Characterizing temporal distinguishability of an *N*-photon state by a generalized photon bunching effect with multiphoton interference

Z. Y. Ou*

Department of Physics, Indiana University–Purdue University Indianapolis, 402 North Blackford Street, Indianapolis, Indiana 46202, USA

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The complementarity principle of quantum mechanics relates qualitatively the visibility of quantum interference with path indistinguishability. A quantitative study was recently presented [Z. Y. Ou, Phys. Rev. A **74**, 063808 (2006)]. Following the formalism of this study, we investigate another scheme for characterizing quantitatively the degree of temporal distinguishability of an *N*-photon state, based on constructive quantum interference between an *N*-photon and a single-photon state. This scheme is related to a generalized photon bunching effect in the form of a "bump," in contrast to the "dip" for the destructive interference effect in the previous study. Generalization to other more complicated cases is straightforward and is much simpler than for the scheme of destructive interference in the previous study. A degree of (in)distinguishability is defined and can be determined experimentally by the measurement of the size of the constructive multiphoton interference effect.

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I. INTRODUCTION

The complementarity principle of quantum mechanics was first proposed by Bohr [1] to deal with the wave-particle duality of quantum particles. On the one hand, it successfully explained the peculiar quantum behavior of particles in interference. On the other hand, it provides only a qualitative description of the quantum interference process. The problem stems from the lack of a quantitative definition of distinguishability. Efforts were made to find such a definition with some success [2–5].

The above-mentioned discussions of the complementarity principle were mostly limited to fundamental conceptual study and to interference involving only one particle. However, recent interest in quantum information led to investigation of the quantum interference of multiple particles [6], especially in the context of linear optical quantum computing with qubits realized by photons [7]. An issue thus arises about distinguishability among the photons that may degrade the quantum interference effects, leading to poor performance of the quantum operations. So it is desirable to study photon distinguishability quantitatively and find its relation to the multiphoton quantum interference effect.

The first investigation of the effect of photon distinguishability on multiphoton interference was made by Grice and Walmsley [8] with an analysis of a two-photon polarization Hong-Ou-Mandel interferometer [9]. A more complicated four-photon case was studied by Ou *et al.* [10,11] and later by Tsujino *et al.* [12,13] with concerns about the distinguishability between two pairs of photons.

Recently, the current author generalized the above discussion to a system of an arbitrary number of photons [14]. A general formalism was presented to address the question of how to distinguish different temporal distributions of an *N*-photon state, that is, whether the *N* photons are all indistinguishably in one temporal mode or whether some of them are well separated from others. A degree of temporal distinguishability is quantitatively defined and a destructive multiphoton interference method is proposed that relies on a quantum state projection measurement [15–17] to measure it experimentally. Subsequent experimental demonstrations [18,19] confirmed some of the predictions. It was shown [14] that the visibility of interference is proportional to the number of indistinguishable photons in a simple situation. But since the visibility is bounded by 1, accurate measurement of the visibility is required to distinguish various scenarios of different photon distributions, especially when the photon number is large. The accuracy problem is compounded by the fact that destructive interference in this scheme shows up in the form of a "dip" in which the maximum interference effect occurs at the minimum of the measured quantity. So it requires a long recording time for good accuracy. Furthermore, the scheme of quantum state projection measurement [15–17] is complicated in structure and requires phase shifters with precise values. And because of the complexity, the extension of the discussion to the general cases is nontrivial [20].

Another scheme was recently discussed in Ref. [20] and implemented [21] that relies on a generalized Hong-Ou-Mandel interference effect with an asymmetric beam splitter [22,23]. This scheme needs fewer optical elements and thus significantly simplifies the optical arrangement. But since this new scheme is still based on destructive interference, which shows up in the form of a "dip," it suffers the same problem as before, that is, low count rate at maximum interference effect. Another disadvantage in this new scheme is that we need to control the precise value of the transmissivity of the beam splitter, which depends on the total photon number, in order to achieve complete destructive multiphoton interference.

On the other hand, a photon pair bunching effect was demonstrated recently to characterize the temporal distinguishability between two pairs of photons [10,11]. In this

*zou@iupui.edu

case, constructive four-photon interference for two pairs of photons leads to a fivefold increase in four-photon coincidence when the two pairs are indistinguishable, whereas the state of two separated pairs produces only threefold increase. The observed effect is in the form of a "bump," in contrast to the dip for destructive interference in previous schemes. The enhancement factor is expected to be bigger for larger photon number because of the Bose statistics. The largeness of the measured quantity at maximum effect leads to a good accuracy in distinguishing different scenarios of photon states.

In this paper, we generalize the study of the photon bunching effect in Refs. [10,11] to arbitrary photon number. We find that the enhancement effect in photon bunching is due to constructive interference and can be used to characterize the temporal distinguishability of photons, as defined in Ref. [14]. The scheme is based on the photon bunching effect and shows up as a bump as contrasted with the dip in the scheme in Ref. [14]. The enhancement factor in the bump is not sensitive to the experimental parameters and the optical arrangement is relatively simple. This scheme seems to overcome all the shortcomings of previous schemes. The paper is arranged as follows: In Sec. II, we discuss stimulated emission as a photon bunching effect due to constructive interference and exploit it for characterizing the temporal distinguishability of incoming photons. We also discuss its analogy with a beam splitter. This is a simple few-mode analysis. In Sec. III, we perform a more rigorous multimode analysis and confirm the results from the simple few-mode analysis. In Sec. IV, we consider other more general scenarios of photon temporal distribution and derive the corresponding enhancement factor. We end the paper with a summary.

II. STIMULATED EMISSION AS A MULTIPHOTON CONSTRUCTIVE INTERFERENCE EFFECT

Recently, it was pointed out [24] that stimulated emission can be interpreted as a result of multiphoton constructive interference: when N input photons are indistinguishable from the photon emitted by the excited atom, constructive interference leads to a factor of N enhancement in the atomic emission rate from spontaneous emission. The enhanced emission is due to stimulated emission. On the other hand, if the input photons are completely distinguishable from the photon emitted by the atom, no enhancement occurs and the atom undergoes only spontaneous emission.

If the input photons are partially indistinguishable from the emitted photon, only the indistinguishable part will give rise to the stimulated emission. Therefore, such a scheme can be used to characterize quantitatively the degree of distinguishability of the input photons. To see how this works, we consider an excited atom modeled as a phase-insensitive quantum amplifier with small gain [25]:

$$\hat{a}_s^{(\mathrm{out})} = G\hat{a}_s + g\hat{a}_0^\dagger, \tag{1}$$

where $\hat{a}_0$ represents all the internal modes of the amplifier and is usually independent of the signal mode $\hat{a}_s$ and in vacuum. To preserve the commutation relation, we need $|G|^2-|g|^2=1$ and, for small gain, $|g|\ll 1$. The related evolution operator for the system has the form of

$$\hat{U} = \exp(\eta\hat{a}_s^\dagger\hat{a}_0^\dagger - \mathrm{H.c.}) \approx 1 + (g\hat{a}_s^\dagger\hat{a}_0^\dagger + \mathrm{H.c.}) \tag{2}$$

with $g \approx \eta$.

With a vacuum input of $|0\rangle$, we have the output state

$$|\Phi\rangle_{\mathrm{out}}^{(0)} = \hat{U}|0\rangle \approx |0\rangle + g|1\rangle_s \otimes |1\rangle_0. \tag{3}$$

This gives the spontaneous emission probability of $|g|^2$. When the input is an N-photon state $|N\rangle_s \otimes |0\rangle_0$, we have

$$\begin{aligned} |\Phi\rangle_{\mathrm{out}}^{(1)} &\approx |N\rangle_s|0\rangle_0 + g(\hat{a}_s^\dagger|N\rangle_s) \otimes (\hat{a}_0^\dagger|0\rangle_0) \\ &= |N\rangle_s|0\rangle_0 + g\sqrt{N+1}|N+1\rangle_s \otimes |1\rangle_0. \end{aligned} \tag{4}$$

The probability becomes $(N+1)|g|^2$. The stimulated emission helps to enhance the emission rate by a factor of $N+1$.

In Eq. (4), the input N photons are all in the same mode as the mode $\hat{a}_s$ of the amplifier. However, if some of the input photons are in different modes from the mode $\hat{a}_s$ of the amplifier, these photons are not coupled to the amplifier and cannot stimulate the emission of the amplifier. Mathematically, we have the input as $|m\rangle_s|N-m\rangle_{s'}|0\rangle_0$ and the output state as

$$\begin{aligned} |\Phi\rangle_{\mathrm{out}}^{(1)'} &\approx |m\rangle_s|N-m\rangle_{s'}|0\rangle_0 \\ &\quad + g(\hat{a}_s^\dagger|m\rangle_s) \otimes (|N-m\rangle_{s'}) \otimes (\hat{a}_0^\dagger|0\rangle_0) \\ &= |m\rangle_s|N-m\rangle s'|0\rangle_0 \\ &\quad + g\sqrt{m+1}|m+1\rangle_s|N-m\rangle_{s'}|1\rangle_0. \end{aligned} \tag{5}$$

The enhancement factor is now $m+1$. In the special cases when $m=0,N$, we recover Eqs. (3) and (4), respectively. Therefore, spontaneous emission corresponds to the case when the input photons are completely distinguishable from the photon emitted from the amplifier, whereas stimulated emission occurs when the input photons are indistinguishable from the photon emitted by the amplifier.

Notice that the enhancement factor $m+1$ is linearly related to the number of indistinguishable photons. Thus, by observing the size of the enhancement, we can quantitatively characterize the degree of distinguishability. However, the mode of the amplifier is somewhat complicated, which makes this scheme hard to implement.

We can circumvent this problem with linear optics. As discussed before, the enhancement effect in stimulated emission is due to photon indistinguishability and is the result of constructive multiphoton interference, which can then be mimicked by a lossless beam splitter, as shown in Fig. 1.

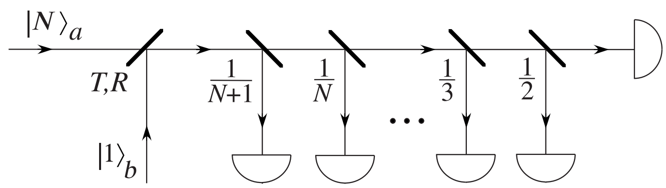

FIG. 1. Generalized photon bunching effect for characterizing the distinguishability of photons.

Reference [24] showed that the result in the scheme of Fig. 1 for $N+1$ indistinguishable photons is the same as the stimulated emission process described by Eq. (4) with an enhancement factor of $N+1$, compared to the situation when the N photons are distinguishable from the single photon at the other side of the beam splitter.

For the case when $m \neq N$, i.e., the case when some of the input N photons are distinguishable, we may use a similar input state as in Eq. (5), i.e., $|N-m\rangle_{a'} \otimes |m\rangle_a |1\rangle_b$. In this state, the $N-m$ photons are distinguishable from the m photons and the single photon from the other side. Since the outcome from the state $|m\rangle_a|1\rangle_b$ is the same as the result of stimulated emission with an input state of $|m\rangle$ and the state $|N-m\rangle_{a'}$ has no enhancement effect, the overall enhancement factor is simply $m+1$, exactly the same as in the case of stimulated emission given in Eq. (5).

The enhancement effect with a beam splitter in Fig. 1 is a generalized photon bunching effect and can be similarly used for characterizing the degree of photon distinguishability.

However, the above analysis is a few-mode analysis. It covers only the situation when the m photons and the $N-m$ photons are completely separated and distinguishable. To fully prove its validity and cover some intermediate cases of partial indistinguishability among the N photons, we need to consider a multimode analysis.

III. MULTIMODE ANALYSIS OF THE MULTIPHOTON BUNCHING EFFECT

In this multimode analysis, we will concentrate only on the problem of temporal or spectral mode distinguishability and assume other modes, such as spatial and polarization modes, are perfectly matched. However, the generalization of the discussion to other modes is straightforward.

In Ref. [14], we provided a general formalism for describing an N-photon state of arbitrary temporal distribution. An arbitrary N-photon state of a wide spectral range in a single spatial and polarization mode is expressed for the multimode analysis as

$$|\Phi_N\rangle = \mathcal{N}_N^{-1/2} \int d\omega_1 d\omega_2 \cdots d\omega_N \Phi_N(\omega_1, \ldots, \omega_N) \times a^\dagger(\omega_1)\hat{a}^\dagger(\omega_2)\cdots\hat{a}^\dagger(\omega_N)|0\rangle, \quad (6)$$

where the normalization factor $\mathcal{N}_N$ is given by

$$\mathcal{N}_N = \int d\omega_1 \cdots d\omega_N \Phi_N^*(\omega_1, \ldots, \omega_N) \sum_{\mathbb{P}} \Phi_N(\mathbb{P}\{\omega_1, \ldots, \omega_N\}). \quad (7)$$

Here the N-photon wave function Φ_N is usually given for some specific photon source.

For example, an N-photon state can be formed by projecting N identical single-photon sources (such as quantum dots) via beam splitters into one spatial mode, as shown in Fig. 2. The Φ_N function can be easily derived in the form of [14]

$$\Phi_N(\omega_1, \ldots, \omega_N) = \phi(\omega_1)e^{i\omega_1 T_1} \cdots \phi(\omega_N)e^{i\omega_N T_N}, \quad (8)$$

where $\phi(\omega)$ describes the spectral (temporal) profile of the identical single photons in a single temporal mode defined by

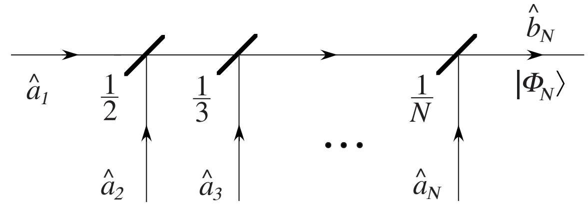

FIG. 2. Generation of an N-photon state by projecting N single-photon sources into one spatial mode via beam splitters.

the field operator: $\hat{A}_j \equiv \int d\omega\, \phi(\omega)\hat{a}_j(\omega)$ $(j=1,\ldots,N)$. T_j is the delay with reference to the first beam splitter in Fig. 1 when the photons are input to port a of the beam splitter.

In particular, as explained in Ref. [14], if the Φ_N function satisfies the permutation relations

$$\Phi_N(\omega_1, \ldots, \omega_N) = \Phi_N(\mathbb{P}_{\{n_i\}}\{\omega_1, \ldots, \omega_N\}), \quad (9)$$

where the permutation $\mathbb{P}_{\{n_i\}}$ $(i=1,2,\ldots,k)$ applies only to a subgroup of $\{\omega_1, \omega_2, \ldots, \omega_N\}$, and the orthogonal relations

$$\int d\omega_1 \cdots d\omega_N \Phi_N^*(\omega_1, \ldots, \omega_N)\Phi_N(\mathbb{P}_{\text{rest}}\{\omega_1, \ldots, \omega_N\}) = 0, \quad (10)$$

where $\mathbb{P}_{\text{rest}}$ are all the permutations excluding $\mathbb{P}_{\{n_i\}}$ $(i=1,2,\ldots,k)$, the $n_i (i=1,2,\ldots,k)$ photons in each group are indistinguishable and are in the same mode, but photons of different groups are distinguishable from each other. For the wave function in Eq. (8), the above situation occurs when $T_1 = T_2 = \cdots = T_{n_1}$, $T_{n_1+1} = T_{n_1+2} = \cdots$, etc., and $|T_1 - T_{n_1+1}| \gg 1/\Delta\omega, \ldots$, etc., with $\Delta\omega$ as the bandwidth of the $\phi(\omega)$ function.

However, the Φ_N function is not unique as defined in Eq. (6). Due to an exchange symmetry in $\hat{a}^\dagger(\omega_1)\cdots\hat{a}^\dagger(\omega_N)$ in the integral, a wave function defined by exchanging two variables as

$$\Phi_N^{ex} = [\Phi_N(\omega_1, \omega_2, \ldots, \omega_N) + \Phi_N(\omega_2, \omega_1, \ldots, \omega_N)]/2 \quad (11)$$

describes the same N-photon state. As a matter of fact, a wave function defined as the sum of permutations of any arbitrary group,

$$\Phi_N^{\{n_i\}} = (1/n_i!) \sum_{\mathbb{P}_{\{n_i\}}} \Phi_N(\mathbb{P}_{\{n_i\}}\{\omega_1, \omega_2, \ldots, \omega_N\}), \quad (12)$$

also describes the same quantum state. In particular, if the group is the whole set of the N variables, we have the symmetric wave function

$$\Phi_N^{\text{sym}} = (1/N!) \sum_{\mathbb{P}} \Phi_N(\mathbb{P}\{\omega_1, \omega_2, \ldots, \omega_N\}) \quad (13)$$

for the same N-photon state in Eq. (6).

Since the symmetrization process described above only increases the permutation symmetry and also destroys the orthogonal relation in Eq. (10), the permutation condition in Eq. (9) is only a necessary but not a sufficient condition for some of the N photons to be completely indistinguishable, and the orthogonal relation is a sufficient but not necessary condition for photons to be distinguishable from each other. So if some wave function Φ_N satisfies both the conditions in Eqs. (9) and (10), we can claim only that photons between different groups are totally distinguishable from each other due to Eq. (10), but we cannot be sure that the photons in each individual group are completely indistinguishable from each other because of the symmetrization process.

The above dilemma in describing mathematically the (in)distinguishability of an N-photon state is reflected in the inability to experimentally distinguish by N-photon measurement alone the different scenarios of temporal distribution of an N-photon state, that is, to find whether the N photons are all indistinguishably in one single mode or whether some of them may be totally separate from others. This inability is demonstrated in Ref. [14].

On the other hand, if we introduce as a reference another photon that is in a different spatial or polarization mode from the photons in the N-photon state, as we will see in the following, we can avoid the above dilemma and furthermore characterize quantitatively the degree of temporal (in)distinguishability of the N-photon state experimentally.

Let us consider the scheme in Fig. 1 where an N-photon state and a single-photon state enter a beam splitter from two separate sides (labeled as a and b). The $(N+1)$-photon state then has the form of

$$|\Phi_{N,1}\rangle = \int d\omega_0 d\omega_1 d\omega_2 \cdots d\omega_N \Phi_{N,1}(\omega_0;\omega_1,\ldots,\omega_N) \times \hat{b}^\dagger(\omega_0) a^\dagger(\omega_1)\hat{a}^\dagger(\omega_2)\cdots\hat{a}^\dagger(\omega_N)|0\rangle/\sqrt{\mathcal{N}}, \quad (14)$$

with the normalization factor $\mathcal{N}$ given by

$$\mathcal{N} = \int d\omega_0 d\omega_1 \cdots d\omega_N \Phi^*_{N,1}(\omega_0;\omega_1,\ldots,\omega_N) \times \sum_{\mathbb{P}} \Phi_{N,1}(\omega_0;\mathbb{P}\{\omega_1,\ldots,\omega_N\}), \quad (15)$$

where the operator $\mathbb{P}$ is the permutation on the indices of $1,2,\ldots,N$ only, and the sum is over all possible permutations. $\hat{a}$ and $\hat{b}$ represent the input modes a and b of the beam splitter, respectively.

Let us consider the situation when the single photon from input port b overlaps temporally with m photons from the N input photons at input port a and the rest of the $(N-m)$ photons in side a are completely distinguishable in time from the $(m+1)$ photons. Now the permutation symmetry relations

$$\Phi_{N,1}(\omega_0;\omega_1,\ldots,\omega_N) = \Phi_{N,1}(\mathbb{P}\{\omega_0;\omega_1,\ldots,\omega_m\},\omega_{m+1},\ldots,\omega_N) \quad (16)$$

for all permutation operations $\mathbb{P}$ and the orthogonal relations

$$\int d\omega_0 d\omega_1 \cdots d\omega_N \Phi^*_{N,1}(\omega_0;\omega_1,\ldots,\omega_N) \times \Phi_{N,1}(\mathbb{P}(k,j)\{\omega_0;\omega_1,\ldots,\omega_N\}) = 0, \quad (17)$$

for all the permutations $\mathbb{P}(k,j)$ with $k \le m$, $j \ge m+1$ that move ω_k to the jth variable position in $\Phi_{N,1}(\omega_0;\omega_1,\ldots,\omega_N)$, are sufficient conditions for indistinguishability among the $m+1$ photons and for temporal distinguishability between the $m+1$ photons and the remaining $N-m$ photons. This is so because the symmetrization process cannot be applied to the ω_0 variable and, if it is applied to other variables, it will break the permutation relations in Eq. (16).

Now let us calculate the outcome of the $(N+1)$-photon coincidence measurement in Fig. 1. For the sake of argument and generality, we start with a symmetrized but arbitrary wave function

$$\Phi^{\rm sym}_{N,1} = (1/N!)\sum_{\mathbb{P}} \Phi_{N,1}(\omega_0;\mathbb{P}\{\omega_1,\omega_2,\ldots,\omega_N\}) \quad (18)$$

for the $(N+1)$-photon state in Eq. (14) with the normalization constant $\mathcal{N}$ given by

$$\mathcal{N} = N!\int d\omega_0 d\omega_1 \cdots d\omega_N |\Phi^{\rm sym}_{N,1}(\omega_0;\omega_1,\ldots,\omega_N)|^2, \quad (19)$$

which is the same as in Eq. (15).

The $(N+1)$-photon coincidence rate of the $N+1$ detectors in Fig. 1 is proportional to a time integral of the correlation function of [27]

$$\Gamma^{(N+1)}(t_0,t_1,\ldots,t_N) = \langle\Phi_{N,1}|\hat{E}_1^{(o)\dagger}(t_N)\cdots\hat{E}_1^{(o)\dagger}(t_1)\hat{E}_1^{(o)\dagger}(t_0) \times \hat{E}_1^{(o)}(t_0)\hat{E}_1^{(o)}(t_1)\cdots\hat{E}_1^{(o)}(t_N)|\Phi_{N,1}\rangle. \quad (20)$$

where

$$\hat{E}_1^{(o)}(t) = \sqrt{T}\hat{E}_a(t) + \sqrt{R}\hat{E}_b(t)$$

$$\text{with } \hat{E}_c(t) = (1/\sqrt{2\pi})\int d\omega\, \hat{c}(\omega)e^{-i\omega t} \quad (c=a,b). \quad (21)$$

Let us first evaluate $\hat{E}_1^{(o)}(t_0)\hat{E}_1^{(o)}(t_1)\cdots\hat{E}_1^{(o)}(t_N)|\Phi_{N,1}\rangle$, which has the form of

$$\hat{E}_1^{(o)}(t_0)\hat{E}_1^{(o)}(t_1)\cdots\hat{E}_1^{(o)}(t_N)|\Phi_{N,1}\rangle = T^{N/2}R^{1/2}\sum_{k=0}^{N} P_{0k}\{\hat{E}_b(t_0)\hat{E}_a(t_1)\cdots\hat{E}_a(t_N)\}|\Phi_{N,1}\rangle, \quad (22)$$

where P_{0k} exchanges the variables t_0 and t_k. It is straightforward to show that, for the state in Eq. (14) and the symmetrized wave function in Eq. (18), we have

$$\hat{E}_b(t_0)\hat{E}_a(t_1)\cdots\hat{E}_a(t_N)|\Phi_{N,1}\rangle = \frac{\mathcal{N}^{-1/2}N!}{(2\pi)^{(N+1)/2}}\int d\omega_0\cdots d\omega_N \Phi^{\rm sym}_{N,1}(\omega_0;\omega_1,\ldots,\omega_N) \times e^{-i(\omega_0 t_0+\cdots+\omega_N t_N)}|0\rangle \equiv \mathcal{N}^{-1/2}N!\,\mathcal{G}(t_0;t_1,\ldots,t_N)|0\rangle, \quad (23)$$

where the $\mathcal{G}$ function is completely symmetric with respect to $t_1,\ldots,t_N$:

$$\mathcal{G}(t_0;t_1,\ldots,t_N) = \mathcal{G}(t_0;\mathbb{P}\{t_1,\ldots,t_N\}). \quad (24)$$

The overall $(N+1)$-photon coincidence probability is proportional to a time integral of the Γ function in Eq. (20):

$$P_{N+1} = \int dt_0 dt_1 \cdots dt_N \Gamma^{(N+1)}(t_0, t_1, \ldots, t_N). \quad (25)$$

With Eqs. (20), (22), and (23), we obtain

$$P_{N+1} = T^N R \mathcal{N}^{-1} (N!)^2 \int dt_0 dt_1 \cdots dt_N \times \sum_{k,j} \mathbb{P}_{0k}\{\mathcal{G}^*(t_0;t_1, \ldots, t_N)\} \mathbb{P}_{0j}\{\mathcal{G}(t_0;t_1, \ldots, t_N)\} = T^N R \left(\sum_{k=j} + \sum_{k\neq j} \right). \quad (26)$$

It is straightforward to find that the first term in Eq. (26) is

$$\sum_{k=j} = \frac{(N!)^2}{\mathcal{N}} \sum_{k=0}^{N} \int dt_0 dt_1 \cdots dt_N |\mathbb{P}_{0k}\{\mathcal{G}(t_0, t_1, \ldots, t_N)\}|^2 = \frac{N!}{\mathcal{N}}(N+1)! \int dt_0 dt_1 \cdots dt_N |\mathcal{G}(t_0;t_1, \ldots, t_N)|^2, \quad (27)$$

where we switched the integral variables t_0 and t_k. Using the definition of the $\mathcal{G}$ function in Eq. (23), we have simply

$$\int dt_0 dt_1 \cdots dt_N |\mathcal{G}(t_0;t_1, \ldots, t_N)|^2 = \int d\omega_0 \cdots d\omega_N |\Phi_{N,1}^{\text{sym}}(\omega_0;\omega_1, \ldots, \omega_N)|^2 = \mathcal{N}/N!\,. \quad (28)$$

For the second term in Eq. (26), we may use the symmetry relation in Eq. (24) and find that the integral is the same for all the terms in the sum:

$$\sum_{k\neq j} = \frac{(N!)^2}{\mathcal{N}} N(N+1) \int dt_0 dt_1 \cdots dt_N \mathcal{G}^*(t_0;t_1, \ldots, t_N) \times \mathcal{G}(t_1;t_0,t_2, \ldots, t_N) = \frac{N!}{\mathcal{N}}(N+1)!\, N \int d\omega_0 \cdots d\omega_N \Phi_{N,1}^{\text{sym}*}(\omega_0;\omega_1, \ldots, \omega_N) \times \Phi_{N,1}^{\text{sym}}(\omega_1;\omega_0, \ldots, \omega_N), \quad (29)$$

Finally, Eq. (26) becomes

$$P_{N+1} = T^N R (N+1)!\,(1 + N\mathcal{D}_{N,1}) = P_{N+1}^{\text{cl}}(1 + N\mathcal{D}_{N,1}) \quad (30)$$

with

$$\mathcal{D}_{N,1} \equiv \frac{\int dt_0 dt_1 \cdots dt_N \mathcal{G}^*(t_0;t_1, \ldots, t_N)\mathcal{G}(t_1;t_0, \ldots, t_N)}{\int dt_0 dt_1 \cdots dt_N |\mathcal{G}(t_0;t_1, \ldots, t_N)|^2} = \frac{\int d\omega_0 \cdots d\omega_N \Phi_{N,1}^{\text{sym}*}(\omega_0;\omega_1, \ldots, \omega_N)\Phi_{N,1}^{\text{sym}}(\omega_1;\omega_0, \ldots, \omega_N)}{\int d\omega_0 \cdots d\omega_N |\Phi_{N,1}^{\text{sym}}(\omega_0;\omega_1, \ldots, \omega_N)|^2}. \quad (31)$$

Here the classical probability $P_{N+1}^{\text{cl}} \equiv T^N R (N+1)!$ is defined when the single photon at port b is completely distinguishable from the N photons at port a, i.e., when $\mathcal{D}_{N,1}=0$. Thus the enhancement factor is

$$\frac{P_{N+1}}{P_{N+1}^{cl}} = 1 + N\mathcal{D}_{N,1}. \quad (32)$$

Note that when $\mathcal{D}_{N,1}=1$ we have the maximum enhancement factor of $N+1$, indicating complete indistinguishability, whereas when $\mathcal{D}_{N,1}=0$ there is no enhancement effect, due to complete distinguishability. In general, $0 \leq \mathcal{D}_{N,1} \leq 1$, and it describes the intermediate situation of partial (in)distinguishability. Thus the quantity $\mathcal{D}_{N,1}$ gives the degree of distinguishability between the single photon in port b and the N photons in port a and can be determined experimentally by measuring the enhancement factor $P_{N+1}/P_{N+1}^{\text{cl}}$. This degree of photon indistinguishability is directly reflected in the degree of mode match between the single photon and the N photons as described in the integral in Eq. (31).

Let us now consider some special cases of partial distinguishability. We shall assume that the function $\Phi_{N,1}(\omega_0;\omega_1,\omega_2, \ldots, \omega_N)$ in Eq. (18) satisfies Eqs. (16) and (17). As discussed earlier, this is the situation when the single photon from input port b overlaps temporally with m photons from the N input photons at input port a and the rest of the $(N-m)$ photons in side a are completely distinguishable in time from the $(m+1)$ photons. The numerator in Eq. (31) can be calculated as follows:

$$\int d\omega_0 \cdots d\omega_N \Phi_{N,1}^{\text{sym}*}(\omega_0;\omega_1, \ldots, \omega_N)\Phi_{N,1}^{\text{sym}}(\omega_1;\omega_0, \ldots, \omega_N) = \frac{1}{N!^2} \sum_{\mathbb{P},\mathbb{P}'} \int d\omega_0 \cdots d\omega_N \Phi_{N,1}^*(\omega_0;\mathbb{P}\{\omega_1, \ldots, \omega_N\}) \times \Phi_{N,1}(\omega_1;\mathbb{P}'\{\omega_0,\omega_2, \ldots, \omega_N\}). \quad (33)$$

Now let us break the arbitrary permutation $\mathbb{P}$ into $\mathbb{P}_{N-1}P_{1k}$ $(k=1, \ldots, N)$ with $\mathbb{P}_{N-1}$ as the permutation on the remaining indices $2,3, \ldots, N$, that is, $\Sigma_{\mathbb{P}} = \Sigma_{\mathbb{P}_{N-1}} \Sigma_{P_{1k}}$. Since $\Sigma_{\mathbb{P}'}$ covers all permutations in $\{\omega_0, \omega_2, \ldots, \omega_N\}$ and certainly covers the subset $\{\omega_2, \ldots, \omega_N\}$, all the $(N-1)!$ terms in the sum $\Sigma_{\mathbb{P}_{N-1}}$ are equal if we change the integral variables accordingly. So Eq. (33) becomes

$$\int d\omega_0 \cdots d\omega_N \Phi_{N,1}^{\text{sym}*}(\omega_0;\omega_1,\ldots,\omega_N)\Phi_{N,1}^{sym}(\omega_1;\omega_0,\ldots,\omega_N)$$

$$= \frac{1}{N!N}\sum_{P_{1k},\mathbb{P}'}\int d\omega_0 \cdots d\omega_N \Phi_{N,1}(\omega_1;\mathbb{P}'\{\omega_0,\omega_2,\ldots,\omega_N\})$$

$$\times \Phi_{N,1}^*(\omega_0;P_{1k}\{\omega_1,\ldots,\omega_N\}). \tag{34}$$

In the sum of $\Sigma_{P_{1k}}$, if $k \le m$, using the permutation symmetry relation in Eq. (16), we have $\Phi_{N,1}^*(\omega_0;P_{1k}\{\omega_1,\ldots,\omega_N\}) = \Phi_{N,1}^*(\omega_1;\omega_0,\ldots,\omega_N)$ but if $k>m$, the integral is zero according to the orthogonal relation in Eq. (17). So Eq. (34) becomes

$$\int d\omega_0 \cdots d\omega_N \Phi_{N,1}^{\text{sym}*}(\omega_0;\omega_1,\ldots,\omega_N)\Phi_{N,1}^{sym}(\omega_1;\omega_0,\ldots,\omega_N)$$

$$= \frac{m}{N!N}\sum_{\mathbb{P}'}\int d\omega_0 \cdots d\omega_N \Phi_{N,1}(\omega_0;\mathbb{P}'\{\omega_1,\omega_2,\ldots,\omega_N\})$$

$$\times \Phi_{N,1}^*(\omega_0;\omega_1,\ldots,\omega_N)$$

$$= \frac{m}{N}\frac{\mathcal{N}}{N!}. \tag{35}$$

Here we switched the integral variables $\omega_0 \leftrightarrow \omega_1$ and used the definition in Eq. (15). With Eqs. (28) and (35), we obtain the quantity $\mathcal{D}_{N,1}$ in Eq. (31) as

$$\mathcal{D}_{N,1} = m/N, \tag{36}$$

and the enhancement factor in Eq. (32) is then

$$\frac{P_{N+1}}{P_{N+1}^{\text{cl}}} = 1 + m, \tag{37}$$

which confirms the prediction by the few-mode analysis in Sec. II.

Furthermore, the multimode analysis can cover some intermediate cases of partial indistinguishability. For the illustration of the principle, we consider the simple N-photon state generated in Fig. 2 where the N-photon wave function has the form of Eq. (8). Moreover, let us assume the N photons are divided into k groups with the same T_{n_i} for each subgroup $\{n_i\}$ $(i=1,\ldots,k)$ but $|T_{n_i}-T_{n_j}| \gg 1/\Delta\omega$ $(i \neq j)$ for different subgroups. From previous discussion, we can depict this scenario in Fig. 3(a). For the single-photon state input from port b, we assume it is from a single-photon source identical to the ones depicted in Fig. 2 so that it is given by

$$|1\rangle_b = \int d\omega_0 \phi(\omega_0) e^{i\omega_0 T_0}\hat{b}^\dagger(\omega_0)|0\rangle, \tag{38}$$

where T_0 is some arbitrary delay that can be varied. If $T_0 = T_i$, the single-photon input in port b is indistinguishable from the n_i photons in the ith group of the N photons in port a. The $(N+1)$-photon wave function is then

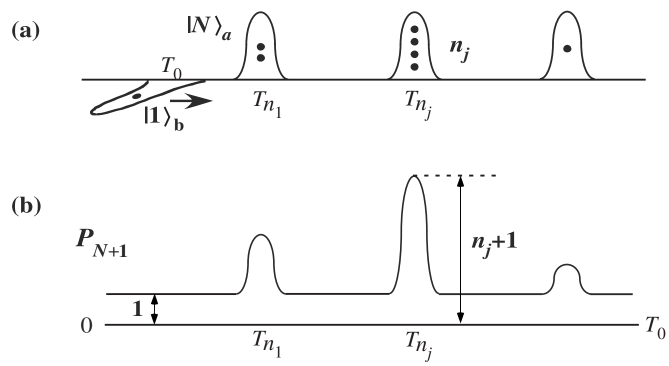

FIG. 3. (a) Temporal distribution with well-separated groups of N photons and (b) the corresponding normalized P_{N+1} as the delay T_0 of the single photon is scanned.

$$\Phi_{N,1}(\omega_0;\omega_1,\ldots,\omega_N)$$

$$= \phi(\omega_0)e^{i\omega_0 T_0}\phi(\omega_1)e^{i\omega_1 T_1}\cdots\phi(\omega_N)e^{i\omega_N T_N}. \tag{39}$$

For this wave function, we find the quantity $\mathcal{D}_{N,1}$ in Eq. (31) with some manipulations as

$$\mathcal{D}_{N,1} = \frac{1}{N}\sum_{j=1}^{k} n_j |g(T_0 - T_j)/g(0)|^2 \tag{40}$$

with $g(T) \equiv \int d\omega |\phi(\omega)|^2 e^{i\omega T}$. The enhancement factor in Eq. (32) is then

$$\frac{P_{N+1}}{P_{N+1}^{\text{cl}}} = 1 + \sum_{j=1}^{k} n_j |g(T_0 - T_j)/g(0)|^2. \tag{41}$$

The meaning of the above equation is depicted in Fig. 3(b), where as we scan the delay T_0 the enhancement factor shows some bumps. The size of a specific bump is exactly n_j+1 corresponding to the enhancement factor when $T_0 = T_j$ or the single photon overlaps with the n_j photons in the N-photon state. The width of the bump is determined by the function $|g(T)|^2$.

Now the experimental procedure to measure the distinguishability of the N-photon state is clear. As depicted in Fig. 3, we scan the relative delay of the single photon in port b with respect to the N-photon state in port a. Whenever the single photon scans through n_j indistinguishable photons, the $N+1$ coincidence count shows a bump of size n_j relative to the baseline. In this way, we can characterize the temporal distinguishability of the N-photon state.

IV. MORE GENERAL CASE OF $|N_a, M_b\rangle$

We can discuss the more general input state of $|N_a, M_b\rangle$, i.e., N photons input at port a and M photons at port b of the beam splitter in Fig. 1, in a similar manner to the state of $|N_a, 1_b\rangle$ in previous sections.

When all the $N+M$ photons are in a single temporal mode, from the quantum theory of a lossless beam splitter [26], we may find the probability of finding all $N+M$ photons in one output side of the beam splitter as

$$P_{N+M} = T^N R^M (N+M)!/N!M!. \tag{42}$$

But when the incoming N photons are completely distinguishable from the M photons, the $N+M$ photons act like classical particles and follow the probability law. The corresponding classical probability is then

$$P^{\text{cl}}_{N+M} = T^N R^M. \tag{43}$$

So the enhancement factor due to quantum interference is

$$\frac{P_{N+M}}{P^{\text{cl}}_{N+M}} = \frac{(N+M)!}{N!\,M!}. \tag{44}$$

A special case is for $N=M=2$, which gives an enhancement ratio of 6. This is the photon pair bunching effect experimentally demonstrated by Ou *et al.* [10].

The multimode analysis starts with the $(N+M)$-photon state:

$$|\Phi_{N,M}\rangle = (1/\sqrt{\mathcal{N}_{N,M}})\int d\omega_{b1}\cdots d\omega_{bM} d\omega_{a1}\cdots d\omega_{aN} \times \Phi_{N,M}(\omega_{b1},\ldots,\omega_{bM};\omega_{a1},\ldots,\omega_{aN}) \times \hat{b}^\dagger(\omega_{b1})\cdots b^\dagger(\omega_{bM}) a^\dagger(\omega_{a1})\cdots \hat{a}^\dagger(\omega_{aN})|0\rangle, \tag{45}$$

where the normalization function is

$$\mathcal{N}_{N,M} \equiv \int d\omega_{b1}\cdots d\omega_{bM} d\omega_{a1}\cdots d\omega_{aN} \times \sum_{\mathbb{P}_N,\mathbb{P}_M} \Phi_{N,M}(\mathbb{P}_M\{\omega_{b1},\ldots,\omega_{bM}\};\mathbb{P}_N\{\omega_{a1},\ldots,\omega_{aN}\}) \times \Phi^*_{N,M}(\omega_{b1},\ldots,\omega_{bM};\omega_{a1},\ldots,\omega_{aN}). \tag{46}$$

The symmetrized wave function is then

$$\Phi^{\text{sym}}_{N,M}(\omega_{b1},\ldots,\omega_{bM};\omega_{a1},\ldots,\omega_{aN}) \equiv (1/N!\,M!)\sum_{\mathbb{P}_N,\mathbb{P}_M} \times \Phi_{N,M}(\mathbb{P}_M\{\omega_{b1},\ldots,\omega_{bM}\};\mathbb{P}_N\{\omega_{a1},\ldots,\omega_{aN}\}). \tag{47}$$

We now use $N+M$ detectors at one of the output ports of the beam splitter to measure $(N+M)$-photon coincidence. The result is similar to Eq. (20) but the number of time variables increases to $N+M$. For the $(N+M)$-photon state in Eq. (45), we have

$$\hat{E}^{(o)}_1(t_1)\cdots \hat{E}^{(o)}_1(t_N) E^{(o)}_1(t_{M+1})\cdots \hat{E}^{(o)}_1(t_{N+M})|\Phi_{N,M}\rangle = T^{N/2}R^{M/2}\sum_j \mathbb{C}^{(j)}_{M,N}\{\hat{E}_b(t_1)\cdots E_b(t_M)\hat{E}_a(t_{M+1})\cdots \hat{E}_a(t_{N+M})\} \times |\Phi_{N,M}\rangle, \tag{48}$$

where $\mathbb{C}^{(j)}_{M,N}$ is a combination operation on the $N+M$ variables $t_1,\ldots,t_{N+M}$ that regroups them into two subgroups with M variables in one and N in the other. There are totally $C^M_{N+M}=(N+M)!/N!M!$ different terms in the sum in Eq. (48). Similar to Eq. (23), we have if we use the symmetric wave function in Eq. (47)

$$\begin{aligned}\hat{E}_b(t_1)\cdots E_b(t_M)\hat{E}_a(t_{M+1})\cdots \hat{E}_a(t_{N+M})|\Phi_{N,M}\rangle &= \frac{N!\,M!\,\mathcal{N}^{-1/2}_{N,M}}{(2\pi)^{(N+M)/2}}\int d\omega_1\cdots d\omega_{N+M}\Phi^{sym}_{N,M}(\omega_1,\ldots,\omega_M;\omega_{M+1},\ldots,\omega_{N+M})e^{-i(\omega_1 t_1+\cdots+\omega_{N+M}t_{N+M})}|0\rangle \\ &\equiv N!\,M!\,\mathcal{N}^{-1/2}_{N,M}\mathcal{G}(t_1,\ldots,t_M;t_{M+1},\ldots,t_{N+M})|0\rangle,\end{aligned} \tag{49}$$

where the $\mathcal{G}$ function so defined satisfies a permutation symmetry similar to Eq. (24):

$$\mathcal{G}(t_1,\ldots,t_M;t_{M+1},\ldots,t_{N+M}) = \mathcal{G}(\mathbb{P}_M\{t_1,\ldots,t_M\};\mathbb{P}_N\{t_{M+1},\ldots,t_{N+M}\}). \tag{50}$$

Similarly to Eq. (26), the overall $(N+M)$-photon coincidence probability is proportional to

$$P_{N+M} = \frac{T^N R^M (N!\,M!)^2}{\mathcal{N}}\int dt_1\cdots dt_{N+M}\left(\sum_{k=j} + \sum_{k\neq j}\right)\mathcal{G}^*(\mathbb{C}^{(k)}_{M,N})\mathcal{G}(\mathbb{C}^{(j)}_{M,N}), \tag{51}$$

where $\mathbb{C}^{(k)}_{M,N}$ operates on $\{t_1,\ldots,t_M;t_{M+1},\ldots,t_{N+M}\}$. Similarly to Eqs. (27)–(29), we can derive after using Eq. (50)

$$P_{N+M} = T^N R^M (N+M)!\left(1 + \sum_{j=1}^{K}\mathcal{D}^{(j)}_{N,M}C^j_M C^j_N\right), \tag{52}$$

where $K=\min\{M,N\}$ and

$$\mathcal{D}_{N,M}^{(j)} \equiv \frac{\int dt_1 \cdots dt_{N+M} \mathcal{G}^*(t_1, \ldots, t_M; t_{M+1}, \ldots, t_{N+M}) \mathcal{G}(\overbrace{t_{M+1}, \ldots, t_{M+j}}^{j}, t_{j+1}, \ldots, t_M; \overbrace{t_1, \ldots, t_j}^{j}, t_{M+j+1}, \ldots, t_{N+M})}{\int dt_1 \cdots dt_{N+M} |\mathcal{G}(t_1, \ldots, t_M; t_{M+1}, \ldots, t_{N+M})|^2}$$

$$= \frac{\int d\omega_1 \cdots d\omega_{N+M} \Phi_{N,M}^{\text{sym}*}(\omega_1, \ldots, \omega_{N+M}) \Phi_{N,M}^{\text{sym}}(\overbrace{\omega_{M+1}, \ldots, \omega_{M+j}}^{j}, \omega_{j+1}, \ldots, \omega_M; \overbrace{\omega_1, \ldots, \omega_j}^{j}, \omega_{M+j+1}, \ldots, \omega_{N+M})}{\int d\omega_1 \cdots d\omega_{N+M} |\Phi_{N,M}^{\text{sym}}(\omega_1, \ldots, \omega_M; \omega_{M+1}, \ldots, \omega_{N+M})|^2}. \quad (53)$$

The quantities $\mathcal{D}_{N,M}^{(j)}$ $(j=1, \ldots, K)$, like $\mathcal{D}_{N,1}$ in Eq. (31), describe the mode match between the N photons at port a and the M photons at port b and can be thought of as some degrees of indistinguishability of photons. But with $K>1$ we now have more than one quantity for the description.

Similarly to Eq. (30), if the classical probability is defined when the M photons at port b are completely distinguishable from the N photons at port a, i.e., when $\mathcal{D}_{N,M}^{(j)}=0$ for $j=1, \ldots, K$, then we have the enhancement factor due to quantum interference as

$$\frac{P_{N+M}}{P_{N+M}^{\text{cl}}} = 1 + \sum_{j=1}^{K} \mathcal{D}_{N,M}^{(j)} C_M^j C_N^j. \quad (54)$$

So the enhancement factor is directly related to the quantities $\mathcal{D}_{N,M}^{(j)}$ $(j=1, \ldots, K)$ of the degrees of indistinguishability. Note that, by setting $M=1$ in Eq. (54), we recover the formula in Eq. (32).

In the special case when $\mathcal{D}_{N,M}^{(j)}=1$ for $j=1, \ldots, K$, the enhancement factor becomes

$$\frac{P_{N+M}}{P_{N+M}^{\text{cl}}} = 1 + \sum_{j=1}^{K} C_M^j C_N^j = \frac{(N+M)!}{N!\,M!}. \quad (55)$$

This is exactly the same as in the single-mode analysis presented earlier in Eq. (44).

The photon bunching enhancement factor in Eq. (55) is for all the photons involved to be indistinguishable. When some of the photons are distinguishable, the enhancement factor will decrease. A somewhat general scenario is depicted in Fig. 4 when some of the N photons at input port a are indistinguishable from some of the M photons at side b. Here we break the N photons and the M photons into $k+1$ groups, respectively, namely, $N=n_1+\cdots+n_k+n_{k+1}$ and $M=m_1+\cdots+m_k+m_{k+1}$. In these groups, n_i photons are indistinguishable from m_i photons with $i=1,2, \ldots, k$ and $\{n_i, m_i\}$ group of photons are distinguishable from $\{n_j, m_j\}$ group of photons with $i \neq j$. Furthermore, n_{k+1} photons are distinguishable from m_{k+1} photons.

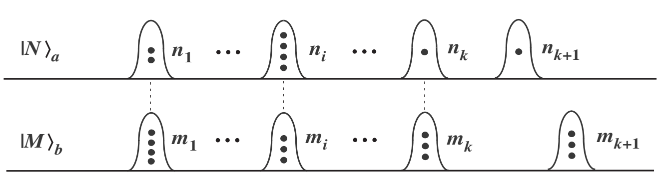

FIG. 4. Temporal distributions for photons from input sides a and b, respectively.

As in Sec. III, we may consider the multimode description of the scenario and proceed to use Eq. (54) for evaluating the enhancement factor. But this approach is obviously even more complicated than in Sec. III. Since we have confirmed in Sec. III the validity of the few-mode approach in Sec. II, we may apply it to the scenario in Fig. 4.

In analogy to the case of stimulated emission described in Eq. (5), we may write the input state to the beam splitter as

$$|\Phi\rangle_{in} = |n_{k+1}\rangle_a^{(k+1)} \otimes |m_{k+1}\rangle_b^{(k+1)} \prod_{i=1}^{k} \otimes (|n_i\rangle_a^{(i)} |m_i\rangle_b^{(i)}), \quad (56)$$

where we use $\otimes$ and the superscript (i) to separate and label the states of distinguishable photons.

Since $|n_i\rangle_a |m_i\rangle_b$ is the same state as $|N_a, M_b\rangle$, which gives rise to the enhancement factor in Eqs. (44) and (55), it will contribute a factor of $(n_i+m_i)!/n_i!m_i!$ to the overall enhancement factor, which is then

$$\frac{P_{\{n_i,m_i\}}}{P_{N+M}^{\text{cl}}} = \prod_{i=1}^{k} \frac{(n_i+m_i)!}{n_i!\,m_i!}. \quad (57)$$

Note that, since $|n_{k+1}\rangle_a$ and $|m_{k+1}\rangle_b$ are distinguishable states, they have no contribution to the enhancement factor.

As mentioned earlier, the derivation of $\mathcal{D}_{N,M}^{(j)}$ in Eq. (53) is complicated in a multimode analysis for the scenario in Fig. 4 and will not be given here, but the result is presented as the quantity $\mathcal{T}_{k-l}/\mathcal{T}_k$ in Appendix B of Ref. [20] and has the form

TABLE I. Enhancement factor for input of two a photons and two b photons.

$2a2b$	$2a1b+1b$	$1ab+1ab$	$1ab+a+b$
6	3	4	2

TABLE II. Enhancement factor for input of three a photons and two b photons.

$3a2b$	$2a2b+a$	$3a1b+b$	$2a1b+ab$	$1a2b+2a$	$2a1b+a+b$	$ab+a+ab$	$ab+a+a+b$
10	6	4	6	3	3	4	2

$$\mathcal{D}_{N,M}^{(j)} = \sum_{\substack{i_1\ldots i_k \\ i_1+\cdots+i_k=j}}^{j} C_{m_1}^{i_1}\cdots C_{m_k}^{i_k} C_{n_1}^{i_1}\cdots C_{n_k}^{i_k}/C_M^j C_N^j \quad (58)$$

in terms of the notations in this paper. Substituting the above into Eq. (54), we obtain a result exactly the same as Eq. (57). Thus we have further confirmed the validity of the few-mode analysis that leads to Eq. (57).

Compared to the visibility formula in Eq. (134) of Ref. [14] for the analysis of the same (in)distinguishability scenario, the enhancement factor in Eq. (57) is much simpler. From Eq. (57), we find that the bigger the photon number is, the larger the enhancement factor is, and it is largest for the case of $N=M$. The largeness of the enhancement factor will make the measurement process easier. Although the enhancement factor in Eq. (57) does not depend on T,R, the actual value of the probability in Eq. (42) does and it is maximum when $T=N/(N+M)$.

In Tables I–III, we list the enhancement factors for various scenarios of the input states $|2_a,2_b\rangle, |3_a,2_b\rangle, |3_a,3_b\rangle$, respectively. These tables are in contrast to the corresponding visibility tables in Ref. [14]. The enhancement factors given here are numbers larger than 1 whereas the visibilities in Ref. [14] are all smaller than 1. Thus measurement of the enhancement factors will have some advantages over that of the visibilities in terms of the signal size. Furthermore, we find that the values here are more spread out and this also makes it easier to tell them apart experimentally.

The scenarios of $2a2b$ and $1ab+1ab$ in Table I were demonstrated experimentally and discussed in Refs. [10] and [11], respectively, and indeed lead to six and four times enhancement, respectively. The scenario of $3a3b, 2a2b+1ab, ab\times 3$ in Table III was studied experimentally in Ref. [18] with a measurement scheme given in Ref. [14]. It should be straightforward to demonstrate the enhancement factors for these three scenarios with the current detection scheme.

It should be noted that the simple few-mode argument employed here applies only to the extreme cases when there is either complete indistinguishability or complete distinguishability among the $N+M$ photons, as depicted in Fig. 4. When there is some partial (in)distinguishability, we need to resort to the formula in Eq. (54).

To illustrate the application of the enhancement formula in Eq. (54), let us consider the experimental case in Ref. [10]. The four-photon wave function in the specific form of two pairs of photons from parametric down-conversion is given by

$$\Phi_{2,2}(\omega_1,\omega_2;\omega_3,\omega_4) = \Phi(\omega_1,\omega_3)\Phi(\omega_2,\omega_4), \quad (59)$$

where $\Phi(\omega,\omega')$ is the two-photon wave function for each pair of photons in parametric down-conversion and satisfies $\Phi(\omega,\omega')=\Phi(\omega',\omega)$. Here we use a notation that is consistent in this paper. From Eq. (47), we find that the symmetric wave function for this case is

$$\begin{aligned}\Phi_{2,2}^{\mathrm{sym}}&(\omega_1,\omega_2;\omega_3,\omega_4)\\ &= \frac{1}{4}[\Phi(\omega_1,\omega_3)\Phi(\omega_2,\omega_4) + \Phi(\omega_1,\omega_4)\Phi(\omega_2,\omega_3)\\ &\quad + \Phi(\omega_2,\omega_3)\Phi(\omega_1,\omega_4) + \Phi(\omega_2,\omega_4)\Phi(\omega_1,\omega_3)]\\ &= \frac{1}{2}[\Phi(\omega_1,\omega_3)\Phi(\omega_2,\omega_4) + \Phi(\omega_1,\omega_4)\Phi(\omega_2,\omega_3)].\end{aligned} \quad (60)$$

Substituting the above in Eq. (54) for the case of $M=N=2$, we obtain after some manipulations

$$\frac{P_4}{P_4^{cl}} = 1 + 4\frac{\mathcal{A}+3\mathcal{E}}{2\mathcal{A}+2\mathcal{E}} + 1 = 4 + \frac{4\mathcal{E}}{\mathcal{A}+\mathcal{E}}, \quad (61)$$

where

$$\mathcal{A} = \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 |\Phi(\omega_1,\omega_3)\Phi(\omega_2,\omega_4)|^2 \quad (62)$$

and

$$\begin{aligned}\mathcal{E} = \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 &\Phi^*(\omega_1,\omega_3)\Phi^*(\omega_2,\omega_4)\\ &\times\Phi(\omega_1,\omega_2)\Phi(\omega_3,\omega_4).\end{aligned} \quad (63)$$

So Eq. (61) gives the enhancement factor for the intermediate case between $2a2b$ and $1ab+1ab$, corresponding to partial overlap between the two pairs of photons. $P_4/P_4^{\mathrm{cl}}=6$ for $\mathcal{E}/A=1$ corresponds to the case when the two pairs are completely indistinguishable, or the $2a2b$ scenario in Table I, whereas $P_4/P_4^{\mathrm{cl}}=4$ for $\mathcal{E}/A=0$ is the case when the two pairs are completely separated, or the $1ab+1ab$ scenario in Table I. Other values of $\mathcal{E}/A$ $(0<\mathcal{E}/A<1)$ cover the case of partial overlap between the two pairs of photons. Note that Eq.

TABLE III. Enhancement factor for input of three a photons and three b photons.

$3a3b$	$3a2b+b$	$3a1b+2b$	$2a2b+ab$	$2a2b+a+b$	$2a1b+1a2b$	$2a1b+1ab+b$	$2a1b+a+b+b$	$ab\times 3$	$ab\times 2+a+b$	$ab+b+a+a+b$
20	10	4	12	6	9	6	3	8	4	2

(61) was derived before in Ref. [11] as Eq. (4.20) for the same wave function in Eq. (59).

V. SUMMARY

In this paper, we discussed a generalized photon bunching effect which may involve an arbitrary number of photons. This bunching effect is a result of constructive multiphoton interference and is responsible for stimulated emission of an excited atom. Furthermore, we find that the bunching effect can be used to characterize temporal distinguishability of photons: various scenarios of photon temporal distribution give different enhancement factors. We derive the enhancement factor for a general wave function and relate it to some quantities defined as the degrees of (in)distinguishability of photons. Thus, by measuring the enhancement factor, we will be able to determine experimentally the degree of temporal (in)distinguishability. We apply the formula to some special cases and calculate the enhancement factor analytically.

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