s, s')$  uniformly for all  $(x, x')$  and  $(s, s')$ . This explains why the general expression (85) reduces to (38) for short times  $t$  satisfying  $D_t(x, x'; s, s') \leq 1$ ; then,  $|g_t(x, x'; s, s')| \ll 1$  and the phase factor inside the integral in (85) can be neglected; by performing the resulting Gaussian integral, one gets  $R_t(x, x'; s, s') \approx R_0(s, s')$ . In the special case  $\alpha=1$ , the integral in (85) can be evaluated exactly for all times  $t$ . This leads to  $R_t = R_0 e^{-(g_t)^2 - i\phi_t'}$ . Here and in what follows  $\phi_t'$ ,  $\phi_t''$ , etc., denote real phases irrelevant for decoherence. Replacing the latter value of  $R_t$  into (37), the factor  $e^{-(g_t)^2}$  can be dropped by invoking  $g_t^2 \ll D_t$  again. We are thus led to

$$e^{-D_t(x, x'; s, s')} R_t(x, x'; s, s') \approx e^{-D_t(x, x'; s, s') - i\phi_t''} R_0(x, x') \quad (\alpha=1), \quad (88)$$

which is now valid for all times  $t \ll T_S, T_P$ . The integral (85) can be evaluated exactly for  $\alpha=2$  as well. By (87) and the same arguments as above, for  $\alpha>1$  the stronger condition  $g_t(x, x'; s, s')^2 \ll (\lambda_{\text{th}}/\Delta_{\text{th}})^2 D_t(x, x'; s, s')$  holds. Using also the restriction  $|x|, |x'| \leq \Delta_{\text{eff}} \approx \Delta_{\text{th}}$  coming from the factor in

front of the integral in (85), one obtains for all times  $t \ll T_S, T_P$

$$e^{-D_t(x, x'; s, s')} R_t(x, x'; s, s') \approx \frac{e^{-D_t(x, x'; s, s') - i\phi_t''} R_0(x, x')}{[1 + 4g_t^2(x, x'; s, s')]^{1/4}} \quad (\alpha=2). \quad (89)$$

Notice that this equation is consistent with  $R_t \approx R_0$  at times  $t$  satisfying  $D_t \leq 1$ .

Proceeding toward the inequality (87) we rewrite (86) as

$$\frac{(8\pi^2)^\alpha \hbar^4}{\lambda_{\text{th}}^{2\alpha}} g_t(x, x'; s, s')^2 = \left( \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{\gamma}(\omega) \int_0^t d\tau \times \cos(\omega\tau) [x'_s(-\tau)^\alpha - x_s(-\tau)^\alpha] \right)^2, \quad (90)$$

where  $\hat{\gamma}(\omega) = \hat{\gamma}(-\omega) \geq 0$  is the Fourier transform of  $\gamma(t)$ ; see (77). By using  $\gamma_0 = \gamma(0) = \int d\omega \hat{\gamma}(\omega)/2\pi$  and the Cauchy-Schwarz inequality, one gets

$$\frac{(8\pi^2)^\alpha \hbar^4}{\lambda_{\text{th}}^{2\alpha}} g_t(x, x'; s, s')^2 \leq \gamma_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{\gamma}(\omega) \left( \int_0^t d\tau \cos(\omega\tau) \times [x'_s(-\tau)^\alpha - x_s(-\tau)^\alpha] \right)^2. \quad (91)$$

The integral over  $\omega$  on the right-hand side of (91) can be bounded with the help of (C11) by

$$\frac{\hbar\beta}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\widehat{\text{Re}h})(\omega) \left[ \text{Re} \int_0^t d\tau e^{-i\omega\tau} [x'_s(-\tau)^\alpha - x_s(-\tau)^\alpha] \right]^2. \quad (92)$$

Comparing (92) with (34), we bound the last quantity by  $\hbar^3 \beta D_t(x, x'; s, s')$  and have thus established the inequality (87).

## VIII. CONCLUSION

Let us summarize the main results of this paper. We have investigated a model for a quantum measurement in which the entanglement produced by the interaction between the measured quantum object and the pointer is simultaneous with decoherence of distinct pointer readouts; the apparatus (pointer and bath) is taken initially in a metastable local thermal equilibrium, not correlated with the object. Our model has four parameters: the object-pointer coupling constant  $\epsilon$ , the thermal variance  $\langle B^2 \rangle$  of the bath coupling agent, the temperature  $T = (k_B \beta)^{-1}$  of the bath, and the exponent  $\alpha$  in the pointer-bath Hamiltonian (3). One may construct out of the first three parameters two relevant dimensionless constants. The first one is the entanglement time  $\tau_{\text{ent}} = \Delta(\epsilon \delta s \hbar \beta)^{-1}$  in units of the thermal time  $\hbar \beta$ . Here  $\delta s$  is the separation between neighboring eigenvalues of the measured

observable and  $\Delta$  the uncertainty in the initial pointer position. That entanglement time  $\tau_{\text{ent}}$  describes the efficiency of the pointer-bath interaction (a coupling is efficient if  $\tau_{\text{ent}}$  is small). More precisely,  $\tau_{\text{ent}}$  is the time after which pointer positions corresponding to distinct eigenvalues  $s$  begin to be resolved. The second dimensionless combination is the coupling energy  $\eta = \langle B^2 \rangle^{1/2} \Delta^\alpha \beta$  in units of  $k_B T$ , which measures

the strength of the pointer-bath coupling. We have found that, after a certain time  $t_{\text{dec}}$ , the object-pointer state is close to a statistical mixture of separable states  $\sum_s p_s |s\rangle\langle s| \otimes \rho_P^s$ , with  $p_s = \langle s | \rho_S | s \rangle$ ,  $\rho_S$  the object initial state, and  $\rho_P^s$  a distinguished pointer state depending on  $s$ . The decoherence time  $t_{\text{dec}} = \hbar \beta \tau_{\text{dec}} \ll T_S, T_P$  is given by [we ignore here numerical factors, given explicitly in (51), (55), and (57)]

$$\tau_{\text{dec}} \propto (\eta^{-1/\alpha} \tau_{\text{ent}})^\gamma, \quad \gamma = \begin{cases} \frac{\alpha}{\alpha + 1} & \text{if } t_{\text{dec}} \lesssim t_B \quad (\text{interaction-dominated regime}), \\ \frac{2\alpha}{2\alpha + 1} & \text{if } t_{\text{dec}} \gtrsim T_B \text{ for an Ohmic bath (Markov)}, \\ 1 & \text{if } t_{\text{dec}} \gtrsim T_B \text{ for a super-Ohmic bath (Markov)}. \end{cases} \quad (93)$$

For reasonably strong pointer-bath coupling and not too strong object-pointer coupling, the decoherence time  $t_{\text{dec}}$  (needed for the transformation of linear superpositions into statistical mixtures) can be so small that the whole measurement is performed without producing a Schrödinger cat state as an intermediate step. Two distinct regimes ought to be identified in (93): in the *interaction-dominated regime*,  $t_{\text{dec}}$  is shorter than the characteristic time  $t_B$  after which the bath correlation function  $h(t)$  differs significantly from its value  $\langle B^2 \rangle$  at  $t=0$ ; in the opposite *Markov regime*, one must wait more than the bath correlation time  $T_B$ —i.e., the largest decay time of  $h(t)$ —to obtain the required statistical mixture. While  $t_{\text{dec}}$  presents a universal behavior in the interaction-dominated regime (it depends on the bath through the single parameter  $\eta$ ), in the Markov regime it is determined by the small-frequency behavior of  $\text{Im}h(t)$ ,  $(\text{Im}h)(\omega) \sim -i\hat{\gamma}\omega^m$ . Larger values of  $t_{\text{dec}}$  are found for larger  $m$ 's, with a significant change of behavior between  $m=1$  (Ohmic bath) and  $m > 1$  (super-Ohmic bath); see (93). In both time regimes,  $t_{\text{dec}}$  strongly depends on the nonlinearity exponent  $\alpha$ , as illustrated in Figs. 3 and 4. Smaller decoherence times are obtained for larger  $\alpha$ 's save for the Markov regime if  $m > 1$  and  $\eta \gg \hbar\beta/T_B$  or if  $m=1$  and  $\eta^2 \tau_{\text{ent}} \gtrsim \hbar\beta/T_B$ , where the reverse statement holds. The linearization of the pointer-bath interaction with respect to the pointer position (dipole approximation) may then lead to an overestimation of  $t_{\text{dec}}$  in the interaction-dominated regime or for super-Ohmic baths in the Markov regime. For a bath at very low temperature, (93) still holds with  $\tau_{\text{dec}}$  and  $\eta$  replaced by  $t_{\text{dec}}/t_B$  and  $\eta_D = \langle B^2 \rangle^{1/2} \Delta^\alpha t_B / \hbar$ , save for the Ohmic case where  $t_{\text{dec}}/t_{\text{ent}}$  becomes logarithmically small in  $t_{\text{dec}}/t_B$ .

Several generalizations of our results may be of interest. The first one concerns measurements of observables with continuous or dense spectra. One must then allow for a finite resolution  $\delta s$  in the measurement result. Unlike in the case of discrete nondegenerate spectra studied in this work, the “final” object-pointer state will not be a separable state because coherences for pairs  $(s, s')$  of close eigenvalues ( $|s - s'| \leq \delta s$ ) are damped on a smaller time scale exceeding the time

duration of the measurement. A second generalization concerns the bath, assumed in this paper to consist of independent degrees of freedom. As stated in Sec. II, it can be shown that the validity of the QCLT extends to baths of interacting degrees of freedom if the correlator  $\langle B_\mu B_\nu \rangle$  decays more rapidly than  $1/|\mu - \nu|$  (see [23] for a related version of the QCLT in this context). This implies that our results apply to a broad class of baths including certain interacting spin chains.

It would be interesting to investigate concrete models for the object and pointer involving projective measurements in the “no-cat” regime (decoherence fast compared with entanglement), in connection with recent experiments in solid-state physics. We should also mention that our results can be of interest in a broader context. Actually, we have studied quantitatively another scenario for decoherence. In this “indirect decoherence” scheme, the decay of the quantum coherences of the small system (object) does not result from a direct coupling to the many degrees of freedom of a bath, but rather from a strong coupling to few degrees of freedom of the environment only (here, to the pointer). These few degrees of freedom are in turn coupled to all others bath coordinates and serve as an intermediate in the decoherence process.

### ACKNOWLEDGMENTS

We acknowledge support by the Deutsche Forschungsgemeinschaft (through Grant No. SFB/TR 12) and the Agence Nationale de la Recherche (Project No. ANR-05-JCJC-0107-01). D.S. thanks M. Guta for pointing out Ref. [22] to him and D. Vion for his encouragement to study the low-temperature case.

### APPENDIX A: APPARATUS AS A NONINTERACTING INFINITE GAS

Let us consider a gas of  $\mathcal{N} = N + 1$  noninteracting particles with mass  $m_\nu$ , momentum  $P_\nu$ , and position  $X_\nu$  ( $\nu = 0, \dots, N$ )

submitted to a slowly varying external potential  $V(x)$ . To simplify the discussion we restrict ourselves to a one-dimensional geometry. Let  $M = \sum_\nu m_\nu$  and  $P = \sum_\nu P_\nu$  be the total mass and momentum and  $X = \sum_\nu m_\nu X_\nu / M$  the center-of-

mass position;  $R_{\nu 0} = \mathcal{N}^{-1/2}(X_\nu - X_0)$  and  $P_{\nu 0}$  are the relative positions and their conjugate momenta. Expanding  $V(\underline{X}_\nu)$  as  $V(X) + (X_\nu - X)V'(X)$  and using  $X_\nu - X = \sqrt{\mathcal{N}}(R_{\nu 0} - \sum_\mu m_\mu R_{\mu 0} / M)$ , the Hamiltonian of the gas reads

$$H_{\text{app}} = \underbrace{\frac{P^2}{2M} + \mathcal{N}V(X)}_{H_{\mathcal{P}}} + \underbrace{\sqrt{\mathcal{N}} \sum_{\nu=1}^N \left(1 - \frac{\mathcal{N}m_\nu}{M}\right) R_{\nu 0} V'(X)}_{H_{\mathcal{PB}}} + \underbrace{\sum_{\mu,\nu=1}^N \frac{\ell_{\mu\nu}}{2\mathcal{N}m_\nu} P_{\mu 0} P_{\nu 0}}_{H_{\mathcal{B}}}, \quad (\text{A1})$$

where  $\ell$  is the  $N \times N$  matrix with inverse  $\ell^{-1} = (\delta_{\mu\nu} - m_\nu / M)_{\mu,\nu=1}^N$ . The pointer  $\mathcal{P}$  is the center-of-mass degree of freedom. Its Hamiltonian  $H_{\mathcal{P}}$  is given by the two first terms in (A1). The bath  $\mathcal{B}$  is constituted by the  $N$  relative degrees of freedom. Its Hamiltonian  $H_{\mathcal{B}}$  is the last term in (A1). The third term in (A1) describing the coupling between  $\mathcal{P}$  and  $\mathcal{B}$  has the form (3) if  $V(x) = \mathcal{N}^{-1}(\alpha+1)^{-1}x^{\alpha+1}$  and  $B$  is given by (3) with  $B_\nu = (1 - \mathcal{N}m_\nu / M)R_{\nu 0}$ . If the measured system is strongly coupled to the total momentum  $P$  of the gas, one obtains a tripartite model of the kind discussed in Sec. II (although  $H_{\mathcal{B}}$  does not satisfies all our hypothesis).

wherein  $A$  and  $C$  are any two operators. After a few transformations, (B1) becomes

$$e^A e^{2C} e^A = e^{2A+2C} e^{-[A,[A,C]]/3+2[C,[C,A]]/3+\dots}. \quad (\text{B2})$$

We write  $(\tilde{X}^\alpha)'(0)$  and  $(\tilde{X}^\alpha)''(0)$ , the two first time derivatives at  $t=0$  of the free-evolved observable  $X^\alpha$  [see (9)] and similarly for  $B$ . Let us take  $A = -\beta H_{\mathcal{P}}/2$  and  $C = -\beta(H_{\mathcal{B}} + H_{\mathcal{PB}})/2$ . Then

$$[A,[A,C]] = \frac{\hbar^2 \beta^3}{8} (\tilde{X}^\alpha)''(0)B, \quad (\text{B3})$$

## APPENDIX B: APPROXIMATION FOR THE APPARATUS EQUILIBRIUM STATE

In this appendix we justify the approximation (17) for the initial Gibbs state  $\rho_{\mathcal{PB}}^{(\text{eq})}$  of the apparatus. Moreover, we show that  $\rho_{\mathcal{P}}^0(t) = e^{-itH_{\mathcal{P}}/\hbar} \rho_{\mathcal{P}} e^{itH_{\mathcal{P}}/\hbar} \simeq \rho_{\mathcal{P}}$  when  $t \ll T_{\mathcal{P}}$  for the quasiclassical pointer states  $\rho_{\mathcal{P}}$  considered in Sec. II. A similar result holds for  $\rho_{\mathcal{PB}}^{(\text{eq})}$ .

We recall that  $T_{\mathcal{P}}$  is defined by  $T_{\mathcal{P}} = [M/V''(0)]^{1/2}$ . Taking  $V(x) \simeq V''(0)x^2/2$  and invoking (6), (7), and (9) one easily finds that  $\text{tr}_{\mathcal{P}}[\tilde{X}(t)^2 \rho_{\mathcal{P}}] - \Delta^2$  and  $\text{tr}_{\mathcal{P}}[\tilde{P}(t)^2 \rho_{\mathcal{P}}] - \Delta p^2$  are equal to lowest order in time to  $(-\Delta^2 T_{\mathcal{P}}^{-2} + \Delta p^2 M^{-2})t^2 \simeq \Delta^2 T_{\mathcal{P}}^{-2} t^2$  and  $[-\Delta p^2 T_{\mathcal{P}}^{-2} + V''(0)^2 \Delta^2]t^2 \simeq \Delta p^2 T_{\mathcal{P}}^{-2} t^2$ , respectively. Hence  $T_{\mathcal{P}}$  can be identified with the time scale for significant evolution of  $\tilde{X}(t)$  and  $\tilde{P}(t)$  when the pointer is in the quasiclassical state (7) (Sec. III A). One easily convinces oneself that  $\rho_{\mathcal{P}} = \rho_{\mathcal{P}}(X, P)$  is an operator-valued function of the position and momentum operators  $X$  and  $P$ . Letting  $\rho_{\mathcal{P}}$  evolve under the Hamiltonian  $H_{\mathcal{P}}$  up to time  $t$  amounts to substituting  $X$  by  $\tilde{X}(t)$  and  $P$  by  $\tilde{P}(t)$ ; see (9). This shows that  $\rho_{\mathcal{P}}^0(t) = \rho(\tilde{X}(t), \tilde{P}(t)) \simeq \rho_{\mathcal{P}}$  as  $\tilde{X}(t) \simeq X$  and  $\tilde{P}(t) \simeq P$  for  $t \ll T_{\mathcal{P}}$ .

In order to approximate  $\rho_{\mathcal{PB}}^{(\text{eq})}$  we shall rely on the Baker-Campbell-Hausdorff formula

$$e^A e^C = e^{A+C+[A,C]/2+[A,[A,C]]/12+[C,[C,A]]/12+\dots} \quad (\text{B1})$$

$$[C,[C,A]] = -\frac{\hbar^2 \beta^3}{8} \left( (\tilde{X}^\alpha)'(0) \tilde{B}'(0) - \frac{\alpha^2}{M} X^{2\alpha-2} B^2 \right). \quad (\text{B4})$$

Each time derivative of  $\tilde{X}^\alpha$  (of  $\tilde{B}$ ) gives an extra factor of  $T_{\mathcal{P}}^{-1}$  ( $t_{\mathcal{B}}^{-1}$ ). Therefore, the right-hand side of (B3) is smaller than  $\beta H_{\mathcal{PB}} = \beta X^\alpha B$  by a factor of the order of  $(\hbar\beta/T_{\mathcal{P}})^2 \ll 1$ . By virtue of  $\Delta_{\text{th}} = [\beta V''(0)]^{-1/2}$  and  $T_{\mathcal{P}} = [M/V''(0)]^{1/2}$ , the right-hand side of (B4) is of the order of  $(\hbar\beta)^2 (T_{\mathcal{P}}^{-1} t_{\mathcal{B}}^{-1} + T_{\mathcal{P}}^{-2} \eta_{\text{th}}) \beta H_{\mathcal{PB}}$  with  $\eta_{\text{th}}$  given by (40). Assuming that  $\eta_{\text{th}}$  is at most of order 1 [this is the case in particular if (40) holds true] this indicates that the double commutators (B3) and (B4) are much smaller than  $C$  and can be neglected in (B2) when  $\hbar\beta \ll T_{\mathcal{P}}$  and  $(\hbar\beta)^2 \ll T_{\mathcal{P}} t_{\mathcal{B}}$ . Neglecting these commutators, (B2) reduces to (17) for the aforementioned choices of  $A$  and  $C$ . Notice that our approximation of  $\rho_{\mathcal{PB}}^{(\text{eq})}$  is self-adjoint and is better than  $Z_{\mathcal{PB}}^{-1} e^{-\beta H_{\mathcal{P}}} e^{-\beta(H_{\mathcal{B}}+H_{\mathcal{PB}})}$  (the error is of one order smaller in  $\hbar\beta/T_{\mathcal{P}}$ ). The approximation  $e^{-itH_{\mathcal{P}}/\hbar} \rho_{\mathcal{PB}}^{(\text{eq})} e^{itH_{\mathcal{P}}/\hbar} \simeq \rho_{\mathcal{PB}}^{(\text{eq})}$  for  $t \ll T_{\mathcal{P}}$  is obtained similarly, by using (B1) with  $A = -itH_{\mathcal{P}}/\hbar$  and  $C = -\beta(H_{\mathcal{P}}+H_{\mathcal{B}}+H_{\mathcal{PB}})$ . One can check explicitly by means of similar arguments as in Sec. VII C that the relative errors are small. Indeed, one can show that if one multiplies (B3) or (B4) by the approximate equilibrium state  $Z_{\mathcal{PB}}^{-1} e^{-\beta H_{\mathcal{P}}/2} e^{-\beta(H_{\mathcal{B}}+H_{\mathcal{PB}})} e^{-\beta H_{\mathcal{P}}/2}$ , traces out the bath variables, and takes the matrix elements between  $|x\rangle$  and  $|x'\rangle$ , the matrix elements so obtained are much smaller than (79) if (40) is satisfied and  $\hbar\beta \ll T_{\mathcal{P}}$ .

### APPENDIX C: PROPERTIES OF THE BATH CORRELATION FUNCTION

In this appendix, we establish some general properties of the bath two-point autocorrelation function

$$h(t_1, t_2) = \langle \tilde{B}(t_1) \tilde{B}(t_2) \rangle = h(t_1 - t_2) \quad (\text{C1})$$

and its Fourier transform

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} dt h(t) e^{i\omega t}. \quad (\text{C2})$$

Most (but perhaps not all) of these properties are well known. The average  $\langle \dots \rangle$  in (C1) is taken with respect to the Gibbs state  $\rho_B^{(\text{eq})}$ ,  $\tilde{B}(t)$  is the bath coupling agent in the interaction picture, and  $\langle B \rangle = 0$ ; see (9)–(11). The fact that  $h(t_1, t_2)$  depends only on the time difference  $t_1 - t_2$  is a consequence of the stationarity of  $\rho_B^{(\text{eq})}$  [50].

*Real and imaginary parts.* The real and imaginary parts of  $h(t)$  are given by  $\text{Re}h(t) = \langle \tilde{B}(t)B + B\tilde{B}(t) \rangle / 2$  and  $\text{Im}h(t) = -i \langle [\tilde{B}(t), B] \rangle / 2$ . We write  $(\widehat{\text{Re}h})(\omega)$  and  $(\widehat{\text{Im}h})(\omega)$  their Fourier transforms. Then

$$\text{Re}h(t) = \text{Re}h(-t), \quad \text{Im}h(t) = -\text{Im}h(-t)$$

$$(\widehat{\text{Re}h})(\omega) = (\widehat{\text{Re}h})(-\omega), \quad (\widehat{\text{Im}h})(\omega) = -(\widehat{\text{Im}h})(-\omega). \quad (\text{C3})$$

The imaginary part  $\text{Im}h(t)$  is linked to the linear susceptibility by  $\chi(t) = -2\theta(t)\text{Im}h(t)/\hbar$ , where  $\theta(t)$  denotes the Heaviside function [50]. Such a susceptibility characterizes the response of the bath when its Hamiltonian is perturbed by the time-dependent potential  $V_B(t) = -x^\alpha(t)B$ , where  $x^\alpha(t)$  is a real-valued function of time. More precisely, if  $B(t)$  is the observable  $B$  in the Heisenberg picture (i.e.,  $dB(t)/dt = (i/\hbar)[H_B + V_B(t), B(t)]$ ), then  $\langle B(t) \rangle = \int d\tau \chi(t)x^\alpha(t-\tau)$  up to terms of order  $x^{2\alpha}$ .

*The function  $h(t)$  is of positive type.* This means that for any integer  $n \geq 1$ , complex numbers  $c_1, \dots, c_n$ , and times  $t_1, \dots, t_n$ , one has

$$\sum_{i,j=1}^n c_i^* c_j h(t_i - t_j) \geq 0. \quad (\text{C4})$$

This property can be easily checked on (C1). It is equivalent to  $\hat{h}(\omega) \geq 0$  for any real  $\omega$ . The real part of  $h(t)$  is also of positive type, as  $2(\widehat{\text{Re}h})(\omega) = \hat{h}(\omega) + \hat{h}(-\omega) \geq 0$  for any real  $\omega$ . By using the Cauchy-Schwarz inequality for the Hermitian sesquilinear form  $(A, B) \mapsto \langle A^\dagger B \rangle$ , one shows that  $|h(t)| \leq h(0) = \langle B^2 \rangle$  for any time  $t$ .

*KMS property.* This property is specific to our choice of the Gibbs state for the bath average. It says that  $h(t)$  can be extended to an analytic function in the strip  $\{z \in \mathbb{C}; -\hbar\beta < z < 0\}$ , continuous on  $\{z \in \mathbb{C}; -\hbar\beta \leq z \leq 0\}$ , and such that [47]

$$h(t) = h(-t - i\hbar\beta), \quad t \in \mathbb{R}. \quad (\text{C5})$$

Deforming the path of integration in (C2), one can show that (C5) is equivalent to  $\hat{h}(\omega) = e^{\hbar\beta\omega} \hat{h}(-\omega)$ . In view of (C3), this means that

$$(\widehat{\text{Re}h})(\omega) = \frac{i(\widehat{\text{Im}h})(\omega)}{\tanh(\hbar\beta\omega/2)}. \quad (\text{C6})$$

By replacing in this equation  $(\widehat{\text{Re}h})(\omega)$  and  $(\widehat{\text{Im}h})(\omega)$  by their Fourier integrals, expanding and identifying each power of  $\omega$ , and using the parity properties (C3), one finds relations between the integrals  $\int dt t^a h(t)$  for even and odd  $a$ 's. For instance, the identification of the zeroth power in  $\omega$  in (C6) yields

$$\int_{-\infty}^{\infty} dt th(t) = -i \frac{\hbar\beta}{2} \int_{-\infty}^{\infty} dt h(t). \quad (\text{C7})$$

We now assume that  $i(\widehat{\text{Im}h})(\omega) \sim \hat{\gamma}\omega^m$  as  $\omega \rightarrow 0$  with  $m$  a positive odd integer and  $\hat{\gamma} > 0$  (such a choice is motivated in Sec. VI C). By (C6), this entails  $(\widehat{\text{Re}h})(\omega) \sim 2\hat{\gamma}\omega^{m-1}(\hbar\beta)^{-1}$ . Let  $a$  be a non-negative integer,  $a \leq m-2$ . Then

$$\begin{aligned} \int_0^{\infty} dt t^a \text{Re}h(t) &= \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\widehat{\text{Re}h})(\omega) \int_0^{\infty} dt t^a e^{-i(\omega-i\varepsilon)t} \\ &= -(-i)^{a-1} a! \int_{-\infty}^{\infty} \frac{d\omega (\widehat{\text{Re}h})(\omega)}{2\pi \omega^{1+a}}. \end{aligned} \quad (\text{C8})$$

Note that the frequency integral converges for  $a \leq m-2$ , vanishes for even  $a$ 's,  $a < m-2$ , and diverges for  $a \geq m-1$ . For  $a=m-1$ , a similar formula holds,

$$\int_0^{\infty} dt t \text{Re}h(t) = - \int_{-\infty}^{\infty} \frac{d\omega (\widehat{\text{Re}h})(\omega) - (\widehat{\text{Re}h})(0)}{2\pi \omega^2}, \quad (\text{C9})$$

where the diverging frequency integral has been regularized by subtracting  $(\widehat{\text{Re}h})(0)$  to  $(\widehat{\text{Re}h})(\omega)$ . This is equivalent to subtracting  $(\widehat{\text{Re}h})(0)\delta(t)$  from  $\text{Re}h(t)$  in (C8), and this does affect the left-hand side of this equation. The integral on the right-hand side of (C9) converges since  $(\widehat{\text{Re}h})(\omega) - (\widehat{\text{Re}h})(0)$  behaves like  $\omega^2$  as  $\omega \rightarrow 0$ .

*Integration in the complex plane.* Let us denote by  $\gamma(t) = \gamma(-t)$  the primitive of  $\text{Im}h(t)$  vanishing at  $t \rightarrow \pm\infty$ ,

$$\begin{aligned} \gamma(t) &= \int_{-\infty}^t d\tau \text{Im}h(\tau) = - \int_t^{\infty} d\tau \text{Im}h(\tau) \\ &= \int_0^{\infty} \frac{d\omega}{\pi} \hat{\gamma}(\omega) \cos(\omega t). \end{aligned} \quad (\text{C10})$$

The KMS property (C6) and the bound  $\tanh(u) \leq u$  for  $u \geq 0$  imply

$$0 \leq \hat{\gamma}(\omega) = \frac{i(\widehat{\text{Im}h})(\omega)}{\omega} \leq \frac{\hbar\beta}{2} (\widehat{\text{Re}h})(\omega). \quad (\text{C11})$$

Employing (C11) in (C10) we obtain the following inequalities for  $\gamma_0 = \gamma(0)$ :

$$0 \leq \gamma_0 \leq \frac{\hbar\beta}{2} h(0). \quad (\text{C12})$$

We can exploit the KMS property further to obtain the two identities

$$\int_0^{\hbar\beta} dz h(-iz - t) = 2\gamma(t), \quad (\text{C13})$$

$$\int_0^{\hbar\beta} dz_1 \int_0^{z_1} dz_2 h(-iz_2) = \int_0^{\hbar\beta} dz h(-iz)z = \hbar\beta\gamma_0. \quad (\text{C14})$$

To show (C13), we deform the contour of integration in the complex plane to get

$$\int_0^{-i\hbar\beta} dz h(z - t) = \int_0^\infty d\tau [h(\tau - t) - h(\tau - t - i\hbar\beta)] \quad (\text{C15})$$

and then use (C5), (C3), and (C10). The second equality in (C14) is established in a similar way, relying also on (C7). Finally, we note that the left-hand side of (C14) reads

$$\hbar\beta \int_0^{\hbar\beta} dz_2 h(-iz_2) - \int_0^{\hbar\beta} dz_2 h(-iz_2)z_2. \quad (\text{C16})$$

Therefore, the first equality in (C14) is a consequence of the second one and of (C13).

#### APPENDIX D: WICK'S THEOREM

We show in this appendix that Wick's theorem (12) implies formulas (70) and (82)—i.e.,

$$F_{t,y}[k,l] = \langle \tilde{U}_{t,0}[k]^\dagger \tilde{U}_{t,0}[l] \rangle_y \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau [k(\tau) - l(\tau)] \right. \\ \left. \times \langle \tilde{B}(\tau) \rangle_y \right\} = \exp \left\{ -\frac{1}{\hbar^2} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 [k(\tau_1) \right. \\ \left. - l(\tau_1)] [k(\tau_2)h(\tau_2, \tau_1) - l(\tau_2)h(\tau_1, \tau_2)] \right\}, \quad (\text{D1})$$

where  $t$  and  $y$  are real numbers (time and position),  $k$  and  $l$  are (smooth) real functions,  $\langle \cdots \rangle_y$  is the bath average (83),  $h(\tau_1, \tau_2)$  is the bath function (C1), and

$$\tilde{U}_{t,0}[k] = \mathcal{T} \exp \left\{ -\frac{i}{\hbar} \int_0^t d\tau k(\tau) \tilde{B}(\tau) \right\}. \quad (\text{D2})$$

Let us first recall that Wick's theorem (12) can be rephrased as the following recursive relation for the bath  $n$ -point functions (10),  $n > 2$ :

$$h_n(t_1, \dots, t_n) = \sum_{1 \leq j < i} h(t_j, t_i) h_{n-2}(t_1, \dots, t_n) \\ + \sum_{i < j \leq n} h(t_i, t_j) h_{n-2}(t_1, \dots, t_n), \quad (\text{D3})$$

wherein  $i$  is a fixed integer between 1 and  $n$  and the time arguments of the  $(n-2)$ -point functions are the times appearing in the  $n$ -point function on the left-hand side except for  $t_i$  and  $t_j$ . Wick's theorem (D3) holds for any value of  $N$  if the bath consists of  $N$  harmonic oscillators linearly coupled to the pointer (Sec. VI D). For the more general baths considered in this work, its validity relies on the limit  $N \gg 1$  and is a consequence of the additivity of the bath coupling agent  $B$  and Hamiltonian  $H_B$  in single-degree-of-freedom contributions and of the QCLT of Ref. [22]. This theorem provides a mapping between the correlation functions  $h_n(t_1, \dots, t_n)$  and the correlation functions of a certain bath of harmonic oscillators in thermal equilibrium. In such a mapping,  $B_\nu$  is identified with the position of the  $\nu$ th oscillator [22].

We now proceed to proving (D1). Let us first consider the case  $y=0$ . To shorten the notation, we write  $dk(\tau)$  in place of  $k(\tau)d\tau$ . The two members of (D1) being equal at time  $t=0$ , it is enough to prove that they satisfy the same first-order time differential equation. Hence, we need to show that  $F_{t,0}[k,l] = \langle \tilde{U}_{t,0}[k]^\dagger \tilde{U}_{t,0}[l] \rangle$  satisfies

$$\frac{\partial F_{t,0}[k,l]}{\partial t} = -\frac{1}{\hbar^2} [k(t) - l(t)] \int_0^t [dk(\tau)h(\tau, t) \\ - dl(\tau)h(t, \tau)] F_{t,0}[k,l]. \quad (\text{D4})$$

But  $i\hbar \partial \tilde{U}_{t,0}[k] / \partial t = k(t) \tilde{B}(t) \tilde{U}_{t,0}[k]$ ; hence, (D4) is equivalent to

$$\langle \tilde{U}_{t,0}[k]^\dagger \tilde{B}(t) \tilde{U}_{t,0}[l] \rangle_0 = \frac{i}{\hbar} \int_0^t [dk(\tau)h(\tau, t) - dl(\tau)h(t, \tau)] \\ \times \langle \tilde{U}_{t,0}[k]^\dagger \tilde{U}_{t,0}[l] \rangle_0. \quad (\text{D5})$$

To show (D5), let us expand the two time-ordered exponentials on left-hand side of (D5). Invoking also (D3),  $\langle \tilde{B}(t) \rangle_0 = 0$ , and setting  $h_0=1$ , this left-hand side reads

$$\begin{aligned} & \sum_{n+m \geq 1}^N \frac{i^n (-i)^m}{\hbar^{n+m}} \int_{0 \leq \tau_n \leq \dots \leq \tau_1 \leq t} dk(\tau_1) \cdots dk(\tau_n) \int_{0 \leq t_m \leq \dots \leq t_1 \leq t} dl(t_1) \cdots dl(t_m) \\ & \times ((1 - \delta_{n,0}) \sum_{p=1}^n h(\tau_p, t) h_{n+m-1}(\tau_n, \dots, \tau_{p+1}, \tau_{p-1}, \dots, \tau_1, t_1, \dots, t_m) \\ & + (1 - \delta_{m,0}) \sum_{q=1}^m h(t, t_q) h_{n+m-1}(\tau_n, \dots, \tau_1, t_1, \dots, \tau_{q-1}, \tau_{q+1}, \dots, t_m)). \end{aligned} \tag{D6}$$

One may perform variable substitutions in the integrals in such a way that the time arguments of the first  $(n+m-1)$ -point function become  $(\tau_{n-1}, \dots, \tau_1, t_1, \dots, t_m)$  and those of the second become  $(\tau_n, \dots, \tau_1, t_1, \dots, t_{m-1})$ . Doing so and resumming the series, we find that (D6) reduces to (D5).

Turning to the case  $y \neq 0$ , we invoke (74) to write

$$\langle \tilde{U}_{t,0}[k]^\dagger \tilde{U}_{t,0}[l] \rangle_y = \frac{Z_0}{Z_y} \langle \tilde{U}_{-i\hbar\beta,0}[y^\alpha] \tilde{U}_{t,0}[k]^\dagger \tilde{U}_{t,0}[l] \rangle_0, \tag{D7}$$

wherein  $\tilde{U}_{-i\hbar\beta,0}[y^\alpha]$  is obtained by choosing the complex time  $-i\hbar\beta$  and the constant function  $y^\alpha$  in (D2). By expanding the three time-ordered exponentials in the right-hand side of (D7) and using Wick's theorem (D3), one obtains in a similar way as above

$$\begin{aligned} \frac{\partial}{\partial t} \langle \tilde{U}_{t,0}[k]^\dagger \tilde{U}_{t,0}[l] \rangle_y &= -\frac{1}{\hbar^2} [k(t) - l(t)] \left( \int_0^t [dk(\tau) h(\tau, t) \right. \\ & \left. - dl(\tau) h(t, \tau)] + iy^\alpha \int_0^{\hbar\beta} dz h(-iz, t) \right) \\ & \times \langle \tilde{U}_{t,0}[k]^\dagger \tilde{U}_{t,0}[l] \rangle_y. \end{aligned} \tag{D8}$$

Therefore, the functional

$$\begin{aligned} G_{t,y}[k, l] &= \langle \tilde{U}_{t,0}[k]^\dagger \tilde{U}_{t,0}[l] \rangle_y \exp \left\{ \frac{iy^\alpha}{\hbar^2} \int_0^t d\tau \int_0^{\hbar\beta} dz [k(\tau) \right. \\ & \left. - l(\tau)] h(-iz, \tau) \right\} \end{aligned} \tag{D9}$$

satisfies the same time differential equation (D4) as  $F_{t,0}[k, l]$ . Moreover, it is equal to 1 for  $t=0$ . Thus  $G_{t,y}[k, l] = F_{t,0}[k, l]$  for any  $t, y, k$ , and  $l$ . Setting  $k=0$  in this equation and differentiating with respect to  $l(\tau)$  at  $l=0$  yields

$$-\frac{iy^\alpha}{\hbar^2} \int_0^{\hbar\beta} dz h(-iz, \tau) - \frac{i}{\hbar} \langle \tilde{B}(\tau) \rangle_y = -\frac{i}{\hbar} \langle \tilde{B}(\tau) \rangle_0 = 0. \tag{D10}$$

This identity and (C13) imply (83). We may now replace (D10) into (D9) to get (D1). Let us stress that, although (D1) and (D10) coincide with the lowest-order results of perturbative expansions in  $k$  and  $l$ , these formulas are in fact valid to all orders in  $k$  and  $l$  in the limit  $N \gg 1$  as a consequence of the QCLT.

[1] J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, NJ, 1983).  
 [2] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1955).  
 [3] D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.-O. Stamatescu, and H. D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, Berlin, 1996).  
 [4] W. H. Zurek, Phys. Rev. D **24**, 1516 (1981).  
 [5] W. H. Zurek, Phys. Today **44** (10), 36 (1991); e-print arXiv:quant-ph/0306072 (updated version).  
 [6] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).  
 [7] M. B. Plenio and P. L. Knight, Rev. Mod. Phys. **70**, 101 (1998), and references therein.  
 [8] M. C. Fischer, B. Gutierrez-Medina, and M. G. Raizen, Phys. Rev. Lett. **87**, 040402 (2001).

[9] P. E. Toschek and C. Wunderlich, Eur. Phys. J. D **14**, 387 (2001).  
 [10] M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J.-M. Raimond, and S. Haroche, Phys. Rev. Lett. **77**, 4887 (1996).  
 [11] C. J. Myatt, B. E. King, Q. A. Turchette, C. A. Sackett, D. Kielpinski, W. M. Itano, C. Monroe, and D. J. Wineland, Nature (London) **403**, 269 (2000).  
 [12] P. Borri, W. Langbein, S. Schneider, U. Woggon, R. L. Sellin, D. Ouyang, and D. Bimberg, Phys. Rev. Lett. **87**, 157401 (2001); A. Vagov, V. M. Axt, T. Kuhn, W. Langbein, P. Borri, and U. Woggon, Phys. Rev. B **70**, 201305(R) (2004).  
 [13] G. Ithier, E. Collin, P. Joyez, P. J. Meeson, D. Vion, D. Esteve, F. Chiarello, A. Shnirman, Y. Makhlin, J. Schrieffer, and G. Schön, Phys. Rev. B **72**, 134519 (2005).

- [14] J. Claudon, A. Fay, L. P. Lévy, and O. Buisson, *Phys. Rev. B* **73**, 180502(R) (2006).
- [15] K. Hornberger, S. Uttenthaler, B. Brezger, L. Hackermüller, M. Arndt, and A. Zeilinger, *Phys. Rev. Lett.* **90**, 160401 (2003).
- [16] P. Sonntag and F. Hasselbach, *Phys. Rev. Lett.* **98**, 200402 (2007).
- [17] F. Haake and M. Żukowski, *Phys. Rev. A* **47**, 2506 (1993).
- [18] A. E. Allahverdyan, R. Balian, and T. M. Nieuwenhuizen, *Phys. Rev. A* **64**, 032108 (2001).
- [19] A. E. Allahverdyan, R. Balian, and T. M. Nieuwenhuizen, *Europhys. Lett.* **61**, 452 (2003).
- [20] E. Wigner, *Am. J. Phys.* **31**, 6 (1963).
- [21] D. Braun, F. Haake, and W. T. Strunz, *Phys. Rev. Lett.* **86**, 2913 (2001); W. T. Strunz, F. Haake, and D. Braun, *Phys. Rev. A* **67**, 022101 (2003).
- [22] M. Ohya and D. Petz, *Quantum Entropy and Its Use* (Springer-Verlag, Berlin, 1993).
- [23] A. F. Verbeure, in *Encyclopedia of Mathematical Physics*, edited by J.-P. Francoise, G. L. Naber, and T. S. Tsou (Elsevier, Amsterdam, 2006), Vol. 4, p. 130.
- [24] A. O. Caldeira and A. J. Leggett, *Ann. Phys.* **149**, 374 (1983); *Physica A* **121**, 587 (1983).
- [25] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993).
- [26] D. Spehner and F. Haake, *J. Phys. A* **41**, 072002 (2008).
- [27] W. H. Zurek, *Phys. Rev. Lett.* **90**, 120404 (2003).
- [28] M. Gross and S. Haroche, *Phys. Rep.* **93**, 301 (1982).
- [29] D. Polder, M. F. H. Schuurmans, and Q. H. F. Vreken, *Phys. Rev. A* **19**, 1192 (1979).
- [30] F. Haake, H. King, G. Schröder, J. Haus, and R. Glauber, *Phys. Rev. A* **20**, 2047 (1979).
- [31] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, 1995).
- [32] S. J. Freedman and J. F. Clauser, *Phys. Rev. Lett.* **28**, 938 (1972).
- [33] A. Aspect, J. Dalibard, and G. Roger, *Phys. Rev. Lett.* **49**, 1804 (1982).
- [34] S. Gröblacher, T. Paterek, R. Kaltenbaek, C. Brukner, M. Żukowski, M. Aspelmeyer, and A. Zeilinger, *Nature (London)* **446**, 871 (2007).
- [35] Our calculations can be generalized to a Hamiltonian  $H_{\mathcal{P}B}$  of the form  $Bf(X)$ , where  $f$  is a nice real function. In particular, a cutoff function at large distances can be introduced by choosing  $f(x) \approx 0$  for large  $x$ 's and  $f(x) \approx x^\alpha$  otherwise, so that the pointer coupling agent  $f(X)$  remains bounded.
- [36] L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1996).
- [37] T. Gorin, C. Pineda, and T. H. Seligman, *Phys. Rev. Lett.* **99**, 240405 (2007).
- [38] E. Lutz and H. A. Weidenmüller, *Physica A* **267**, 354 (1999); E. Lutz, *Physica E (Amsterdam)* **9**, 369 (2001).
- [39] D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, NJ, 1951).
- [40] F. Haake and R. Reibold, *Phys. Rev. A* **32**, 2462 (1985).
- [41] D. A. Lidar and K. B. Whaley, in *Irreversible Quantum Dynamics*, edited by F. Benatti and R. Floreanini, Lecture Notes in Physics, Vol. 622 (Springer, Berlin, 2003), p. 83; e-print arXiv:quant-ph/0301032.
- [42] To show that the constant is uniform in time, it must be proven that for all  $0 \leq a, b \leq \alpha$  one has  $\sup_{t \geq 0} t^{2\alpha-a-b} I_{a,b,t} / I_{\alpha,\alpha,t} < \infty$  with  $I_{a,b,t} = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \tau_1^a \tau_2^b \text{Re} h(\tau_1 - \tau_2)$ . This can be done by using the asymptotic analysis developed in Secs. VI B and VI C.
- [43] An upper bound  $t_{\text{dec}} \leq \tilde{t}_{\text{dec}}$  may be obtained by substituting  $t_{\text{ent}}(s, s')$  in (44) by its maximum value  $t_{\text{ent}} = \Delta / (\epsilon \delta s)$ ,  $c_\alpha(s, s')$  by its minimum value  $c_\alpha^{\text{min}}$ , and  $t_{\text{dec}}(s, s')$  by  $\tilde{t}_{\text{dec}}$ . Indeed, as the right-hand side of (44) increases with  $t_{\text{dec}}(s, s')$  the solution  $\tilde{t}_{\text{dec}}$  obtained after these substitutions is larger than  $t_{\text{dec}}$ .
- [44] In fact, the small-time behavior (50) can be obtained directly, without relying on our previous general results, by setting  $H = H_{\mathcal{P}S} + H_{\mathcal{P}B}$  in (8) and determining the dynamics in a similar way as in Ref. [21].
- [45] K. Hepp and E. H. Lieb, *Helv. Phys. Acta* **46**, 573 (1973).
- [46] V. Gorini and A. Kossakowski, *J. Math. Phys.* **17**, 1298 (1976).
- [47] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics* (Springer, Berlin, 1997), Vol. 2.
- [48] G. M. Palma, K.-A. Suominen, and A. K. Ekert, *Proc. R. Soc. London, Ser. A* **452**, 567 (1996).
- [49] D. Spehner and F. Haake, *J. Phys.: Conf. Ser.* **84**, 012018 (2007).
- [50] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-photon Interactions: Basic Processes and Applications* (Wiley, New York, 1992).