Nonclassical correlation in a multipartite quantum system: Two measures and evaluation

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There is a commonly recognized paradigm in which a multipartite quantum system described by a density matrix having no product eigenbasis is considered to possess nonclassical correlation. Supporting this paradigm, we define two entropic measures of nonclassical correlation of a multipartite quantum system. One is defined as the minimum uncertainty about a joint system after we collect outcomes of particular local measurements. The other is defined by taking the maximum over all local systems about the minimum distance between a genuine set and a mimic set of eigenvalues of a reduced density matrix of a local system. The latter measure is based on an artificial game to create mimic eigenvalues of a reduced density matrix of a local system from eigenvalues of a density matrix of a global system. Numerical computation of these measures for several examples is performed.

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I. INTRODUCTION

There has been a long-standing discussion on the definition of quantumness in a quantum state of a multipartite system. One definition is, of course, entanglement which is considered equivalent to inseparability according to the separability paradigm [1–3]. The separability paradigm suggests, as is well-known in this field, a classification of density matrices of a system consisting of subsystems $1,\ldots,m$. Separable density matrices are those of the form

$$\rho_{\mathrm{sep}}^{[1,\ldots,m]} = \sum_k w_k \rho_k^{[1]} \otimes \cdots \otimes \rho_k^{[m]} \tag{1}$$

with positive weights w_k ($\Sigma_k w_k = 1$) and density matrices $\rho_k^{[\cdot]}$ of subsystems. Inseparable density matrices are those that cannot be represented in this form. A system (consisting of remote subsystems) represented by a separable density matrix is regarded as a classically correlated system because local operations and classical communications (LOCC; see, e.g., [4]) can create it from scratch: It can be prepared remotely when distant persons (Alice, Bob, …) receive instructions from a common source (Clare).

A bipartite system is a typical system to investigate. Bipartite separable density matrices are those of the form

$$\rho_{\mathrm{sep}}^{[A,B]} = \sum_k w_k \rho_k^{[A]} \otimes \rho_k^{[B]} \tag{2}$$

with positive weights w_k ($\Sigma_k w_k = 1$). Bipartite inseparable density matrices are those that cannot be represented in this form. One supporting evidence for the paradigm is that a bipartite system represented by a separable density matrix does not violate Bell's inequality [2,3].

Detection methods of inseparability have opened a large research field, many of which are based on the Peres-Horodecki test [2,5] using positive but not completely positive linear maps.

It is still a challenging issue to find classes of density matrices possessing nonlocal nature, other than the class of inseparable density matrices. Bennett *et al.* [6] discussed a certain nonlocality about locally nonmeasurable separable states. Ollivier and Zurek [7] later introduced a measure called quantum discord defined as a discrepancy of two expressions of mutual information that should be equivalent to each other in a classical information theory. Another branch of study on quantumness was started by Oppenheim and the Horodecki family [8]; this was extensively studied by a group consisting of the Horodecki family and other authors [9]. They introduced a protocol called closed LOCC (CLOCC). This protocol allows only local unitary operations, attaching ancillas in separable pure states, and operations to send subsystems through a complete dephasing channel. They also defined a measure of quantumness named quantum deficit as a discrepancy between the information that can be localized by applying CLOCC operations and the total information of the system. The present work is in the stream of these studies on quantumness in correlation.

One way to evoke a discussion on the validity of the separability paradigm is to look at the persistent question why a pseudopure state

$$\rho_{\mathrm{ps}} = p|\psi\rangle\langle\psi| + (1-p)\mathbb{1}/d, \tag{3}$$

with $|\psi\rangle$ an entangled pure state and d the dimension of the Hilbert space, is often regarded as a classically correlated state for small probability p because of its separability proved by Braunstein *et al.* [10]. The state ρ_{ps} can be regarded as a state possessing quantumness in correlation if we choose another paradigm than the separability paradigm. It is thus a rather conceptual question and is in relation to the following discussion.

Suppose that we have a system consisting of two subsystems (local systems) and cannot eliminate a local and/or

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global superposition by local unitary operations. A system described by $\rho_{\rm ps}$ for any $p \neq 0$ is a typical example. Then, can correlation between those local systems be regarded as classical one? An answer is found in the paper by Oppenheim *et al.* [8] (see also [9]) in which they made use of a class of states having a biproduct eigenbasis for a certain classical-nonclassical separation.

Definition 1. Let us consider a joint system consisting of subsystems A and B with Hilbert space dimensions $d^{[A]}$ and $d^{[B]}$, respectively. A complete orthonormal basis (CONB) consisting of eigenvectors, $\{|e_i^{[A,B]}\rangle\}_{i=1}^{d^{[A]}d^{[B]}}$, of a density matrix of the system is a biproduct eigenbasis if and only if it is given by the direct product as $\{|e_i^{[A,B]}\rangle\}_i = \{|e_j^{[A]}\rangle\}_{j=1}^{d^{[A]}} \times \{|e_k^{[B]}\rangle\}_{k=1}^{d^{[B]}}$ where $\{|e_j^{[A]}\rangle\}_j$ and $\{|e_k^{[B]}\rangle\}_k$ are eigenbases of individual subsystems.

If a density matrix $\rho^{[A,B]}$ has a biproduct (BP) eigenbasis $\{|e_i^{[A,B]}\rangle\}_i = \{|e_j^{[A]}\rangle\}_{j=1}^{d^{[A]}} \times \{|e_k^{[B]}\rangle\}_{k=1}^{d^{[B]}}$, it can be written in the form

$$\rho_{\rm BP}^{[A,B]} = \sum_{jk} c_{jk} |e_j^{[A]}\rangle\langle e_j^{[A]}| \otimes |e_k^{[B]}\rangle\langle e_k^{[B]}| \tag{4}$$

with coefficients $0 \leq c_{jk} \leq 1$ $(\Sigma_{jk} c_{jk} = 1)$. A state represented by this density matrix is called a properly classically correlated state or, shortly, classically correlated state [9]. It is also called a classical-classical state [18]. The state that cannot be represented by a density matrix in the above form is called a nonclassically correlated state.

It is trivial to extend the above definition to a multipartite system. A density matrix having a (fully) product (FP) eigenbasis is written in the form

$$\rho_{\rm FP}^{[1,\ldots,m]} = \sum_{j,\ldots,x=1,\ldots,1}^{d^{[1]},\ldots,d^{[m]}} c_{j,\ldots,x} |e_j^{[1]}\rangle \cdots |e_x^{[m]}\rangle\langle e_j^{[1]}| \cdots \langle e_x^{[m]}|, \tag{5}$$

with the local CONBs $\{|e_j^{[1]}\rangle\}_{j=1}^{d^{[1]}}, \ldots, \{|e_x^{[m]}\rangle\}_{x=1}^{d^{[m]}}$ ($d^{[\cdot]}$ is the dimension of the Hilbert space of a local system) and the coefficients $0 \leq c_{j,\ldots,x} \leq 1$ $(\Sigma_{j,\ldots,x} c_{j,\ldots,x} = 1)$ in use. This definition separates the class of density matrices having a product eigenbasis (this is a nonconvex set) from the class of density matrices having no product eigenbasis, as illustrated in Fig. 1. The latter class is characterized by nonvanishing superposition (namely, nonvanishing off-diagonal elements) under local unitary transformations.

In relation to the discussion on biproduct eigenbasis, Groisman *et al.* [11] recently introduced a measure of quantumness given by

$$\mathcal{Q}(\rho^{[A,B]}) = \min_{\rho_{\rm BP}^{[A,B]}} F(\rho^{[A,B]}, \rho_{\rm BP}^{[A,B]}),$$

where the minimum is taken over all density matrices $\rho_{\rm BP}^{[A,B]}$ having a biproduct eigenbasis; F is any properly defined distance function, such as the relative entropy function. $\mathcal{Q}(\rho^{[A,B]})$ must be invariant under local unitary operations and must be zero for $\rho^{[A,B]}$ having a biproduct eigenbasis. They also suggested to use a special density matrix $\rho_{\rm Sch}^{[A,B]}$ (they called it Schmidt state) to define an easily computable

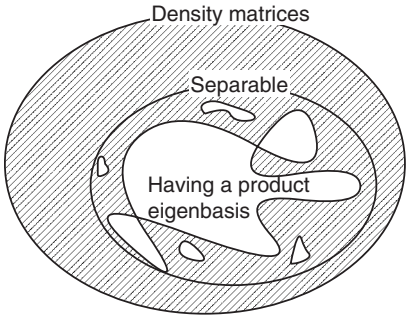

FIG. 1. Class of density matrices having a product eigenbasis (white regions in the class of separable density matrices) and the class of density matrices having no product eigenbasis (shaded region).

measure of quantumness $F(\rho^{[A,B]}, \rho_{\rm Sch}^{[A,B]})$. The density matrix $\rho_{\rm Sch}^{[A,B]}$ is created by keeping only diagonal elements of $\rho^{[A,B]}$ under the special basis diagonalizing $\mathrm{Tr}_B\, \rho^{[A,B]} \otimes \mathrm{Tr}_A\, \rho^{[A,B]}$.

Indeed, it is a natural statement that a measure M of nonclassical correlation should satisfy the following conditions.

(i) $M=0$ if a system is described by a density matrix having a product eigenbasis (i.e., $M=0$ is a necessary but not sufficient condition for a state to have a product eigenbasis).

(ii) M is invariant under local unitary operations.

These conditions are considered to be prerequisite hereafter. In addition, one may test if a measure possesses either of (iii) full additivity, (iv) weak additivity, (v) subadditivity, etc. in the family of additivity properties. These properties are defined in the following way. They are desirable properties for measures of multipartite correlation and not exactly based on the additivity concept for bipartite correlation often seen for entanglement measures [12]. Let us denote a measure of m-partite nonclassical correlation by $M_m(\sigma)$ where σ is the density matrix of an m-partite quantum system. First, the measure is fully additive if and only if

$$M_{m_1 \times m_2}(\sigma_1 \otimes \sigma_2) = M_{m_1}(\sigma_1) + M_{m_2}(\sigma_2)$$

with σ_1 the density matrix of an m_1-partite system and σ_2 the density matrix of an m_2-partite system. Second, the measure possesses weak additivity if and only if

$$M_{m^n}(\sigma^{\otimes n}) = n M_m(\sigma).$$

Third, the measure possesses subadditivity if and only if

$$M_{m_1 \times m_2}(\sigma_1 \otimes \sigma_2) \leq M_{m_1}(\sigma_1) + M_{m_2}(\sigma_2).$$

In this paper, we introduce two measures of nonclassical correlation for a general multipartite system and numerically evaluate them for several examples. One of them, defined in Sec. II, is similar to but different from the measure proposed by Groisman *et al.* and the other one, defined in Sec. III, is totally independent. This paper is organized as follows. In Sec. II, we quantify a nonclassical correlation by a measure defined as the minimum uncertainty with respect to a joint system after we collect outcomes of particular local measurements. This measure satisfies the full additivity condition. The other measure will be introduced in Sec. III, which is defined in the following way: we consider the minimum dis-

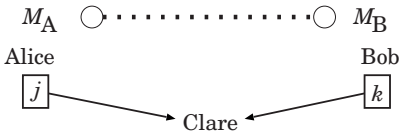

FIG. 2. System for which nonclassical correlation between Alice's part and Bob's part of a quantum state is discussed. A measure of nonclassical correlation is defined as a minimum uncertainty for Clare about the state after receiving their reports.

tance between a genuine set and a mimic set of eigenvalues of a reduced density matrix of a local system on the basis of an artificial game in which one creates mimic eigenvalues of a reduced density matrix of a local system from eigenvalues of a density matrix of a global system. The measure is defined by taking the maximum of this minimum distance over all local systems. It satisfies the subadditivity condition and a slightly stronger condition. We perform numerical computation of the two introduced measures for several examples in Sec. IV and compare them with negativity that is a common entanglement measure based on the separability paradigm. A discussion on definitions of nonclassical correlation is given in Sec. V. Section VI summarizes the results of this paper.

II. MEASURE OF NONCLASSICAL CORRELATION I

We introduce the first of two measures of nonclassical correlation in this section. It is based on the paradigm in which a system described by a density matrix having a product eigenbasis is considered to possess only a classical correlation; in contrast, a system described by a density matrix having no product eigenbasis is considered to possess a nonclassical correlation. We have seen the form of a density matrix having a product eigenbasis for a bipartite system in Eq. (4) and that for a multipartite system in Eq. (5). We will begin with the bipartite case.

A. Bipartite case

To quantify nonclassical correlation between distant subsystems of a bipartite system, we consider the situation illustrated in Fig. 2. Let us introduce Alice, Bob, and Clare. Alice and Bob have subsystems of a system and they are distant from each other. They can send reports to Clare. Alice (Bob) can choose a complete orthonormal basis of her (his) subsystem (basis $\{|e_j^{[A]}\rangle\}_j$ for Alice and basis $\{|e_k^{[B]}\rangle\}_k$ for Bob) for local (projective) measurements. Suppose that Alice and Bob use the observables $M_A=\Sigma_j j|e_j^{[A]}\rangle\langle e_j^{[A]}|$ and $M_B=\Sigma_k k|e_k^{[B]}\rangle\langle e_k^{[B]}|$, respectively, and report the outcomes j and k to Clare. The probability that Clare receives j from Alice and k from Bob is $p_{jk}=\langle e_j^{[A]}|\langle e_k^{[B]}|\rho^{[A,B]}|e_j^{[A]}\rangle|e_k^{[B]}\rangle$. The same process is performed for many copies of the same system shared by Alice and Bob without changing the bases initially chosen. Then, minimum uncertainty (over all possible initial choices of local bases) that Clare has about $\rho^{[A,B]}$ after receiving their reports is

$$D(\rho^{[A,B]}) = \min_{\text{local bases}}\left(-\sum_{jk} p_{jk}\log_2 p_{jk}\right) - S_{\text{vN}}(\rho^{[A,B]}), \quad (6)$$

where $S_{\text{vN}}(\rho^{[A,B]})$ is the von Neumann entropy. We employ this quantity as a measure of nonclassical correlation. It is obvious that $D(\rho^{[A,B]})=0$ for a density matrix with a biproduct eigenbasis. Otherwise, there is a possibility that $D(\rho^{[A,B]})>0$. Thus a bipartite separable density matrix having no product eigenbasis (as well as a bipartite inseparable density matrix) possibly has a nonlocal correlation that may be quantified by $D(\rho^{[A,B]})$. A typical example is a density matrix shown in Eq. (3) for $0<p\leq 1$. For this density matrix, $D(\rho^{[A,B]})>0$ holds. In addition, we should stress that $D(\rho^{[A,B]})$ is invariant under local unitary operations; this is clear from its definition in which we search over all local bases to obtain the minimum.

One may find that the above process involving Alice, Bob, and Clare can be reconstructed in terms of CLOCC operations [8,9] to define the same quantity: Consider the minimum discrepancy between the von Neumann entropy of the original state $\rho^{[A,B]}$ and that of the state Clare can achieve by merging states received from Alice and Bob only after Alice and Bob locally use dephasing operations $\Lambda_D(\rho^{[X]})=\Sigma_{i=1}^{d^{[X]}}\langle e_i|\rho^{[X]}|e_i\rangle|e_i\rangle\langle e_i|$ where X is subsystem A or B, $d^{[X]}$ is the dimension of its Hilbert space, and $\{|e\rangle_i\}_i$ is the CONB of her (his) choice. The minimum is taken over all choices of local CONBs. Then the discrepancy is equal to the measure $D(\rho^{[A,B]})$. In this way, $D(\rho^{[A,B]})$ can be related to the CLOCC protocol that was the base protocol for quantum deficit [9]. We should note that, in general, $D(\rho^{[A,B]})$ is not equal to quantum deficit; this is clear by comparing Eq. (11) of Ref. [9] with the above definition. The measure $D(\rho^{[A,B]})$ is equal to the quantum deficit in the case where the zero-way CLOCC protocol [9] is considered [20].

In addition, the measure $D(\rho^{[A,B]})$ is similar to but not included in the measure of quantumness defined by Groisman *et al.* [11] in the sense that we search over all local bases while their measure is defined by taking a minimum over all classical states (hence, over all possible combinations of eigenvalues and local bases).

B. Multipartite case

It is straightforward to extend the definition, Eq. (6), of D to that for general multipartite systems. Let us consider a density matrix $\rho^{[1,\ldots,m]}$ of an m-partite system. Consider local CONBs $\{|e_j^{[1]}\rangle\}_j,\ldots,\{|e_x^{[m]}\rangle\}_x$. Then, a measure of nonclassical correlation is given by

$$D(\rho^{[1,\ldots,m]}) = \min_{\text{local bases}}\left(-\sum_{j,\ldots,x} p_{j,\ldots,x}\log_2 p_{j,\ldots,x}\right) - S_{\text{vN}}(\rho^{[1,\ldots,m]}) \quad (7)$$

with

$$p_{j,\ldots,x} = \langle e_j^{[1]}|\langle e_k^{[2]}|\cdots\langle e_x^{[m]}|\rho^{[1,\ldots,m]}|e_j^{[1]}\rangle|e_k^{[2]}\rangle\cdots|e_x^{[m]}\rangle.$$

An interpretation of this measure from an operational viewpoint is possible. Let L_1 (L_2) be the average code length of an optimal classical data compression, focusing on only diagonal elements, acting on a register of qudits subsequent to local (global) unitary operations. Then the minimum discrepancy between L_1 and L_2 leads to the definition of the mea-

sure. The value of $D(\rho^{[1,\ldots,m]})$ is zero if $\rho^{[1,\ldots,m]}$ has a (fully) product eigenbasis. In addition, $D(\rho^{[1,\ldots,m]})$ is invariant under local unitary operations as is clear from its definition in which the minimum is taken over all local bases. Furthermore, we can prove that it is fully additive.

The proof of full additivity uses only a general property of an entropy function in a particular form. Suppose that there is a system consisting of subsystems X and Y described by the reduced density matrices σ_X and σ_Y, respectively. Consider an entropy function E of the product density matrix $\sigma_X \otimes \sigma_Y$ and CONBs $\{|x\rangle\}$, $\{|y\rangle\}$ of the reduced density matrices, written as

$$E(\{|x\rangle\},\{|y\rangle\},\sigma_X \otimes \sigma_Y) = -\sum_{|x\rangle|y\rangle} \langle x|\langle y|\sigma_X \otimes \sigma_Y|x\rangle|y\rangle \times \log_2\langle x|\langle y|\sigma_X \otimes \sigma_Y|x\rangle|y\rangle.$$

We rewrite it as

$$E(\{|x\rangle\},\{|y\rangle\},\sigma_X \otimes \sigma_Y) = E(\{|x\rangle\},\sigma_X) + E(\{|y\rangle\},\sigma_Y)$$

with entropies of subsystems

$$E(\{|x\rangle\},\sigma_X) = -\sum_{|x\rangle} \langle x|\sigma_X|x\rangle \log_2\langle x|\sigma_X|x\rangle,$$

$$E(\{|y\rangle\},\sigma_Y) = -\sum_{|y\rangle} \langle y|\sigma_Y|y\rangle \log_2\langle y|\sigma_Y|y\rangle.$$

It is obvious that $E(\{|x\rangle\},\{|y\rangle\},\sigma_X\otimes\sigma_Y)$ is minimized if and only if $E(\{|x\rangle\},\sigma_X)$ and $E(\{|y\rangle\},\sigma_Y)$ are individually minimized. We conclude that $D(\sigma\otimes\tau)=D(\sigma)+D(\tau)$ holds for density matrices σ and τ accordingly. Thus the measure D satisfies the full additivity condition.

C. Numerical method to estimate D

A pure random search of local bases is a practical method to compute an estimated value of $D(\rho^{[1,\ldots,m]})$, defined by Eq. (7), for a small size of a multipartite density matrix $\rho^{[1,\ldots,m]}$. Let us consider the bipartite case for clarity, where the density matrix is $\rho^{[A,B]}$. We employ an *ad hoc* random search algorithm introduced in Fig. 3 in which a Gram-Schmidt process (see, e.g., p. 108 of Ref. [13]) is utilized. In addition, it is needless to say that a similar algorithm can be used to estimate a value of $D(\rho^{[1,\ldots,m]})$ for a general m numerically.

The random CONB generation is equivalent to a generation of random unitary matrices acting on fixed CONBs $\{|j\rangle\}$ and $\{|k\rangle\}$. This is clear from the fact that for any local unitary operations U_A and U_B, we have $U_A|j\rangle=|e_j\rangle$ and $U_B|k\rangle=|e_k\rangle$. The probabilities p_{jk} are the diagonal elements of $U_A^\dagger \otimes U_B^\dagger \rho^{[A,B]} U_A \otimes U_B$. (It is now trivial to generalize this to the multipartite case.) Thus random matrix theories [14] will be hopefully used to refine the algorithm. The present algorithm is still fast enough to estimate a value of $D(\rho^{[1,\ldots,m]})$ for a small number of qubits in a reasonable time. We show numerical results in Sec. IV.

III. MEASURE OF NONCLASSICAL CORRELATION II

The previous measure, D, is defined as a minimum uncertainty about a global system after collecting outcomes of local measurements. Although the definition itself looks quite reasonable, we have to stress the difficulty to find the value of D because we need to try all possible product eigenbases to find the minimum, in principle. Strictly speaking, we need to try an infinite number of product eigenbases, but we would rather find an estimate value by using a random search as we have seen. In this section, we will introduce the second measure derived from an artificial game to mimic a set of eigenvalues of a reduced density matrix of a local system by using a set of eigenvalues of a density matrix of a global system. The advantage of this measure is that it can be calculated deterministically within a finite time. The disadvantage is its artificial definition, which may be insignificant taking account of its advantage.

Algorithm:

Continue to run the following trial:

Trial:

I. Generate a local CONB $\{|e_j^{[A]}\rangle\}_j \times \{|e_k^{[B]}\rangle\}_k$ randomly by using the subroutine (written below) for both subsystems, A and B.

II. Calculate the value of $-\sum_{jk} p_{jk} \log_2 p_{jk}$.

A local CONB that minimizes $-\sum_{jk} p_{jk} \log_2 p_{jk}$ is chosen after many trials. The number of trials is typically 10^5 or 10^6 owing to a limitation of computational resource. The measure $D(\rho^{[A,B]})$ is evaluated with the chosen local CONB.

Subroutine (random generation of a CONB)—Gram-Schmidt Process :

i. Generate a normalized (complex) d-dimensional random vector $|v_1\rangle$ (here, d is the dimension of a reduced density matrix of the subsystem A or B for which we generate a basis).

ii. For $k=2$ to $k=d$, do

1. Generate a (complex) d-dimensional random vector $|v_k\rangle$.

2. For $l=1$ to $l=k-1$, do
$|v_k\rangle \longleftarrow |v_k\rangle - \langle v_l|v_k\rangle|v_l\rangle$.
done

3. Normalize $|v_k\rangle$.

4. if $|v_k\rangle$ is zero then goto i.

done

iii. Output $\{|v_k\rangle\}_{k=1}^d$.

FIG. 3. Algorithm to compute an estimated value of $D(\rho^{[A,B]})$. See the text for the symbols used herein.

A. Bipartite case

Consider an artificial game pertaining to Alice, Bob, and Clare. A system consisting of two parts, A (Alice's part) and B (Bob's part), is described by the density matrix $\rho^{[A,B]}$. Alice and Bob cannot measure the system at all. Clare knows the set of all eigenvalues $\{e_{jk}\}_{jk=11}^{d^{[A]}d^{[B]}}$ where $d^{[A]}$ and $d^{[B]}$ are the dimension of the Hilbert space of Alice's part and that of Bob's part, respectively. Alice (Bob) wants to know the eigenvalues of her (his) part. This setup is illustrated in Fig. 4. Let us concentrate on Alice's strategy. She asks Clare to send all the eigenvalues. Alice will partition $d^{[A]}\times d^{[B]}$ eigenvalues received from Clare into $d^{[A]}$ sets

$$\{a_{1,1},\ldots,a_{1,d^{[B]}}\},\ldots,\{a_{d^{[A]},1},\ldots,a_{d^{[A]},d^{[B]}}\}.$$

To mimic an eigenvalue of the reduced density matrix of A, she sums up all elements of each set to make a set of $d^{[A]}$ mimic eigenvalues:

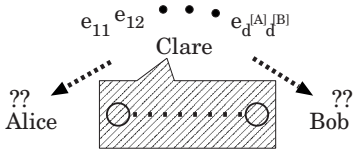

FIG. 4. Illustration of the artificial game in which Alice (Bob) tries to find the eigenvalues of the reduced density matrix of her (his) part by the strategy described in the text. Clare knows the eigenvalues of the total system.

$$\{\tilde{e}_1, \ldots, \tilde{e}_j, \ldots, \tilde{e}_{d^{[A]}}\},$$

where each element is calculated as

$$\tilde{e}_j = \sum_{k=1}^{d^{[B]}} a_{j,k}.$$

The number of possible combinations to partition $d^{[A]} \times d^{[B]}$ eigenvalues into $d^{[A]}$ sets is

$$\binom{d^{[A]} \times d^{[B]}}{d^{[B]}} \binom{(d^{[A]}-1) \times d^{[B]}}{d^{[B]}} \cdots \binom{d^{[B]}}{d^{[B]}}.$$

Let us write the genuine eigenvalues of the reduced density matrix of A by $\{e_j\}_{j=1}^{d^{[A]}}$. The minimum uncertainty for Alice with respect to the set of these eigenvalues in the artificial game may be given by

$$F_A(\rho^{[A,B]}) = \min_{\text{partitionings}} \left| \sum_j (\tilde{e}_j \log_2 \tilde{e}_j - e_j \log_2 e_j) \right|. \quad (8)$$

We may also consider the minimum uncertainty $F_B(\rho^{[A,B]})$ for Bob with respect to the set of eigenvalues for his local part in the same game. The larger one of their minimum uncertainties is then given by

$$G(\rho^{[A,B]}) = \max\{F_A(\rho^{[A,B]}), F_B(\rho^{[A,B]})\}. \quad (9)$$

This function can be used as a measure of nonclassical correlation since $G(\rho^{[A,B]})=0$ if $\rho^{[A,B]}$ has a product eigenbasis. In addition, $G(\rho^{[A,B]})$ is invariant under local unitary operations according to the definition since local unitary operations preserve the eigenvalues of reduced density matrices of individual components and those of the density matrix of the total system.

B. Multipartite case

An extension of the measure G to a multipartite case is straightforward. Let us consider an artificial game to find out eigenvalues of the reduced density matrix of a subpart from eigenvalues of the density matrix of the total system. Suppose that Kate has the kth part of an m-partite quantum system. Let the dimension of the Hilbert space of the kth part be $d^{[k]}$ and that of the Hilbert space of the total system be d_{tot}. She wants to know the eigenvalues $\{e_j^{[k]}\}_{j=1}^{d^{[k]}}$ of the reduced density matrix of the kth part. Kate receives d_{tot} eigenvalues from Tony who knows the eigenvalues of the total system. Kate partitions them into $d^{[k]}$ sets. Summing up elements in individual sets, she has $d^{[k]}$ mimic eigenvalues $\{\tilde{e}_j^{[k]}\}$. Thus a measure of nonclassical correlation for Kate can be

$$F_k(\rho^{[1,\ldots,m]}) = \min_{\text{partitionings}} \left| \sum_{j=1}^{d^{[k]}} (\tilde{e}_j^{[k]} \log_2 \tilde{e}_j^{[k]} - e_j^{[k]} \log_2 e_j^{[k]}) \right|.$$

We may take the maximum over k to have the measure

$$G(\rho^{[1,\ldots,m]}) = \max_k F_k(\rho^{[1,\ldots,m]}). \quad (10)$$

This is equal to zero if $\rho^{[1,\ldots,m]}$ has a (fully) product eigenbasis. In addition, it is invariant under local unitary operations as is clear from the fact that these operations preserve the eigenvalues of a density matrix of the total system and those of the reduced density matrices of individual components. The measure does not satisfy the (full or weak) additivity condition because the number of possible choices of partitioning eigenvalues grows rapidly as the system size grows. We can, however, prove its subadditivity.

The proof for its subadditivity is accomplished in the following way. Suppose the following story: there is a density matrix σ of an m_σ-partite system having the set of eigenvalues, $\{\alpha_a\}_{a=1}^{d_\sigma}$ (here, d_σ is the dimension of the Hilbert space of the system). With this set of eigenvalues, Kate has created a set of $d^{[k]}$ mimic eigenvalues of the reduced density matrix of the kth part. There is another density matrix τ of an m_τ-partite system having the set of the eigenvalues, $\{\beta_b\}_{b=1}^{d_\tau}$. With this set of eigenvalues, Leo has created a set of $d^{[l]}$ mimic eigenvalues of the reduced density matrix of the lth part. Then a new game starts with a joint state $\sigma \otimes \tau$. The game for Kate (Leo) is to make a set of mimic eigenvalues for her (his) part by partitioning the product set of eigenvalues $\{\alpha_a\} \times \{\beta_b\}$ of $\sigma \otimes \tau$. Then, Kate (Leo) may make the same set of mimic eigenvalues as before because she (he) can make the set $\{\alpha_a\}$ ($\{\beta_b\}$) first by partitioning $\{\alpha_a\} \times \{\beta_b\}$. Thus $F_k(\sigma \otimes \tau) \le F_k(\sigma)$ and $F_l(\sigma \otimes \tau) \le F_l(\tau)$ hold. Hence the next inequalities are satisfied.

$$G(\sigma \otimes \tau) \le \max\{G(\sigma), G(\tau)\} \le G(\sigma) + G(\tau).$$

In this way, the subadditivity $G(\sigma \otimes \tau) \le G(\sigma) + G(\tau)$ has been proved. We found that a slightly stronger condition $G(\sigma \otimes \tau) \le \max\{G(\sigma), G(\tau)\}$ is satisfied as a result. This property can be named *submaximizability*.

IV. NUMERICAL RESULTS

A. Examples for bipartite cases

We compare the measures D and G with the negativity [15,16] $\mathcal{N}(\rho^{[A,B]}) = \frac{\|(I \otimes \Lambda_T)\rho^{[A,B]}\| - 1}{2}$ (here the map Λ_T is the transposition map acting on B).

The first example is the two-qubit pseudopure state

$$\rho_{\text{ps}} = p|\psi\rangle\langle\psi| + (1-p)1/4$$

with $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. We used a pure (numerical) random search of local bases, as introduced in Sec. II C, to estimate a value of $D(\rho_{\text{ps}})$. The number of trials of local bases is 4.0×10^4 for each data point (namely, for each p in this example) of $D(\rho_{\text{ps}})$. (The number of trials is 4.0×10^4 for the other examples of two-qubit cases and is 4.0×10^5 for the examples involving an 8×8 density matrix.) Computation of

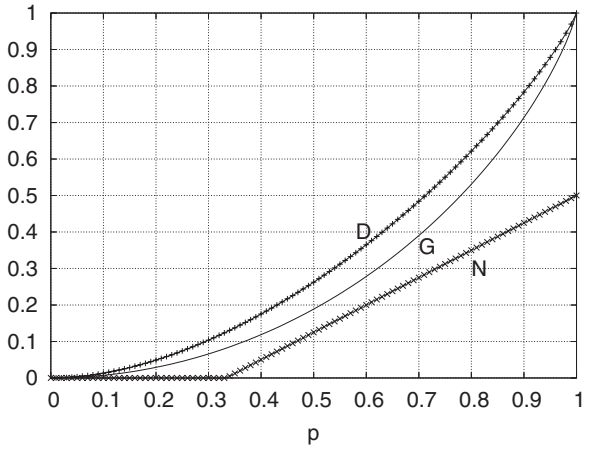

FIG. 5. Plots of $D(\rho_{ps})$, $G(\rho_{ps})$, and $\mathcal{N}(\rho_{ps})$ against p.

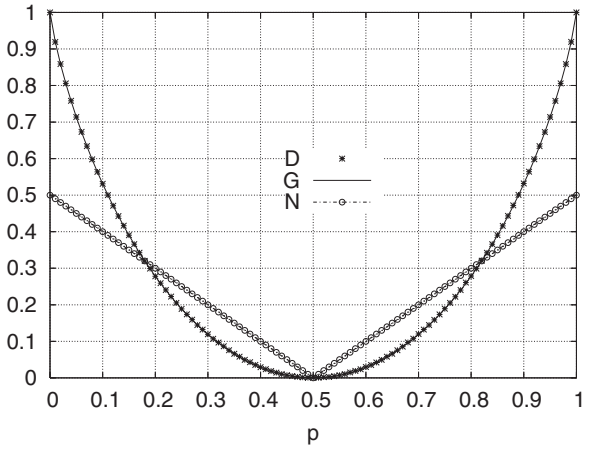

FIG. 6. Plots of $D(\rho_b)$, $G(\rho_b)$, and $\mathcal{N}(\rho_b)$ against p. As we see in the text, $D(\rho_b)=G(\rho_b)$ in this example.

a value of $G(\rho_{ps})$ is, in contrast, performed analytically. The eigenvalues of ρ_{ps} are $(1+3p)/4$ and $(1-p)/4$ with the multiplicity three for the latter one. The eigenvalue of the reduced density matrix of a subpart is 1/2 with the multiplicity two. Thus we have

$$G(\rho_{ps}) = 1 - H\left(\frac{1+p}{2}\right),$$

where

$$H(x) = -x \log_2 x - (1-x)\log_2(1-x)$$

is the binary entropy function $(0 \le x \le 1)$.

Figure 5 shows the plots of $D(\rho_{ps})$, $G(\rho_{ps})$, and $\mathcal{N}(\rho_{ps})$ against p $(0 \le p \le 1)$. The measures D and G reflect nonclassical correlation for $\forall p$ except for $p=0$.

The next example is a mixture of Bell basis states, represented by the density matrix

$$\rho_b = p|b_1\rangle\langle b_1| + (1-p)|b_2\rangle\langle b_2|$$

with $|b_1\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and $|b_2\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. This density matrix has a product eigenbasis if $p=0.5$: $\rho_b(p=0.5) = \frac{1}{2}(|+\rangle|+\rangle\langle+|\langle+| + |-\rangle|-\rangle\langle-|\langle-|)$ with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

With the same basis search method as the previous example, we can estimate the values of $D(\rho_b)$. For this example, however, it can be analytically found: The von Neumann entropy of ρ_b is given by $H(p)$ and the remaining term of Eq. (6) becomes 1 irrespective of the choice of a local basis. Thus

$$D(\rho_b) = 1 - H(p).$$

In addition, analytical computation of $G(\rho_b)$ is easy: the eigenvalues of ρ_b are p, $1-p$, and 0 with the multiplicity two. The eigenvalue of the reduced density matrix of a subpart is 1/2 with the multiplicity two. Thus taking the minimum over partitionings of the eigenvalues of ρ_b, we have

$$G(\rho_b) = 1 - H(p).$$

We find that $D(\rho_b)=G(\rho_b)$ for this particular example.

Figure 6 shows the plots of $D(\rho_b)$, $G(\rho_b)$, and $\mathcal{N}(\rho_b)$ against p $(0 \le p \le 1)$. All of these measures vanish at $p=0.5$ and have positive values for $p \neq 0.5$. Thus the difference among measures is not significant for ρ_b.

The third example is the density matrix (this is also of a 2×2 system):

$$\sigma = \begin{pmatrix} 1/2-p & 0 & 0 & 0 \\ 0 & p & p & 0 \\ 0 & p & p & 0 \\ 0 & 0 & 0 & 1/2-p \end{pmatrix} \tag{11}$$

with $0 \le p \le 1/2$. This may be seen as a mixture of pure states $|00\rangle$, $|11\rangle$, and $(|01\rangle + |10\rangle)/\sqrt{2}$ with certain weights. It is separable for $p \le 1/4$ because the eigenvalues of $(I \otimes \Lambda_T)\sigma$ are $1/2-2p$, p (with the multiplicity two), and 1/2. $D(\sigma)$ is, in contrast, nonzero unless $p=0$ as shown in Fig. 7. In the figure, we also plot

$$G(\sigma) = \min\left\{1 - H\left(\frac{1}{2}+p\right), 1 - H(2p)\right\}.$$

This is derived from the following values: the eigenvalues of σ are 0, $1/2-p$ (with the multiplicity two), and $2p$; the eigenvalue of $\mathrm{Tr}_B\,\sigma$ and that of $\mathrm{Tr}_A\,\sigma$ are both 1/2 with the multiplicity two. Interestingly, the shape of the curve of $G(\sigma)$

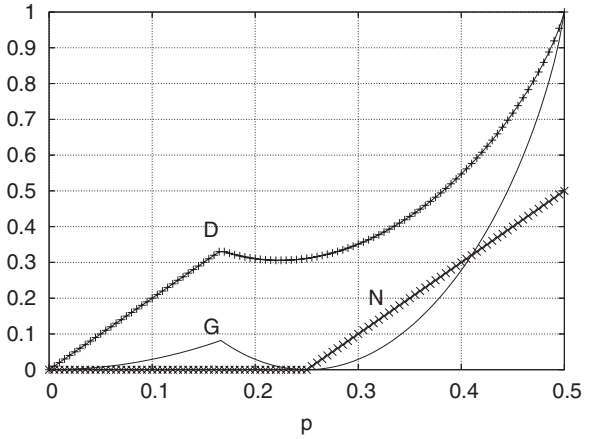

FIG. 7. Plots of $D(\sigma)$, $G(\sigma)$, and $\mathcal{N}(\sigma)$ against p.

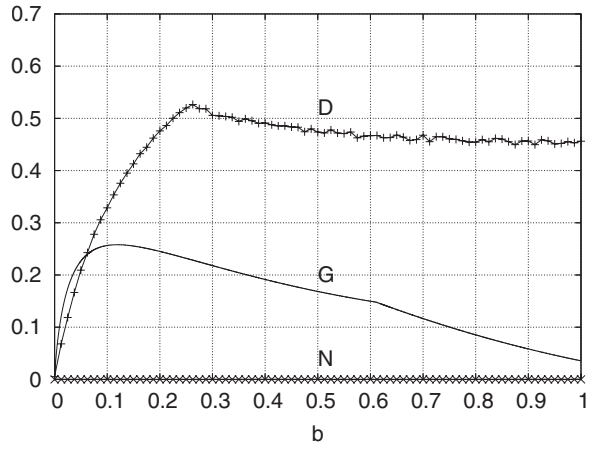

FIG. 8. Plots of $D(\sigma_b^{[A,B]})$, $G(\sigma_b^{[A,B]})$, and $\mathcal{N}(\sigma_b^{[A,B]})$ against b. Note that $\mathcal{N}(\sigma_b^{[A,B]})$ is zero.

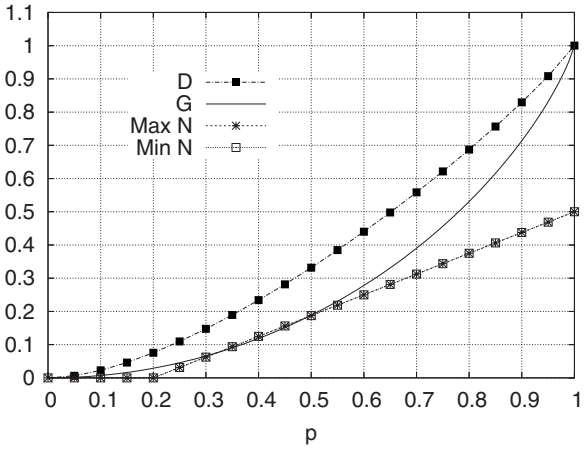

FIG. 9. Plots of $D(\rho_{\mathrm{PGHZ}})$ and $G(\rho_{\mathrm{PGHZ}})$ in comparison to those of the minimum and the maximum of the negativity for ρ_{PGHZ} over all bipartite splittings (as functions of p).

is similar to that of $D(\sigma)$. One drawback in using $G(\sigma)$ is that it vanishes at $p=0.25$ although σ has no product eigenbasis for this value.

As the final example for the bipartite case, we consider the density matrix of a 2×4 system originally introduced by Horodecki [17]:

$$\sigma_b^{[A,B]} = \frac{1}{7b+1}\begin{pmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+b}{2} & 0 & 0 & \frac{\sqrt{1-b^2}}{2} \\ b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & \frac{\sqrt{1-b^2}}{2} & 0 & 0 & \frac{1+b}{2} \end{pmatrix} \tag{12}$$

$(0\le b\le 1)$. This is known to be bound entangled for $0<b<1$, i.e., it is positive after partial transposition although it is inseparable for $0<b<1$. Hence $\mathcal{N}(\sigma_b^{[A,B]})=0$. We find, in contrast, that $D(\sigma_b^{[A,B]})$ and $G(\sigma_b^{[A,B]})$ are nonzero for $b>0$ as shown in Fig. 8. The measure $G(\sigma_b^{[A,B]})$ is also plotted in the figure. This is accomplished by using the definition Eq. (9) together with the following values. We have the eigenvalues of $\sigma_b^{[A,B]}$:

$$\frac{2b+1\pm\sqrt{2b^2-2b+1}}{14b+2}, \quad \frac{b}{7b+1},$$

$$\frac{2b}{7b+1} \text{ (multiplicity two)},$$

$$\text{and } 0 \text{ (multiplicity three)};$$

the eigenvalues of $\mathrm{Tr}_B\,\sigma_b^{[A,B]}$: $(3b+1)/(7b+1)$ and $4b/(7b+1)$; and the eigenvalues of $\mathrm{Tr}_A\,\sigma_b^{[A,B]}$: $(3b+1\pm\sqrt{1-b^2})/(14b+2)$ and $2b/(7b+1)$ with the multiplicity two.

B. Tripartite examples

Numerical computation of D and G defined by Eqs. (7) and (10), respectively, is easy also for a tripartite density matrix $\rho^{[1,2,3]}$ of three qubits. We will compare $D(\rho^{[1,2,3]})$, $G(\rho^{[1,2,3]})$, and the minimum and maximum negativities of $\rho^{[1,2,3]}$ over all bipartite splittings.

Consider the pseudo-GHZ (PGHZ) state:

$$\rho_{\mathrm{PGHZ}} = p|\psi_{\mathrm{GHZ}}\rangle\langle\psi_{\mathrm{GHZ}}| + (1-p)\mathbb{1}/8$$

with $|\psi_{\mathrm{GHZ}}\rangle=(|000\rangle+|111\rangle)/\sqrt{2}$, the Greenberger-Horne-Zeilinger (GHZ) state. For this state, it is possible to find $G(\rho_{\mathrm{PGHZ}})$ analytically: The eigenvalues of ρ_{PGHZ} are $(1+7p)/8$ and $(1-p)/8$ with the multiplicity seven for the latter one; the reduced density matrix of any subpart has the eigenvalue 1/2 with the multiplicity two. Thus $G(\rho_{\mathrm{PGHZ}})=1-H[(1+p)/2]$. We can easily find that this equation holds for any number of qubits (larger than two) for the PGHZ state.

Figure 9 shows the plots of $D(\rho_{\mathrm{PGHZ}})$, $G(\rho_{\mathrm{PGHZ}})$, and the minimum and maximum of the negativity for ρ_{PGHZ} over all bipartite splittings against p $(0\le p\le 1)$. D and G reflect nonclassical correlation for $\forall p$ except for $p=0$.

The density matrix of the final example for a 2×4 bipartite case can be also interpreted as a density matrix of a $2\times2\times2$ tripartite case. Let us consider the density matrix $\sigma_b^{[1,2,3]}$ of three qubits given by the matrix of Eq. (12). Figure 10 shows the plot of $D(\sigma_b^{[1,2,3]})$, the plot of $G(\sigma_b^{[1,2,3]})$, and the plots of the maximum and minimum negativities over all bipartite splittings. The computation of $G(\sigma_b^{[1,2,3]})$ is performed by using the eigenvalues of $\sigma_b^{[1,2,3]}$ (found in the previous subsection) and the sets of eigenvalues of the reduced density matrices of subsystems k ($k=1,2,3$ and the following sets are given in this order): $\{(4b)/(7b+1),(3b+1)/(7b+1)\}$, $\{1/2,1/2\}$, and $\{1/2,1/2\}$. It is found that $D(\sigma_b^{[1,2,3]})$ behaves similarly to the maximum negativity over all bipartite splittings although the convergence values are different. The use of $G(\sigma_b^{[1,2,3]})$ seems to be improper for this example

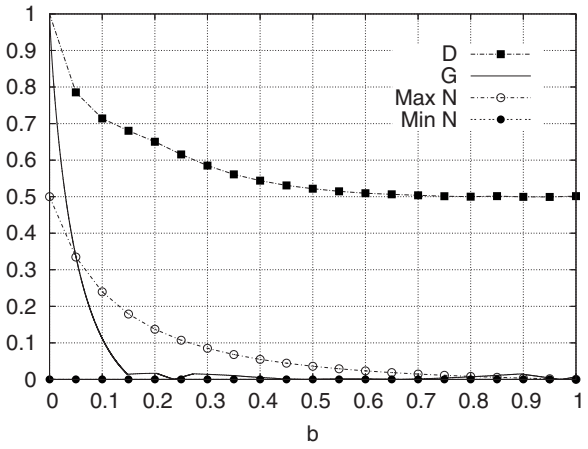

FIG. 10. Plots of $D(\sigma_b^{[1,2,3]})$ and $G(\sigma_b^{[1,2,3]})$ in comparison to those of the minimum and the maximum of the negativity for $\sigma_b^{[1,2,3]}$ over all bipartite splittings (as functions of b).

because its value is very small for $b \gtrsim 0.15$ despite the fact that $\sigma_b^{[1,2,3]}$ has no (fully) product eigenbasis for any b.

V. DISCUSSION

We have considered a nonclassical correlation based on a commonly recognized paradigm other than the separability paradigm. We have introduced two measures of a nonclassical correlation for the paradigm claiming that a multipartite system described by a density matrix having no product eigenbasis possesses a nonclassical correlation.

A motivation to consider a paradigm other than the separability paradigm is connected to a conceptual investigation of the separability paradigm. Let us consider a bipartite system to simplify the discussion. The separability paradigm says that a system described by a bipartite separable density matrix can be prepared remotely when two distant persons (Alice and Bob) receive instructions from a common source. Hence there is no quantumness possessed by the system described by such a density matrix in the context of remote preparation. This assumes that a density matrix is a temporal average of instantaneous density matrices or an ensemble average whose component density matrices are accessible independently because Alice and Bob prepare component states one by one.

Nevertheless, this seems to be a nonmixing process because Alice and Bob have accesses to individual instances or components; this is in contrast to usual processes in ensemble dynamics which are mixing. To make it a mixing process, they should lose their memories about time ordering of instances when the averaging is temporal averaging; in the case of ensemble averaging, they should lose their memories about indices of components. In contrast, Alice and Bob do not have any memory to lose when subparts of a system are distributed to them after a joint preparation of a quantum state. Thus it is questionable to compare the amount and quality of correlation of a remotely prepared state with those of a jointly prepared state because the two contexts are different.

The paradigm that we support in this work assumes that a density matrix having no product eigenbasis possesses nonclassical correlation. A discussion on the state preparation is not involved in its context. It is based on the problem as to whether or not off-diagonal elements of the density matrix of a multipartite system can be completely eliminated by local unitary operations. This paradigm has been widely recognized in the community while it is not considered to be a replacement of the separability paradigm. It is a highly conceptual problem as to which protocol should be a base protocol to think about correlation.

We have studied two measures D and G of nonclassical correlation based on the paradigm. The former is defined as a global uncertainty while the latter is defined as a local uncertainty. These two measures are quite different in their naturalness: the former can be interpreted as the minimum uncertainty for a global observer on the total system after receiving reports on local systems; the latter is based on some quite artificial game and is, of course, unnatural. It is thus unexpected that plotted curves of these measures are sometimes similar to each other as we have seen in numerical results. We may enjoy the advantage to use the measure G, namely its easiness of computation. In contrast to D, which requires a numerical search for computing its value, G can be computed by considering a finite number of eigenvalue partitionings. A drawback is that G is possibly small for a density matrix having many nonzero eigenvalues, as we have seen in Fig. 10 for an example. It is easy to produce mimic eigenvalues of a subsystem close to genuine ones if the number of possible partitionings is large. This drawback is related to the fact that G is not (fully or weakly) additive but subadditive, in contrast to D that is fully additive. A measure with (full or weak) additivity is more reliable to quantify nonclassical correlation as the system size grows.

As we mentioned, there is a similarity between D and the measure defined by Groisman *et al.* [11]. Let us consider a bipartite case. Suppose that we change the definition of D in the way that we choose the basis written as a product of two eigenbases of local systems A and B instead of searching the minimum over all product bases. Then, this redefined measure is included in the distance measures using the discrepancy between the given bipartite density matrix $\rho^{[A,B]}$ and the Schmidt state $\rho_{\mathrm{Sch}}^{[A,B]}$. The measure D is thus close to the measures defined as a discrepancy between $\rho^{[A,B]}$ and some specific density matrix having product eigenbasis (this can be the one that minimizes the distance, or some particular one to simplify the computing process).

Finally, we take a brief look at an ongoing development in measures of nonclassical correlation. One of the measures very recently proposed by Piani *et al.* [18] is designed to quantify nonclassical correlation in the same paradigm as presently employed. It is their measure Δ_{CC} defined in a similar way as that of quantum discord [7]: Δ_{CC} is a discrepancy between the quantum mutual information [19], $I(\rho^{[A,B]})$, calculated for a bipartite density matrix $\rho^{[A,B]}$ and $\min_{\mathcal{M}_A,\mathcal{N}_B} I[(\mathcal{M}_A \otimes \mathcal{N}_B)\rho^{[A,B]}]$ with two measurement maps $\mathcal{M}_A$ and $\mathcal{N}_B$ associated to POVMs. They also showed a straightforward extension of their theory to the multipartite case. An advantage of Δ_{CC} is that it vanishes if and only if $\rho^{[A,B]}$ has a biproduct eigenbasis. A disadvantage is the diffi-

culty of finding the minimum as is similar to the above-discussed disadvantage of D.

There must be a large number of measures to quantify nonclassical correlation for the paradigm; this is reminiscent of the dawn of entanglement measures. It is hoped that the paradigm will be studied extensively to extend another branch of quantum-information science than the branch of the separability paradigm.

VI. SUMMARY

Two measures of nonclassical correlation have been introduced to support the paradigm claiming that a multipartite system described by a density matrix having no product eigenbasis possesses nonclassical correlation. The measure D has been defined as the minimum uncertainty about a joint system after we collect outcomes of particular local measurements. The measure G has been defined in the following way: consider the minimum distance between a set of mimic eigenvalues and a set of genuine eigenvalues of a local system on the basis of an artificial game. The measure is defined by taking the maximum of this minimum distance over all local systems. We have shown that D is fully additive and G is subadditive. Numerical computations of D and G have been performed by using a random search of local bases and a nonprobabilistic search of mimic eigenvalues, respectively.

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[20] The zero-way setting of CLOCC is as follows [9]: there are two parties allowed to communicate under CLOCC only after local complete dephasing possibly subsequent to local unitary operations. In general, the quantum deficit is equal to $\min_{\Lambda\in\mathrm{CLOCC}}[S_{\mathrm{vN}}(\rho'_{\mathrm{Alice}})+S_{\mathrm{vN}}(\rho'_{\mathrm{Bob}})]-S_{\mathrm{vN}}(\rho^{[A,B]})$ with $\rho'=\Lambda(\rho^{[A,B]})$. In case of zero-way CLOCC, the minimum is obtained for the case where Alice or Bob has ρ' totally and the other person has a null system. Then the zero-way case quantum deficit is equal to $\min_{\Lambda\in\text{zero-way CLOCC}} S_{\mathrm{vN}}(\rho')-S_{\mathrm{vN}}(\rho^{[A,B]})$. This is equal to D for the bipartite case.