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Memory effects in spin-chain channels for information transmission

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We investigate the multiple use of a ferromagnetic spin chain for quantum and classical communications without resetting. We find that the memory of the state transmitted during the first use makes the spin chain a qualitatively different quantum channel during the second transmission, for which we find the relevant Kraus operators. We propose a parameter to quantify the amount of memory in the channel and find that it influences the quality of the channel, as reflected through fidelity and entanglement transmissible during the second use. For certain evolution times, the memory allows the channel to exceed the memoryless classical capacity (achieved by separable inputs) and in some cases it can also enhance the quantum capacity.

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Recently, spin chains have been proposed as potential channels for short distance quantum communications (see, for example, Refs. [1,2]). The basic idea is to simply place the state to be transmitted at one end of a spin chain initially in its ground state, allow it to propagate for a specific amount of time, and then receive it at the other end. Generically, while propagating, the information will also inevitably disperse in the chain, and even when a transmission is considered complete (i.e., the state is considered to have been received with some fidelity/probability), some information of the state lingers in the channel. It is thus assumed that a reset of the spin chain to its ground state is made after each transmission [3]. If, on the other hand, a second transmission is performed through the channel without resetting, then the memory of the first transmission should affect the second transmission. A spin chain channel without resetting is thus an interesting physical model of a channel with memory [4].

In this paper, we show that a ferromagnetic spin chain used without resetting is a very different channel than those studied so far in the extensive literature of quantum channels with memory [4–17] . Firstly, the channels usually studied are those with the noise during multiple uses being correlated with each other [5–11], but being independent of the transferred states. In our model, however, the state transmitted during the first use modifies the type of noise during the second use. Secondly, the noise is most often assumed as Markovian correlated [5–10], while this is not the case for us. Thirdly, and most importantly, the channel noises in our case stem from a physical model described by a Hamiltonian. This should stimulate activity in calculating its capacities. To this end, we also introduce a *memory parameter* to quantify the amount of memory. This parameter depends on the distance between the Kraus operators of the second use of the channel with and without memory, so this method can be used to quantify the amount of memory for those channels that admit a description in terms of separate Kraus operators on different uses.

There is also a very important practical issue which motivates our work. The standard way of resetting the chain requires its interaction with a zero temperature environment [18] and this may open up unnecessary avenues for decoherence. Thus one either resets actively by performing a cooling sequence at the chain ends [2] or uses it several times without resetting which automatically raises the question of the effect of memory of one transmission on a subsequent transmission. Multiple usage of a chain of two spins has been studied in [19] to compute the rate of information transmission, but using the swap operators on both spins, a chain of length $N=2$ removes the memory effects. We will compare and contrast our results for the ferromagnetic channel without resetting with some results that have emerged in the recent literature [4–17].

Let us consider a communication system like that of Fig. 1(a) which has a set of sender and receiver registers to store the quantum input and output states respectively and a ferromagnetic open spin chain as a quantum channel. The registers are isolated from the channel, and the Hamiltonian H_{ch} of the chain commutes with S_z (total spin in the z direction) so the number of the excitations in the channel is preserved through the dynamics. Specifically we are going to consider a Heisenberg chain with N spins coupled by the Hamiltonian $H_{ch}=-J\Sigma_{i=1}^{N-1}\sigma_i\cdot\sigma_{i+1}-B\Sigma_{i=1}^{N}\sigma_i^z$ where J and B are the coupling and magnetic field, respectively, and $\sigma_i=(\sigma_i^x,\sigma_i^y,\sigma_i^z)$ is the vector of Pauli operators at site i. To transfer a quantum state from the register S_k (we will restrict our attention to two uses of the channel, so $k=1,2$) to the register R_k we put the state in the channel by applying a swap operator $P_S(k)$which exchanges the state of the register S_k and the first spin of the channel $P_S(k)|\alpha\beta\rangle_{S_k,1}=|\beta\alpha\rangle_{S_k,1}$, then we leave the spin chain to evolve for time τ_k and finally the transmission is completed by applying another swap operator $P_R(k)$ which exchanges the state of site N with the register R_k. The total operator to transfer the quantum state from the sender register S_k to the receiver register R_k is $W(k)=P_R(k)U(\tau_k)P_S(k)$.

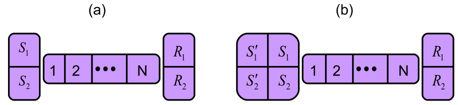

FIG. 1. (Color online) Communication setup includes the sender and receiver registers in both sides of the spin chain. (a) Setup for state transfer and the classical capacity problem. (b) Setup for entanglement distribution and the quantum capacity problem.

The initial state of the system is $\rho(0) = \rho_S(0) \otimes \rho_{ch}(0) \otimes \rho_R(0)$, where $\rho_S(0)$ is an arbitrary initial density matrix of the sender registers, $\rho_{ch}(0) = |\psi_{GS}\rangle\langle\psi_{GS}|$ is the ground state of the chain and $\rho_R(0)$ contains all receiver register spins in the states $|0\rangle$. In the numerical analysis for this paper we have used $N=4$. In fact, for $N=2$ the operations P_S and P_R exclude any memory effects [19] while for $N=3$ the quality of transmission is low [1].

After the first transmission the total state is $\rho(1) = W(1)\rho(0)W^{\dagger}(1)$ which is the case studied in [1]. The received state in the register R_1 is $\rho_{R_1}(1) = \Sigma_{i=1}^{N+1} M_i \rho_{S_1}(0) M_i^{\dagger}$, where M_i's are the following operators:

$$M_m = \begin{pmatrix} 0 & f_{m1}(\tau_1) \\ 0 & 0 \end{pmatrix}, \quad M_N = \begin{pmatrix} 1 & 0 \\ 0 & f_{N1}(\tau_1) \end{pmatrix}, \tag{1}$$

with the index m going from 1 to $N-1$. In the above equation, $f_{m1}(\tau_1) = \langle \mathbf{m} | U(\tau_1) | \mathbf{1} \rangle$, where $|\mathbf{m}\rangle$ represents one flipped spin $|1\rangle$ in site m of the channel and all the other spins in $|0\rangle$. The operator M_{N+1} is a zero matrix which is included here for comparison with the memory case later on. The effect of the operators M_m $(m=1,\ldots,N-1)$ can be combined into one operator, to show that the chain acts as an amplitude damping channel [1]. Except the case of perfect transfer, some information of the first state remains in the state of the channel and the effect of channel is no longer described by the Kraus operators (1). We assume that in the first transmission the state of the sender register S_1 is a general pure state $r|0\rangle + e^{i\phi}\sqrt{1-r^2}|1\rangle$, but it is easy to generalize the results to mixed input states. After the first transmission the state of the channel can be calculated by tracing out the state of the registers from $\rho(1)$. We obtain

$$\rho_{ch}(1) = p_0 |\mathbf{0}\rangle\langle\mathbf{0}| + p_1 |\psi_1\rangle\langle\psi_1|, \tag{2}$$

where

$$p_0 = (1-r^2)|f_{N1}(\tau_1)|^2, \quad p_1 = 1 - p_0,$$

$$|\psi_1\rangle = \frac{1}{\sqrt{p_1}}\left(r|\mathbf{0}\rangle + \sqrt{1-r^2}e^{i\phi}\sum_{n=1}^{N-1} f_{n1}(\tau_1)|\mathbf{n}\rangle \right). \tag{3}$$

The state (2) shows that with probability p_0 the channel is in the state $|\mathbf{0}\rangle$ and acts like an amplitude damping channel but there are some corrections with probability p_1 due to the state $|\psi_1\rangle$. To find the Kraus operators of the channel with the state $|\psi_1\rangle$ one can consider a general density matrix in S_2 where the channel is in the state $|\psi_1\rangle$. By applying the operator $W(2)$ on the state of whole system, the state of the register S_2 is transferred to the register R_2 (albeit with a certain fidelity), so the Kraus operators can be easily derived. We will write down the Kraus operators in a certain way (for simplicity and interpretation), though ours may not be the only way to write the Kraus operators for the channel. Two of the Kraus operators of the channel with the initial state $|\psi_1\rangle$ are as in (1) multiplied by the coefficient $\sqrt{1-r^2/p_1|f_{11}(\tau_1)|^2}$ and the others are some matrices that we shall soon introduce. Thus we can describe the effect of the channel with initial state $|\psi_1\rangle$ as a probabilistic effect, which means that with probability $q = 1 - r^2/p_1|f_{11}(\tau_1)|^2$ the channel affects the inputs like an amplitude damping channel with Kraus operators (1) and with the probability $(1-q)$ the effect of the channel is specified by the following Kraus operators:

$$M'_m = \frac{1}{\sqrt{p_1 - p_1 q}}\begin{pmatrix} A_m\sqrt{1-r^2}e^{i\phi} & f_{m1}(\tau_2)r \\ 0 & B_{mN}\sqrt{1-r^2}e^{i\phi} \end{pmatrix},$$

$$M'_N = \frac{1}{\sqrt{p_1 - p_1 q}}\begin{pmatrix} r & 0 \\ A_N\sqrt{1-r^2}e^{i\phi} & r f_{N1}(\tau_2) \end{pmatrix},$$

$$M'_{N+1} = \frac{1}{\sqrt{p_1 - p_1 q}}\begin{pmatrix} 0 & \sqrt{1-r^2}e^{i\phi}\sqrt{\sum_{k_1k_2}{}' |B_{k_1k_2}|^2} \\ 0 & 0 \end{pmatrix}, \tag{4}$$

where the index m goes from 1 to $N-1$, $\Sigma'_{k_1k_2} = \Sigma_{k_1=1}^{N-1}\Sigma_{k_2=k_1+1}^{N-1}$ and $A_m = \Sigma_{n=2}^{N-1} f_{mn}(\tau_2) f_{n1}(\tau_1)$. $B_{k_1k_2} = \Sigma_{n=2}^{N-1} f_{k_1k_2,Nn}(\tau_2) f_{n1}(\tau_1)$ is the two excitation amplitude transition with $f_{pq,nm} = \langle \mathbf{pq} | e^{-iHt} | \mathbf{nm} \rangle$, and $|\mathbf{nm}\rangle$ means all the spins of the channel are in $|0\rangle$ except the sites n and m. Notice that $B_{k_1k_2}$ includes physical interaction (scattering) between the first and second state.

In order to get a complete description of the channel for the second use we know that with probability p_0 the state of the channel is $|\mathbf{0}\rangle$ (the spin chain is an amplitude damping channel) and with probability $p_1 q$ the state of the chain is $|\psi_1\rangle$ but acts as an amplitude damping channel. Thus with total probability $p_0 + p_1 q$ the spin chain is an amplitude damping channel, otherwise with the probability $p_1(1-q)$ the channel is in the state $|\psi_1\rangle$ and its effect is specified by the Kraus operators (4). Therefore, we have

$$\rho_{R_2}(2) = (p_0 + p_1 q)\xi_{AD}(\rho_{S_2}(0)) + (p_1 - p_1 q)\xi_{Mem}(\rho_{S_2}(0)), \tag{5}$$

where ξ_{AD} is the amplitude damping evolution (1) and ξ_{Mem} is the evolution with Kraus operators (4).

If we consider the memory as a deviation of the channel effect from the memoryless case, then to find a distance between the two evolutions we can consider the distance between the Kraus operators in the two cases. Thus, to quantify this deviation, the following memory parameter is suggested:

$$\Delta = (p_1 - p_1 q)\mathrm{tr}\left\{ \sum_{m=1}^{N+1} (M'_m - M_m)^{\dagger}(M'_m - M_m) \right\}. \tag{6}$$

Notice that we have multiplied the summation of the distances in Eq. (6) by $p_1 - p_1 q$ which is the probability that this evolution takes place. By substituting the exact form of the operators in (6) for the case $\tau_1 = \tau_2 = \tau$, we arrive at

$$\Delta/2 = (1-r^2)(1 - |f_{11}|^2 - |f_{N1}|^2) + (r - \sqrt{p_1 - p_1 q})^2. \tag{7}$$

It is clear that the memory parameter is dependent on the first input of the chain as well as the channel parameter τ_1. The largest deviation from the memoryless case is given for $r=0$, corresponding to the transmission of $|1\rangle$ on the first use. In this case the maximum of Δ is $4(1 - |f_{N1}|^2 - |f_{11}|^2)$. For $|f_{N1}(\tau_1)| = 1$ we have perfect transfer, and for $|f_{11}(\tau_1)| = 1$ the first state is swapped out by the sender into S_2.

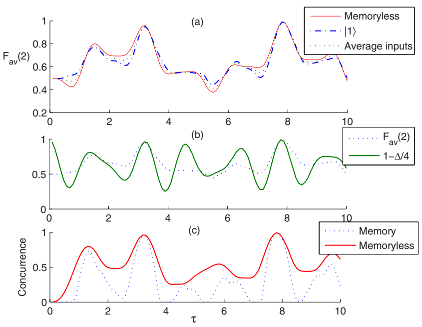

FIG. 2. (Color online) As a function of evolution time τ: (a) Average fidelity in the second use for memoryless channel in comparison with the memory case when the state $|1\rangle$ is transferred and also with average pure input states in the first use. (b) The average fidelity for the second use and the parameter $1-\Delta/4$ after transferring the state $|1\rangle$. (c) The entanglement distribution for both the memory and memoryless channel.

To compare the quality of transmission we can compare the average fidelities. The average fidelity in the kth use of the channel is $F_{av}(k)=\int F(k)d\Omega$, where $F(k)=\mathrm{tr}\{\rho_{S_k}(0)\rho_{R_k}(k)\}$ is the fidelity of the kth transmission and the integration performed over the surface of the Bloch sphere for all pure input states $\rho_{S_k}(0)$. The total description of the channel in the second use, Eq. (5) helps to compute the average fidelity for the second transmission. It is easy to show that

$$F_{av}(2)=(p_0+p_1q)F_{av}(1)+\frac{1-r^2}{6}\sum_{m=1}^{N-1}2\,\mathrm{Re}(A_mB^*_{mN})-\frac{(1-r^2)|A_N|^2}{6}+\frac{2(1-r^2)}{3}(1-|f_{11}|^2-|f_{N1}|^2), \tag{8}$$

where $F_{av}(1)=\frac{1}{2}+\frac{f_{N1}+f^*_{N1}}{6}+\frac{|f_{N1}|^2}{6}$ is the average fidelity for memoryless case, and we have used the identity that $\sum_{m=1}^{N}|A_m|^2=\sum_{k_1=1}^{N-1}\sum_{k_2=k_1+1}^{N}|B_{k_1k_2}|^2=1-|f_{N1}(\tau_1)|^2-|f_{11}(\tau_1)|^2$ to simplify the final result. In Fig, 2(a) the average fidelities for the second use of the channel has been plotted for equal time evolutions $\tau_1=\tau_2=\tau$ (setting $J=1$). In this figure the average fidelity for the memoryless case has been compared with the case where the state $|1\rangle$ has been transferred in the first use and with the case of average inputs in the first transmission. When the average fidelity of the first transmission has a peak, which means almost perfect transmission, the next transmission is also good. In non-optimal times when the first transmission is not good the memory effect can improve the quality of transmission. In Fig. 2(b) the parameter $1-\Delta/4$ (we have used this parameter instead of Δ just for simplicity) and the average fidelity for the second transmission after sending the state $|1\rangle$ in the first use, have been plotted together. When $1-\Delta/4$ take its minimum it means that the amount of the memory in the channel is high, so the average fidelity in the second transmission has a low value because the state of the channel is highly mixed and there is information from the previous transmission in it. In the other

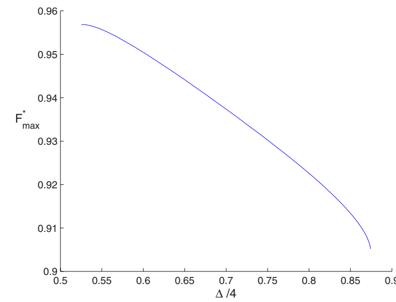

FIG. 3. (Color online) Maximum of the average fidelity in the second use when $|1\rangle$ is sent in the first use versus memory parameter Δ.

case when the parameter $1-\Delta/4$ has a peak it means that after the first transmission the channel has been nearly reset to the initial ground state. But in this case the average fidelity for the second transmission is not necessarily high because the average fidelity also depends from the time evolution τ. For example, in Fig. 2(b) for $\tau\simeq 4.6$, the memory has a low value but the average fidelity is not high because of the non-optimal τ. In this non-optimal time, $|f_{11}|$ has a large value, which means that the information is packed in the first spin and swapped out to the sender register, so the chain reset to its ground state. The same happens for the second transmission, so that the average fidelity is low.

Another problem that can be compared for different uses of the channel is the entanglement distribution. In this case the sender registers are a set of pair registers like Fig. 1(b). Dual registers S'_kS_k $(k=1,2)$ contain a maximally entangled state. In the first transmission the state S_1 is transferred to the register R_1 to create an entangled pair (not necessarily maximal) between S'_1R_1. In the second transmission, without resetting the chain, the state of S_2 is transferred to the register R_2 to create the entanglement between S'_2R_2. In Fig. 2(c) the concurrence as a measure of entanglement [20] for the states $\rho_{S'_1R_1}$ (memoryless) and $\rho_{s'_2R_2}$ (memory case) has been plotted. It shows that the effect of memory is always destructive. The peaks of entanglement are located at times where nearly perfect transmission happens.

Let us now discuss the dependence of the fidelity on Δ. As shown above the quality of state transmission in the second use of the channel depends on the time evolution τ_1 as well. We chose a range of $3.3\leq\tau_1\leq 3.9$ such that the memory parameter is increasing for the case that the state $|1\rangle$ is transferred in the first use. For each value of τ_1 we have compared the maximum average fidelity in a long range of τ_2. In Fig. 3 we have plotted this maximum value of the average fidelity F^*_{av} in the second transmission versus Δ. Fig. 3 is very interesting because it shows that the average fidelity is decreasing when Δ is increased. This shows that the remaining probability amplitude in the chain has a destructive effect on the quality of transfer in the second use of the chain.

Finally we investigate whether the memory effect (taking equal evolution times $\tau_1=\tau_2=\tau$ for simplicity) can enhance either the quantum capacity or the single-shot classical capacity which are both known for the memoryless (amplitude damping) channel [3]. As we will show below, such enhancement is indeed possible, and can be demonstrated even without explicitly calculating the capacities. We compare the Holevo bound for a special equiprobable bipartite input states in memory channel with the classical capacity of separable input states in memoryless channels [3]. Assume that all the four possible equiprobable classical input data are encoded into a special kind of input states,

$$|\phi_1(\theta)\rangle = \cos\theta|++\rangle + \sin\theta|--\rangle$$

$$|\phi_2(\theta)\rangle = \sin\theta|++\rangle - \cos\theta|--\rangle$$

$$|\phi_3(\theta)\rangle = \cos\theta|+-\rangle + \sin\theta|-+\rangle$$

$$|\phi_4(\theta)\rangle = \sin\theta|+-\rangle - \cos\theta|-+\rangle, \qquad (9)$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and all these sates vary from separable states (θ=0) to the maximally entangled one ($\theta = \pi/4$). In Fig. 1(a) one can prepare any of states $|\phi_i\rangle$ in registers S_1 and S_2, and by applying the operator $W(2)W(1)$ this state is received as the state ϱ_i in registers R_1 and R_2. The Holevo bound for input states (9) per use is $C(\tau,\theta) = 1/2\{S(\Sigma_{i=1}^4 p_i\varrho_i) - \Sigma_{i=1}^4 p_i S(\varrho_i)\}$, where $p_i = 1/4$ and $S(\varrho) = -\mathrm{tr}\,\varrho \ln \varrho$is the von Neumann entropy of the state ϱ. To find the optimal input states one can maximize $C(\tau,\theta)$ over the parameter θ. Surprisingly, the maximum $C_{max}(\tau) = max_\theta C(\tau,\theta)$ is always achieved by separable states (θ=0). In Fig. 4(a) we have plotted the $C_{max}(\tau)$ and also the real capacity of memoryless channel with separable input states [3] in terms of τ. The memory helps to increase the classical capacity in non-optimal times. These results for memory spin chain are analogous to those of memory dephasing channel [17].

The coherent information as a lower bound for quantum capacity is $I = S(\xi(\rho)) - S(I \otimes \xi(|\phi\rangle\langle\phi|))$, where ρ is the input and $|\phi\rangle$ is a purification of ρ. In Fig. 1(b) consider two maximally entangled states in registers $S_1'S_1$ and $S_2'S_2$ so the states of unprimed sender registers are $\rho_{S_1,S_2} = I/2 \otimes I/2$. These two states are transferred through the chain by $W(2)W(1)$ and we can consider two maximally entangled states in registers $S_1'S_1$ and $S_2'S_2$ as a purification of transferred states. In Fig.

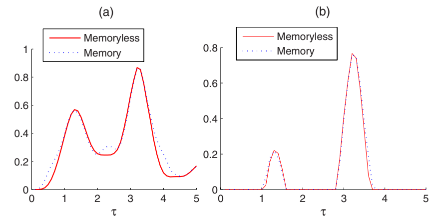

FIG. 4. (Color online) (a) Holevo bound in memory spin chains in compare with classical capacity for separable input states. (b) Coherent information for a special input states in memory channel in compare with quantum capacity for memoryless one.

4(b) we compare the quantum capacity of [3] with the coherent information per use in our model. From the figure we see that though the effect is small, there are certain memory channels (i.e., certain τ) for which even a lower bound to the true quantum capacity exceeds the memoryless quantum capacity.

In conclusion, we have given a characterization of the behavior of a spin chain without resetting. It provides an interesting example of a quantum memory channel, where the memory of the state transmitted during the first use produces a qualitatively different channel in the second use.

We have found the relevant Kraus operators for this model and we have introduced a parameter to quantify the amount of memory in the channel which has broader applicability even outside the domain of spin chain channels. We have shown that the memory effect can enable one to exceed the known classical capacity for separable inputs and the quantum capacity of the memoryless channel. Our study might pave the way for the computation of the full capacities of such a spin chain channel with memory, which will involve joint encodings and a number of uses of the channel in succession.

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