# Decoherent-history approach to the probability distribution of tunneling time

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We show that the probability distribution of tunneling time can be defined, although the decoherence functional does not satisfy the weak decoherence condition. However, the weak decoherence condition holds whenever the measuring device as defined by von Neumann exists as a part of the system. The resultant tunneling time probability distribution does not contain any information on the height of the potential.

The question of the time spent by a particle inside a barrier or tunneling time has attracted considerable attention for many years[1–3]. A mystery still remains at a fundamental level. As a question, "is it possible to define the probability distribution of tunneling time for possible values of $\tau$" is still imperfectly understood. In 1999 Yamada [4] argued that the quantum traversal time, defined by the clocked Schrödinger equation, does not satisfy the weak decoherence condition and the definition of the probability distribution of tunneling time is impossible. In 2002, Sokolovski [5] investigated the possibility of defining meaningful probabilities for a quantity that cannot be represented by a Hermitian operator. These relate to the interaction between the system and its environment, consisting of the positive-operator-valued measures (POVM) [6–8]. However, Sokolovski did not show that the POVM can be used to derive the probability of tunneling time. Later, Yamada [9] derived four tunneling times from the Gell-Mann–Hartle decoherence functionals.

The aim of this paper is to present a derivation of the probability distribution of tunneling time by using the POVM and consider its physical meaning. The derivation reveals the conditional probability theorem. Finally, we show that the weak decoherence condition holds whenever the measuring device exists as a part of the system.

Consider the conventional quantum measurement, as defined by von Neumann [10]. The eigenvalue equation for a dynamical variable $\mathbf{F}$ with the eigenvalue $f_k$ is

$$\mathbf{F}|k\rangle = f_k|k\rangle. \tag{1}$$

Then the probability of the outcome $f_k$ is given as

$$P(f_k) = \text{tr}[|k\rangle\langle k|\mathbf{U}\rho_s\mathbf{U}^+], \tag{2}$$

where $\rho_s$ is the initial density matrix and $\mathbf{U}(t)$ is the time evolution operator. This probability distribution is valid for the instantaneous quantum measurement.

In analogy, we introduce the probability distribution for the continuous measurement of the dynamical quantity $\mathbf{F}$ as

$$P(f_k) = \text{tr}\{\delta(f - F[x(t)])\mathbf{U}\rho_s\mathbf{U}^+\}, \tag{3}$$

where $\delta(\cdot)$ is the Dirac $\delta$ function and $F[x(t)]$ is a functional corresponding to the dynamical variable $\mathbf{F}$. The Dirac $\delta$ function selects a class of paths for which $F[x(t)]$ has the value of $f$. This definition corresponds to the distribution $\sigma_A(a)$ defined by Sokolovski and Connor [11].

Following this method, the probability distribution of tunneling time can be expressed in the form

$$P(\tau) = \text{tr}[\delta(\tau - t_{ab}^{cl}[x(t)])\mathbf{U}\rho_s\mathbf{U}^+] \tag{4}$$

when

$$t_{ab}^{cl}[x(t)] = \int_0^t \Theta_{ab}(x(t'))dt' \tag{5}$$

is the classical traversal time functional and $\Theta_{ab}(x) = 1$ for $a \leq x \leq b$ and 0 otherwise.

Now we show that the decoherence functional is simply obtained from the conditional probability theorem. In particular, the conditional probability $P(A|B)$ which is the probability of some event $A$ given the occurrence of some other event $B$, is defined as

$$P(A|B) = P(A \cap B)/P(B). \tag{6}$$

Suppose event $A$ and $B$ are independent. In this case $P(A \cap B) = P(A)P(B)$ so $P(A|B) = P(A)$ means that the occurrence of $A$ has no influence on the probability of the occurrence of $B$ (corresponding to two orthogonal eigenfunctions).

The traversal time wave function $\psi(x, t|\tau)$ is defined as the probability amplitude for finding the particle at $x$ having spent time in the region $\Omega \equiv [a, b]$ prior to time $t$ for a net duration of $\tau$. It is obvious that the traversal time wave function $\psi(x, t|\tau)$ is obtained by restricting the system's evolution to particular classes of Feynman paths integral approach [12–15]

$$\psi(x, t|\tau) = \int dx_0 \int D[x] \delta(\tau - t_{ab}^{cl}[x(\cdot)]) e^{iS[x(t)]/\hbar} \psi_0(x_0), \tag{7}$$

where $S[x(t)]$ is an action along the path $x(t)$. The summation of all values of $\tau$ must restore $\psi(x, t|\tau)$ to $\Psi(x, t)$,

$$\Psi(x, t) = \int_0^t d\tau \psi(x, t|\tau), \tag{8}$$

where $\Psi(x, t) = \mathbf{U}(t)\psi_0(x)$ is the final state wave function. This is equivalent to partitioning the system's state into a sum of generally nonorthogonal components.

The joint probability of two nonorthogonal events is the probability that a particle has spent time of $\tau$ and $\tau'$ in $\Omega \equiv [a, b]$ and is given as

$$P(\tau \cap \tau') = \text{tr}\{\delta(\tau' - t_{ab}^{cl}[x])\,\delta(\tau - t_{ab}^{cl}[x])\mathbf{U}\rho_s\mathbf{U}^+\} \quad (9)$$

In analogy with the conditional probability, Eq. (6), we introduce a conditional probability distribution $P(\tau|\tau')$ which is the probability that the particle has spent time of $\tau$ in $\Omega \equiv [a,b]$ for having spent time $\tau'$ as

$$P(\tau|\tau') = \frac{P(\tau \cap \tau')}{P(\tau')} = \frac{\text{tr}\{\delta(\tau' - t_{ab}^{cl}[x])\,\delta(\tau - t_{ab}^{cl}[x])\mathbf{U}\rho_s\mathbf{U}^+\}}{\text{tr}\{\delta(\tau' - t_{ab}^{cl}[x])\mathbf{U}\rho_s\mathbf{U}^+\}}. \quad (10)$$

This definition, Eq. (10), is the same as the joint probability relation defined by Steinberg [16].

By following the conditional probability theorem,

$$P(A) = \sum_i P(A|B_i)P(B_i), \quad (11)$$

we can give the probability distribution of tunneling time $P(\tau)$ in the form

$$P(\tau) = \int_0^t d\tau'\, P(\tau|\tau')P(\tau'). \quad (12)$$

Substituting Eq. (4) and Eq. (10) into Eq. (12), we obtain the probability distribution of tunneling time,

$$P(\tau) = \int_0^t d\tau'\, D[\tau;\tau'], \quad (13)$$

where $D[\tau;\tau'] = \text{tr}\{\delta(\tau' - t_{ab}^{cl}[x])\,\delta(\tau - t_{ab}^{cl}[x])\mathbf{U}\rho_s\mathbf{U}^+\}$.

Using Eq. (6), we can write Eq. (13) as

$$D[\tau;\tau'] = \int dx\,\psi(x,t|\tau')\psi^*(x,t|\tau). \quad (14)$$

This is the decoherence functional presented by Yamada [4].

Therefore, the decoherence functional $D[\tau;\tau']$ means the correlation of two events ($\tau$ and $\tau'$) in conjunction. This is purely quantum mechanical effect for nonorthogonal decompositions, Eq. (8). Usually, some physical quantity cannot be represented by a Hermitian operator as the traversal time. The joint probability of two nonorthogonal events does not vanish. It is analogy of the interference of two alternatives whether the particle passes through the upper slit or the lower slit in the two-slit experiment (wavelike property).

As mentioned above, we can briefly examine that the probability distribution of tunneling time $P(\tau)$ can be defined by the form of Eq. (4), although the decoherence functional does not satisfy the weak decoherence condition.

Finally we show that the measuring device, which exists as a part of the system, gives the weak decoherence condition [4],

$$\text{Re}\, D[\tau;\tau'] = \delta(\tau - \tau')P(\tau). \quad (15)$$

Suppose that we want to know the probability distribution of a particle spending time in a given region of space, then it is necessary to use an apparatus that has some interaction with the observed particle. Therefore, we need to extend a single system to a combined system. This combined system is divided into two parts: (i) the observed system, (ii) the apparatus system. We assume that the coupling interaction has the form

$$\mathbf{H}_{int}(t) = g(t)\mathbf{PA}, \quad (16)$$

where $g(t) = g$, for $t > 0$ and 0 otherwise, $\mathbf{P}$ is the generator of translation (momentum operator) for the apparatus, and $\mathbf{A}$ is some operator that we wish to measure, acting on an observed system. This coupling interaction was proposed by von Neumann [10] and developed by Aharonov, Albert, and Vaidman [17].

For measuring the traversal time, it is necessary to turn on the interaction for a time interval $t$. We use the operator $\mathbf{A} = \Theta_{ab}[\mathbf{x}]$. The coupling interaction then takes the form

$$\mathbf{H}_{int}(t) = g(t)\mathbf{P}\Theta_{ab}[\mathbf{x}]. \quad (17)$$

The Hamiltonian of the actual observer has coupling in the form of a weak measurement with the apparatus system. We start with the total Hamiltonian for the whole system

$$\mathbf{H}_{total} = \mathbf{H}_{sys} + \mathbf{H}_A + \mathbf{H}_{int}(t), \quad (18)$$

where $\mathbf{H}_{sys}$ represents the Hamiltonian of the observed particle, $\mathbf{H}_A$ is the Hamiltonian of the apparatus. By following von Neumann, we assume that $\mathbf{H}_A$ and $\mathbf{P}$ commute, $[\mathbf{H}_A, \mathbf{P}] = 0$, so that $\mathbf{H}_A$ may be written as

$$\mathbf{H}_A = \frac{\mathbf{P}^2}{2m}, \quad (19)$$

where $m$ is mass of the apparatus particle. The total Hamiltonian is time dependent and commutes with itself regardless of the time different. So the time-evolution operator can be written as

$$\mathbf{U}(t) = e^{-i/\hbar \int_0^t (\mathbf{H}_{sys} + \mathbf{H}_A + g\mathbf{P}\Theta_{ab}[\mathbf{x}])dt'}. \quad (20)$$

Following Gell-Mann and Hartle decoherence functional [18], a set of history $C_\alpha$ for discrete measurement is

$$C_\alpha = P_\alpha^n(t_n) \cdots P_\alpha^1(t_1), \quad (21)$$

where $P_\alpha^k(t_k)$ are projection operator at time $t_k$. The decoherence functional can be written as

$$D(C_\alpha, C_{\alpha'}) = \text{tr}[C_\alpha \rho C_{\alpha'}^\dagger], \quad (22)$$

where $\rho$ is the initial density matrix.

In analogy, we introduce the history for continuous measurement of the traversal time in the form

$$C(\tau) = \delta(\tau - t_{ab}^{cl}[x(t)])\mathbf{U}(t) = \int dx \int dx'\, P(x,x')|x\rangle\langle x'| \quad (23)$$

where

$$P(x,x') = \langle x|\delta(\tau - t_{ab}^{cl}[x(t)])\mathbf{U}(t)|x'\rangle. \quad (24)$$

This particle path is restricted to those that obey condition $\delta(\tau - t_{ab}^{cl}[x(t)])$.

Then we define the decoherence functional $D[\tau, \tau']$ for continuous measurement as

$$D[\tau, \tau'] = \mathrm{tr}_{s,a}\{\delta(\tau - t^{cl}_{ab}[x])\mathbf{U}\rho_s\rho_a\mathbf{U}^+\delta(\tau' - t^{cl}_{ab}[x])\} \quad (25)$$

where $\rho_s$ and $\rho_a$ are the initial density matrix for the observed system and the apparatus system, respectively. We assume that the mass $m$ of the apparatus system is very large so that the apparatus wave function is localized in space (localized near $R_0$),

$$|\Phi_a\rangle = |R_0\rangle. \quad (26)$$

Then the initial density matrix for the apparatus system is

$$\rho_a = |R_0\rangle\langle R_0|. \quad (27)$$

This interpretation can be expressed explicitly by rearranging Eq. (25) in the form

$$D[\tau, \tau'] = \mathrm{tr}_s[\Delta(\tau, \tau')\rho_s], \quad (28)$$

where

$$\Delta(\tau, \tau') = \mathrm{tr}_a\{\delta(\tau - t^{cl}_{ab}[x])\delta(\tau' - t^{cl}_{ab}[x])\mathbf{U}\rho_a\mathbf{U}^+\}. \quad (29)$$

Equation (29) is the partial trace on the apparatus system. It is the continuous measurement acting on the observed system rather than the instantaneous quantum measurement as a positive operator measures (POMS) [6–8]. Straightforward substitution of Eq. (20) into Eq. (29) leads to

$$\Delta(\tau, \tau') = \int dp\langle p|\delta(\tau' - t^{cl}_{ab}[x])\delta(\tau - t^{cl}_{ab}[x])\mathbf{U}(t)\rho_a\mathbf{U}^+(t)|p\rangle$$

$$= \int dp\,\delta(\tau' - t^{cl}_{ab}[x])\delta(\tau - t^{cl}_{ab}[x])\mathbf{U}(t)\langle p|\rho_a|p\rangle\mathbf{U}^+(t)$$

$$= \int dp\rho_a(p)[\delta(\tau' - t^{cl}_{ab}[x])\delta(\tau - t^{cl}_{ab}[x])$$

$$\times e^{-i/\hbar\int_0^t(\mathbf{H}_{sys}+gp\mathbf{\Theta}_{ab}[\mathbf{x}])dt''}e^{i/\hbar\int_0^t(\mathbf{H}_{sys}+gp\mathbf{\Theta}_{ab}[\mathbf{x}])dt'}] \quad (30)$$

when

$$\rho_a(p) = \langle p|\rho_a|p\rangle = \langle p|R_0\rangle\langle R_0|p\rangle = 1 \quad (31)$$

is the initial density matrix for the apparatus system in $p$ representation. Using the integral representation of the $\delta$ function we can rewrite $\Delta(\tau, \tau')$ as a Fourier transform

$$\Delta(\tau, \tau') = \frac{1}{2\pi\hbar}\int d\alpha\frac{1}{2\pi\hbar}\int d\beta e^{i/\hbar\beta\tau}e^{i/\hbar\alpha\tau'}\Delta(\beta, \alpha; \tau, \tau') \quad (32)$$

where $\Delta(\beta, \alpha; \tau, \tau')$ is

$$\Delta(\beta, \alpha; \tau, \tau')$$
$$= \int dp[e^{-i/\hbar\int_0^t\{\mathbf{H}_{sys}+(gp+\beta)\mathbf{\Theta}_{ab}[\mathbf{x}]\}dt''}e^{i/\hbar\int_0^t\{\mathbf{H}_{sys}+(gp-\alpha)\mathbf{\Theta}_{ab}[\mathbf{x}]\}dt'}]. \quad (33)$$

Switching to new variables $\beta_+ \equiv gp+\beta$ and $\alpha_- \equiv \alpha-gp$, we can rewrite $\Delta(\tau, \tau')$ as

$$\Delta(\tau, \tau') = \int dp e^{i/\hbar p(\tau-\tau')}\left[\frac{1}{2\pi\hbar}\int d\alpha_-\frac{1}{2\pi\hbar}\int d\beta_+ e^{i/\hbar\beta_+\tau}e^{i/\hbar\alpha_-\tau'}\Delta(\beta_+, \alpha_-; \tau, \tau')\right]$$

$$= \delta(\tau - \tau')\frac{1}{2\pi\hbar}\int d\alpha_-\frac{1}{2\pi\hbar}\int d\beta_+ e^{i/\hbar\beta_+\tau}e^{i/\hbar\alpha_-\tau'}\Delta(\beta_+, \alpha_-; \tau, \tau') \quad (34)$$

when

$$\Delta(\beta_+, \alpha_-; \tau, \tau') = [e^{-i/\hbar\int_0^t(\mathbf{H}_{sys}+\beta_+\mathbf{\Theta}_{ab}[\mathbf{x}])dt''}e^{i/\hbar\int_0^t(\mathbf{H}_{sys}-\alpha_-\mathbf{\Theta}_{ab}[\mathbf{x}])dt'}]. \quad (35)$$

Using Eq. (28) and Eq. (35), we have the decoherence functional $D[\tau, \tau']$ in the form of

$$D(\tau, \tau') = \delta(\tau - \tau')\int dx\psi(x,t|\tau')\psi^*(x,t|\tau). \quad (36)$$

Now the interference between different $\tau$ and $\tau'$ disappears and then we obtain the weak decoherence condition.

Equation (36) shows that the weak decoherence condition holds by including the measuring device as a part of the system. This method represents a standard measurement when the system is decoupled from the apparatus system. It leads to a constraint on the observed system. According to the interpretation of quantum theory, the measurement gives the result $\tau$ with the probability distribution of tunneling time $P(\tau)$, after the measurement is performed.

By following Eq. (36) the probability distribution of tunneling time depends on the traversal wave function $\psi(x,t|\tau)$ which satisfies the clocked Schrödinger equation [19]. Usually the $\psi(x,t|\tau)$ depends on $V(x)$, the potential of the quantum system. Then the probability distribution of tunneling time depends on the potential $V(x)$, too. For the rectangular potential $V(x) = V_0\mathbf{\Theta}_{ab}(x)$ we have

$$\psi(x,t|\tau) = e^{i/\hbar V_0 \tau}\psi_{free}(x,t|\tau), \qquad (37)$$

where $\psi_{free}(x,t|\tau)$ is the traversal wave function for free propagation. Then the probability distributions of tunneling time for the rectangular potential and that of the traversal time for free propagation are completely the same. So the resultant tunneling time probability distribution does not contain any information on the height of the potential.

In summary, the probability distribution of tunneling time $P(\tau)$ can be defined by using the conditional probability theorem. Thus, the decoherence functional $D[\tau, \tau']$ does not

satisfy the weak decoherence condition. It implies that the weak decoherence condition cannot be used for validating the question "is the probability distribution $P(\tau)$ definable?" However, the weak decoherence condition holds whenever the measuring device exists as a part of system. Then the probability distribution of tunneling time is definable owing to the effect of the measuring device, monitoring the position of the observed system under consideration. When the measurement gives the value $\tau$ of the tunneling time at time $t$, the classical traversal time functional $t_{ab}^{cl} = \tau$ and coherence is destroyed.

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