

Binary search trees for generalized measurements

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Generalized quantum measurements (positive operator valued measures or probability operator measures) are important for optimally extracting information for quantum communication and computation. The standard realization via the Neumark extension requires extensive resources in the form of operations in an extended Hilbert space. For an arbitrary measurement, we show how to construct a binary search tree with a depth logarithmic in the number of possible outcomes. This could be implemented experimentally by coupling the measured quantum system to a probe qubit which is measured, and then iterating.

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I. INTRODUCTION

A crucial element of quantum information processing (QIP) and communication is measurement of a quantum system to access its information content, hence determining optimal measurements is important. As QIP steps out from the pages of theory and into the laboratory, one has to find implementations given the usual limited resources of the real world. The most general measurement one can perform on a quantum system is given by a positive operator valued measure (POVM) [1] which can be considered as a projective measurement on an extended Hilbert space of which the original state resides on a (proper) subspace. Its realization via the Neumark extension [2,3] requires, broadly speaking, that the extended Hilbert space should have as many dimensions as there are possible outcomes of the POVM. This has been described for atoms or ions [4,5] and for linear optics [6], and POVMs have been realized on optically encoded quantum information [7–10].

For many physical systems, however, it is difficult or impossible to find enough extra dimensions, let alone perform operations on the extended system; hence, a more efficient method is desirable. Recently, a method was discovered which allows an arbitrary POVM to be performed by adding only a single extra dimension to a system, essentially checking the measurement outcomes one by one [11]. However, when the number of possible measurement outcomes becomes large, more time efficient measurement strategies, also carrying a minimal dimensional overhead, would be useful. It is clear that a sequence of partial conditional measurements implements a final effective POVM with many elements [3,12]. Here we show, given any POVM, how to construct a suitable binary search tree of two-outcome POVMs by coupling the original system with a single qubit [13]. This way, a measurement with $N=2^t$ outcomes can be implemented in t steps, resulting in a significant speedup. Current experimental realizations, e.g., that in [9], could easily be adapted to this method.

II. GENERALIZED MEASUREMENT

A quantum measurement is often considered to be a projection in a complete basis of the d -dimensional Hilbert

space. However, many experimental measurements are not well-described by this. More generally, we only require of a measurement that the outcome probabilities are positive and sum to one and satisfy convex linearity over mixtures of states. This leads to the framework of generalized quantum measurements, where a measurement is represented by a set of positive operators $\{M_j\}_{j=1}^N$, $\langle\psi|M_j|\psi\rangle\geq 0\forall|\psi\rangle$, which sum to unity, $\sum_j M_j = \mathbb{1}$. Each outcome j is associated with an operator M_j and occurs with probability $p_j = \text{Tr}[M_j\rho]$, where ρ is the measured state. Hence a generalized measurement is usually called a positive operator valued measure (POVM) or probability operator measure (POM).

The Neumark extension provides a way of performing any POVM via projective measurements, albeit in an extended space. Without loss of generality, assume that each measurement operator M_j is proportional to a one-dimensional projector $M_j = |\psi_j\rangle\langle\psi_j|$ where $|\psi_j\rangle$ is not necessarily normalized [14]. If N is the number of outcomes and d the dimension of the Hilbert space, then $N \geq d$ will hold. If we form the $d \times N$ rectangular array $(M)_{jk} = \langle k|\psi_j\rangle$, where $\{|k\rangle\}$ is the computational basis, then the completeness relation implies that the columns of (M) are orthonormal N -dimensional vectors. Hence (M) can be completed to an N -dimensional unitary matrix U_M whose j th row represents a state $|\psi_j^{ext}\rangle$ in an N -dimensional extended Hilbert space. The normalized projector $|\psi_j^{ext}\rangle\langle\psi_j^{ext}|$ corresponds to outcome j for the original system. This procedure corresponds to applying U_M^\dagger to the extended Hilbert space in which the original system is embedded, and then making a projective measurement in the computational basis. If N is large, it may be infeasible to manipulate the required extended quantum system all at once, and we will therefore look at a way to reduce the number of ancillary dimensions by making sequential measurements.

III. SEQUENTIAL MEASUREMENT

The $\{M_j\}$ are sufficient to determine the measurement probabilities, but the postmeasurement state is not uniquely defined. However, for any *realization* we can find Kraus operators $\{m_j\}$, where $M_j = m_j^\dagger m_j \forall j$, which tell us how the quantum state is affected [3,15]. If outcome j is obtained, then the quantum state ρ transforms as

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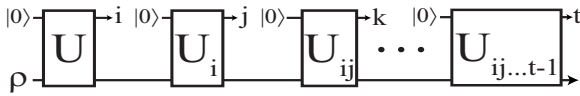


FIG. 1. Sequential POVM. A probe system in the state $|0\rangle$ interacts with the measured system ρ via unitaries. The probe is measured in the computational basis. Subsequent measurements, determined by U_x , are conditional upon the preceding results i, j, k, \dots . The probe is reset before each measurement.

$$\rho \rightarrow \rho_j = \frac{m_j \rho m_j^\dagger}{\text{Tr}[m_j \rho m_j^\dagger]}. \quad (1a)$$

A subsequent measurement, in general depending on the outcome j , acts on this transformed state.

A series of measurements can be viewed as an effective single generalized measurement, the sequence of outcomes determining the cumulative measurement operator. If the sequence j_1, j_2, \dots, j_t of outcomes has $\{M_{j_1}^1, M_{j_2}^2, \dots, M_{j_t}^t\}$ and $\{m_{j_k}^k\}$ as corresponding measurement and Kraus operators, then the final effective measurement operator and Kraus operator are given by

$$M_{j_1, j_2, \dots, j_t} = m_{j_1, j_2, \dots, j_t}^\dagger m_{j_1, j_2, \dots, j_t}, \quad (1b)$$

$$m_{j_1, j_2, \dots, j_t} = m_{j_t}^t m_{j_{t-1}}^{t-1} \dots m_{j_1}^1. \quad (1c)$$

Here we assume that the measurement operators and the Kraus operators are $d \times d$ operators, i.e., the measurements map the system to a Hilbert space of the same dimension [16]. Hence a sequence of measurements requires nondestructive measurement, e.g., indirect measurement of a system by measuring a probe after it has interacted with the system.

A series of binary outcome measurements is shown in Fig. 1. The simplest probe is a two-level system (qubit), giving a binary measurement, a d -level probe allowing a d -outcome measurement. A unitary operator couples the probe with the system, e.g., via a coupling Hamiltonian over a set period. This in general entangles the state of the probe with the state of the system. Measuring the probe performs an indirect measurement of the system. From the Stinespring dilation [17], this effectively implements a completely positive map with Kraus operators given by $b_j = \langle j|U|0\rangle$, where $\{|j\rangle\}$ is the computational basis of the probe. Outcome j corresponds to the measurement operator $B_j = b_j^\dagger b_j$ and the conditional post-measurement state is $\rho_j = b_j \rho b_j^\dagger / \text{Tr}[b_j \rho b_j^\dagger]$. By choosing suitable unitaries, any binary outcome POVM can be implemented at each stage.

Conditioned on the result of the first measurement, a second measurement is performed, a third, and so on (Fig. 1). This builds up a binary measurement tree with each pair of branches representing a different binary POVM, depending on the previous results [18]. Each node represents the effective measurement operator [given by Eq. (1c)] obtained at that point. Hence, after t measurements, the effective POVM may have as many as $N = d^t$ elements at the lowest level for a d -level probe.

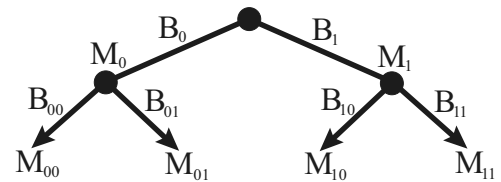


FIG. 2. Four-outcome POVM. The effective measurement operators $\{M_{ij}\}$ are given by the B measurements at each step. We want to determine what binary measurements B are required to implement $\{M_{ij}\}$.

IV. BINARY MEASUREMENT TREES

It is easy to build up POVMs with many elements from a binary measurement tree. However, given an arbitrary POVM with elements M_j , constructing such a measurement tree which implements it is not so obvious. Here we show how it may be done.

It is instructive to look at the simplest nontrivial binary POVM tree with $t=2$ (Fig. 2). Let B_i and B_{ij} denote the binary measurement operators performed at the first and second stages, and M_i, M_{ij} denote the cumulative measurement operators. The following should hold, where $j, k=0, 1$:

$$M_j = B_j,$$

$$I = B_0 + B_1 = B_{j_0} + B_{j_1},$$

$$M_j = M_{j_0} + M_{j_1},$$

$$m_{jk} = b_{jk} m_j.$$

Let us take $b_0 = m_0$, $b_1 = m_1$ and use the ansatz $b_{ij} = m_{ij} \tilde{b}_i^{-1}$ where the Moore-Penrose pseudoinverse \tilde{A}^{-1} of an operator A is uniquely defined by [19]

$$A \tilde{A}^{-1} A = A,$$

$$\tilde{A}^{-1} A \tilde{A}^{-1} = \tilde{A}^{-1},$$

$$A \tilde{A}^{-1} = (A \tilde{A}^{-1})^\dagger,$$

$$\tilde{A}^{-1} A = (\tilde{A}^{-1} A)^\dagger.$$

The b_{ij} so constructed are POVM operators which solve the task.

First, since M_i is a positive operator, $M_i = \sum_{k=1}^r \lambda_{ik} |e_{ik}\rangle \langle e_{ik}|$, with positive eigenvalues λ_{ik} and corresponding eigenvectors $|e_{ik}\rangle$; r is the rank of M_i . The M_{ij} are positive operators and $\sum_j M_{ij} = M_i$, so the null space of M_i is contained in the null spaces of M_{ij} ; hence, $M_{ij} = \sum_{k,l=1}^r \phi_{ij,kl} |e_{ik}\rangle \langle e_{il}|$ for some $\phi_{ij,kl}$. Similarly, $m_{ij} = \sum_{k,l=1}^r \varphi_{ij,kl} |e_{ik}\rangle \langle e_{il}|$ for some $\varphi_{ij,kl}$.

We can expand $b_i = m_i = \sum_k \sqrt{\lambda_{ik}} V_i |e_{ik}\rangle \langle e_{ik}|$ for some unitary V_i , similarly $\tilde{b}_i^{-1} = \sum_k 1/\sqrt{\lambda_{ik}} |e_{ik}\rangle \langle e_{ik}| V_i^\dagger$. Hence we can see that

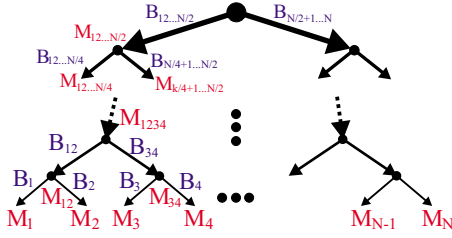


FIG. 3. (Color online) Binary POVM tree. An arbitrary POVM is given by operators $M_j = m_j^\dagger m_j$. A sequence of binary outcomes with measurement operators $B_\mu, B_\nu, \dots, B_\xi$ leads to the measurement $M_j = m_j^\dagger m_j$ where $m_j = b_\xi \cdots b_1 b_\mu$. Half of the possible results in the branches below are eliminated at each step.

$$(m_{ij} \tilde{b}_i^{-1}) b_i = \sum_{kl=1}^r \varphi_{ij,kl} |e_{ik}\rangle \langle e_{il}| \sum_{s=1}^r \frac{1}{\sqrt{\lambda_{is}}} |e_{is}\rangle \langle e_{is}| V_i^\dagger \\ \times \sum_{t=1}^r \sqrt{\lambda_{it}} V_i |e_{it}\rangle \langle e_{it}| = m_{ij}.$$

In general, completeness of $\{B_{ij}\}_j$ requires us to modify our original ansatz by adding an extra operator,

$$b_{ij} = m_{ij} \tilde{b}_i^{-1} + a_j g_i, \quad (2)$$

where $g_i = \sum_{j=r+1}^d |e_{ij}\rangle \langle e_{ij}| V_i^\dagger$ is an isometry on the null space of b_i^\dagger and the coefficients satisfy $\sum_j |a_j|^2 = 1$. We have defined g_i so that $g_i b_i = 0$. With this slight modification, it is easy to show that $\sum_k B_{jk} = \mathbb{1}$.

For a general POVM $\{M_k\}$ with K elements, we first pad the set with null operators until it contains N elements for $N = 2^t$, $t = \lceil \log_2 K \rceil$ (Fig. 3). In a convenient change of notation, the cumulative POVM at the j th level consists of 2^j operators M_x where x is a sequence of 2^{t-j} numbers indicating which of the possible outcomes $M_x = \sum_{i=1}^{2^{t-j}} M_{x_i}$ sit in the corresponding branches below. A binary POVM $\{B_{x_a}, B_{x_b}\}$ splits each node into two possible branches, each containing half of the remaining outcomes. We now determine the binary POVMs B which take us from a higher to lower branch.

At the first level, $B_{12\dots N/2} = \sum_{i=1}^{N/2} M_i = M_{12\dots N/2}$ and $B_{N/2+1\dots N} = \sum_{i=N/2+1}^N M_i = M_{N/2+1\dots N}$. At the second level, from the previous section we have

$$b_{12\dots N/4} = m_{12\dots N/4} \tilde{b}_{12\dots N/2}^{-1} + g_{12\dots N/2},$$

$$b_{N/4+1\dots N/2} = m_{N/4+1\dots N/2} \tilde{b}_{12\dots N/2}^{-1} + g_{12\dots N/2},$$

$$b_{N/2+1\dots 3N/4} = m_{N/2+1\dots 3N/4} \tilde{b}_{N/2+1\dots N}^{-1} + g_{N/2+1\dots N},$$

$$b_{3N/4+1\dots N} = m_{3N/4+1\dots N} \tilde{b}_{N/2+1\dots N}^{-1} + g_{N/2+1\dots N},$$

where we have absorbed the normalization of the g_x operators. At subsequent levels, we can express the required binary POVMs as

$$b_{x_a} = m_{x_a} \tilde{m}_{x_a x_b}^{-1} + g_{x_a x_b}, \quad (3a)$$

$$b_{x_b} = m_{x_b} \tilde{m}_{x_a x_b}^{-1} + g_{x_a x_b}, \quad (3b)$$

where $x_a x_b$ is the concatenation of the strings x_a and x_b . At the last level $b_1 = m_1 \tilde{m}_{12}^{-1} + g_{12}$ and $b_2 = m_2 \tilde{m}_{12}^{-1} + g_{12}$. Note that the unitary freedom $m_x \rightarrow V_x m_x$ leaves the observed probabilities invariant but simply rotates the postselected states after each measurement.

For an N element POVM, we need only a probe qubit and $\lceil \log_2 N \rceil$ rounds of binary measurements. Let us determine the number of operations required to implement this measurement compared to other methods. For a measurement with N outcomes on a d -dimensional quantum system, the standard Neumark extension requires an $N \times N$ unitary transform. This can be realized with $N(N-1)/2$ operations between pairs of basis states [20], followed by a projective measurement in the N -dimensional space. The realization using just a single extra degree of freedom [11] requires a $(d+1) \times (d+1)$ unitary transform to be implemented a maximum of $N-d$ times, giving in total a maximum of $(N-d)(d+1)d/2$ operations [21]. The binary search requires a $2d \times 2d$ transform to be implemented $\lceil \log_2 N \rceil$ times, that is, $\lceil \log_2 N \rceil d(2d-1)$ pairwise interactions, a significant speedup if N is large.

V. EXAMPLE: TETRAD MEASUREMENT

As an example of the method, consider the symmetric informationally complete POVM of a single qubit, the so-called tetrad measurement, with measurement operators [22]

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} |0\rangle,$$

$$|\psi_1\rangle = \frac{1}{\sqrt{6}} (|0\rangle + \sqrt{2} e^{2\pi i/3} |1\rangle),$$

$$|\psi_2\rangle = \frac{1}{\sqrt{6}} (|0\rangle + \sqrt{2} e^{4\pi i/3} |1\rangle),$$

$$|\psi_3\rangle = \frac{1}{\sqrt{6}} (|0\rangle + \sqrt{2} |1\rangle).$$

Although the tetrad POVM can be performed in one projective step with the addition of just one extra qubit, we use it to demonstrate the binary tree approach.

At the first stage, we are free to choose which two final measurement operators to group together, for instance,

$$M_{03} = B_{03} = M_0 + M_3 = \frac{1}{3} \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 2 \end{pmatrix},$$

$$M_{12} = B_{12} = M_1 + M_2 = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

We are also free to choose the Kraus operators $m_x = \sqrt{M_x}$, e.g.,

$$m_{03} = \sqrt{\lambda_+}|e_+\rangle\langle e_+| + \sqrt{\lambda_-}|e_-\rangle\langle e_-|,$$

$$m_{12} = \sqrt{\lambda_-}|e_+\rangle\langle e_+| + \sqrt{\lambda_+}|e_-\rangle\langle e_-|,$$

where the eigenvalues and eigenvectors are

$$\lambda_{\pm} = (1 \pm \sqrt{3})/2,$$

$$|e_{\pm}\rangle = (\pm\sqrt{3} \pm \sqrt{3}|0\rangle + \sqrt{3 \mp \sqrt{3}}|1\rangle)/\sqrt{6}.$$

Although M_{03} and M_{12} share eigenbases, we need a full 4×4 Neumark extension binary POVM so that the postmeasurement state is ready for the next stage. We couple the system via U to an auxiliary probe qubit prepared in the state $|0\rangle_a$. Then, projecting the probe onto states $|0\rangle_a$ and $|1\rangle_a$ corresponds to operations ${}_a\langle 0|U|0\rangle_a = m_{03}$ and ${}_a\langle 1|U|0\rangle_a = m_{12}$ on the system. A suitable coupling U is constructed by making a 4×4 Neumark extension of the two-column matrix with its first two rows given by m_{03} , and the last two rows by m_{12} . In the basis $\{|e_{\pm}\rangle|j\rangle_a\}$, one possible U is

$$U = \begin{pmatrix} \sqrt{\lambda_+} & 0 & \sqrt{\lambda_-} & 0 \\ 0 & \sqrt{\lambda_-} & 0 & \sqrt{\lambda_+} \\ \sqrt{\lambda_-} & 0 & -\sqrt{\lambda_+} & 0 \\ 0 & \sqrt{\lambda_+} & 0 & -\sqrt{\lambda_-} \end{pmatrix}. \quad (5)$$

In this example, the positive operators m_{jk} are invertible so the b_j for the next step are easily obtained as

$$b_0 = \sqrt{M_0}\sqrt{M_{03}}^{-1},$$

$$b_1 = \sqrt{M_1}\sqrt{M_{12}}^{-1},$$

$$b_2 = \sqrt{M_2}\sqrt{M_{12}}^{-1},$$

$$b_3 = \sqrt{M_3}\sqrt{M_{03}}^{-1},$$

which give

$$B_0 = b_0^\dagger b_0 = \frac{1}{2} \begin{pmatrix} 1 + \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 1 + \sqrt{\frac{2}{3}} \end{pmatrix},$$

$$B_1 = b_1^\dagger b_1 = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$

$$B_2 = b_2^\dagger b_2 = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$

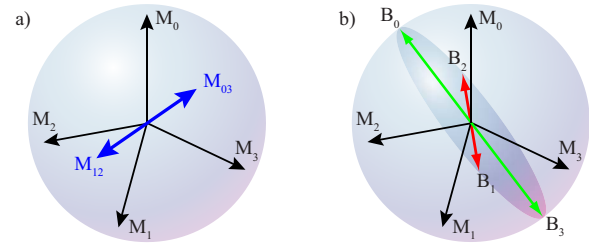


FIG. 4. (Color online) Tetrad POVM. (a) The first binary measurement is a partial filtering with operators $\{M_{03}, M_{12}\}$. (b) The second (projective) measurements depend on the outcome of the first measurement. If M_{03} (M_{12}) was obtained, then $\{B_0, B_3\}$ ($\{B_1, B_2\}$) is measured. The second binary measurements are projective measurements in the plane perpendicular to the directions of the first measurement and the B_j lie in the direction of the projection of the M_j upon this plane.

$$B_3 = b_3^\dagger b_3 = \frac{1}{2} \begin{pmatrix} 1 - \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 1 - \sqrt{\frac{2}{3}} \end{pmatrix}.$$

The b_j are rank one operators but are not Hermitian. We can visualize the sequence of measurements on the Bloch ball (Fig. 4).

VI. CONCLUSION

In conclusion, we provide a constructive proof of the universality of sequential two-outcome POVMs. We show how to construct binary measurement trees to implement any generalized measurement through a sequence of indirect binary POVMs requiring only an extra auxiliary qubit. This avoids having to manipulate extended Hilbert spaces (larger than twice the dimension of the measured system) and reduces the number of required operations when the number of outcomes becomes large. The number of steps is logarithmic in the number of measurement outcomes.

For some experimental implementations, e.g., linear optics, POVMs on a single qubit can be carried out using a path degree of freedom as the probe qubit, and partially polarizing beam splitters and wave plates as binary POVMs. This has the added advantage over the method of [11] that interferometric alignment is not required between separate paths (e.g., as in the implementation of the tetrad measurement in [9]). No feed-forward is necessary though the trade-off is the requirement for extra detectors. Also for the trapped ions, the full set of universal unitary gates between qubits is available; hence, there is no restriction in principle on the size of the measured system and the type of POVM. The binary tree approach reduces the number of extra levels per ion or the number of auxiliary ions. This is an issue for Paul traps as the number of ions which can be trapped in a single trapping zone is limited.

A potential candidate for the binary POVM is cavity QED where the system to be measured is the cavity field and the probe qubits Rydberg atoms passing through the cavity [23].

This allows in principle the measurement of an infinite dimensional system (a single mode of the cavity) by a sequence of atomic probes. For practical purposes, the system may be restricted to a finite dimensional Fock subspace. The key issues are the availability of the necessary interactions and the ability to apply real-time feed-forward control of these interactions. By tuning the atomic transition, a resonant or dispersive interaction, or a combination of the two, can be induced between the probe and system. In the strong interaction regime, nonlinear coupling for different cavity photon number states may be possible through polaritonic states [24]. It is an interesting question what class of measurement operators can be implemented using a restricted set of interactions and control.

We should note that this method is equally amenable to the implementation of any completely positive (CP) map defined by the Kraus operators corresponding to the measure-

ment outcomes. The measurement outcomes represent an unraveling of the evolution, and if we discard these results, on average we implement the desired CP map. To implement an arbitrary CP map, the minimum number of steps is the logarithm of the rank of the Choi matrix [25].

We could also use the binary tree construction to analyze information flow between system and environment (the role of feed-forward) in the context of open quantum systems and non-Markovian evolution.

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