Three fully polarized fermions close to a *p*-wave Feshbach resonance

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We study the three-body problem for three atomic fermions, in the same spin state, experiencing a resonant interaction in the *p*-wave channel via a Feshbach resonance represented by a two-channel model. The rate of inelastic processes due to recombination to deeply bound dimers is then estimated from the three-body solution using a simple prescription. We obtain numerical and analytical predictions for most of the experimentally relevant quantities that can be extracted from the three-body solution: the existence of weakly bound trimers and their lifetime, the low-energy elastic and inelastic scattering properties of an atom on a weakly bound dimer (including the atom-dimer scattering length and scattering volume), and the recombination rates for three colliding atoms towards weakly bound and deeply bound dimers. The effect of "background" nonresonant interactions in the open channel of the two-channel model is also calculated and allows one to determine which three-body quantities are "universal" and which on the contrary depend on the details of the model.

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I. INTRODUCTION

Fermionic superfluidity with *p*-wave pairing is related to a large class of subjects in very different areas of physics including condensed matter, astrophysics, and particle physics [1,2]. As already observed in ^{3}He experiments, the phase diagram in these systems can be very rich [3]. Moreover, the possible observation of quantum phase transitions together with the existence of exotic topological defects in *p*-wave superfluids bring a lot of interest in their study.

Presently, there is some hope that *p*-wave superfluidity and its intriguing properties can be observed with ultracold atoms [4]. Indeed, thanks to the concept of Feshbach resonance [5], it is possible to tune the interatomic interaction and to achieve strongly correlated regimes in ultracold dilute atomic gases. First realized in the *s*-wave channel with bosonic species [6,7], the Feshbach resonance is currently used for achieving BEC-BCS crossover experiments for the two-component Fermi gas in a regime of temperatures where the system can be superfluid [8–23]. In one-spin component Fermi gases, as a consequence of the Pauli exclusion principle, two-body scattering processes are forbidden in the *s*-wave channel and at low temperatures are dominant in the *p*-wave channel. The two-body cross section which is usually negligible in this channel can be greatly enhanced using a *p*-wave Feshbach resonance. This resonant regime is now obtained for ^{40}K [24–27] and ^{6}Li atoms [28–30]. The production of *p*-wave shallow dimers in ultracold ^{6}Li [28] and ^{40}K gases [27] opens very interesting perspectives for the realization of a superfluid *p*-wave phase.

In these experiments, an external magnetic field tunes the energy of a two-body *p*-wave bound state in a closed channel and for a small detuning with respect to the open channel, a resonance occurs in a two-body *p*-wave scattering process. Moreover, due to the presence of a magnetic field, the interaction strength depends on the orbital channels considered—a major difference with respect to what happens in superfluid ^{3}He [25]. As a consequence, the question of the symmetry of the low-temperature ground state in one component fermionic species is nontrivial. Studies of this many-body problem are essentially mean-field and depending on the experimental realizations, they predict the occurrence of p_x+ip_y (axial), p_x (polar), or intermediate phases [31–36]. These predictions lead to possible studies of quantum phase transitions in such systems. However, the main issue in the achievement of a *p*-wave superfluid concerns atom losses which are large in present experiments [24–30]. The question of whether or not it is possible that the system thermalizes is then a crucial point.

Concerning *s*-wave resonant Fermi systems, few-body studies have proven to be very successful in understanding properties of the superfluid gas in the BEC-BCS crossover [37–41]. These studies explain the large lifetime of the system observed at resonance and also predict the dimer-dimer scattering length which is involved in the equation of state for the dilute BEC phase. Surprisingly, although general many-body properties are rather well known in *p*-wave superfluids—thanks to contributions from the condensed matter community, few-body properties in these systems have been less studied [42,43]. However, following the example of the works done in the *s*-wave channel, few-body problems for *p*-wave pairwise potential are valuable for a determination of properties in the strongly interacting dilute gas beyond a mean-field analysis. As an example, we note that consequences of the existence of trimers first found in the present work have already been taken into account for an estimation of the lifetime of *p*-wave shallow dimers [44].

In this paper, we consider three identical fermions close to a *p*-wave Feshbach resonance. We determine their low-energy scattering properties together with the possible existence of trimers. Our study is also a first step toward an understanding of the atom losses observed in present experiments [24,26–29]. The paper is organized as follows. In Sec. II, we recall basic properties, for an isotropic short-range

interaction, of resonant two-body p-wave scattering processes [45]. In the resonant regime, two parameters are needed for a description of the low-energy two-body properties: the scattering volume $\mathcal{V}_s$ and also the p-wave equivalent of the effective range parameter hereafter denoted by α. For large and positive values of $\mathcal{V}_s$ there exists a shallow p-wave dimer of internal angular momentum 1, that is, with threefold degeneracy. For a potential with a compact support of radius b, we show that at resonance ($\mathcal{V}_s=\infty$) the effective range parameter cannot reach arbitrarily small values and $\alpha b \geq 1$. Consequently, unlike what happens in s-wave resonances, there is no scale invariance at low energy and a unitary regime cannot be obtained via a p-wave resonance [46]. In Sec. III, we introduce the main model Hamiltonian that we use in this work. It is a two channel model of the p-wave Feshbach resonance [30] where free atoms in the open channel interact with a molecular p-wave state in the closed channel, of threefold degeneracy provided that one neglects the effect of the dipole-dipole interaction in presence of the Feshbach magnetic field. The interchannel coupling amplitude, as a function of the relative distance of the two atoms, is a Gaussian of range b which mimics the van der Waals range of a more realistic two-body potential. We first briefly determine the two-body collisional properties of this model. At large coupling the resonance is broad $\alpha b \sim 1$ and for $\mathcal{V}_s$ large and positive the shallow dimer is essentially in the open channel. In the opposite regime for a weak coupling, the resonance is narrow, $\alpha b \gg 1$ and the shallow dimer is almost entirely in the closed channel. In Sec. IV, we derive an integral equation for the three-body problem. We consider solutions of total angular momentum $J=1$ and by using the rotational symmetry of the Hamiltonian, we reduce the problem in each involved symmetry sector (odd or even) to a one-dimensional integral equation. In both sectors, we predict the existence of one trimer for sufficiently broad resonances. These trimers can exist in a regime where there is no shallow dimer (for large and negative values of the scattering volume) and are interesting examples of Borromean states [47], since we find that they are not linked to an Efimov effect, contrarily to Ref. [43]. We also determine the atom-dimer scattering length a_{ad} as a function of the effective range parameter α and the potential range b, for different values of the scattering volume. At resonance ($\mathcal{V}_s=\infty$), a_{ad} takes large values (that is significantly larger than the potential range b) only in the vicinity of the threshold of existence of a trimer. The recombination rate of three incoming atoms into a shallow dimer and one outgoing atom is computed; it is shown analytically to vary as $\mathcal{V}_s^{5/2}$ for large values of the scattering volume, away from the trimer formation threshold; this differs from the $\mathcal{V}_s^{8/3}$ law put forward in Ref. [42] on the basis of a dimensional analysis ignoring a possible contribution of the effective range parameter α, but is still compatible with the numerics of Ref. [42]; finally, the recombination rate is shown analytically to present a Fano profile as a function of α close to this trimer threshold. In Sec. V, we calculate the losses due to the recombination into deeply bound dimers. Since these losses are not present in our model Hamiltonian, we estimate them from the probability that three atoms are within a volume of the order of b^3 and we obtain the lifetime of trimer states, the losses due to atom-dimer inelastic scattering and the three-body recombination rate toward deep molecular states from asymptotically free atoms. Finally we make the model more realistic by including an attractive interaction in the open channel, in addition to the coupling with the closed channel, in Sec. VI: we recalculate the trimer energies, the atom-dimer scattering length, and the recombination rate to weakly bound dimers, and we physically explain the impact on these quantities of a nonresonant interaction in the open channel. We conclude in Sec. VII.

II. BASIC PROPERTIES OF THE TWO-BODY p-WAVE SCATTERING

A. The scattering amplitude

We consider in this section two particles of mass m in the same spin state and in the center-of-mass frame, scattering in free space via a rotationally invariant short-range interaction potential. We assume for simplicity that the interaction potential scatters only in the p-wave channel, so that at large distances, where the effect of the potential is negligible, the scattering wave function of energy $E=\hbar^2k^2/m$, $k>0$, takes the form

$$\psi_{\mathbf{k}}(\mathbf{r}) \simeq e^{i\mathbf{k}\cdot\mathbf{r}} + 3f(k)\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}\frac{d}{dr}\left(\frac{e^{ikr}}{ikr}\right), \tag{1}$$

where $\mathbf{r}$ is the relative position of the two particles, $\pm\mathbf{k}$ are their incoming wave vectors, and we have introduced the unit vectors $\hat{\mathbf{r}}=\mathbf{r}/r$ and $\hat{\mathbf{k}}=\mathbf{k}/k$. The function $f(k)$ is the so-called reduced scattering amplitude since the angular dependence of the scattered wave has been pulled out. We note that Eq. (1) becomes exact (that is, one can replace $\simeq$ by $=$) for a compact support interaction potential, when $\mathbf{r}$ is out of the support of the potential.

In this subsection, we briefly review some basic properties of this p-wave scattering amplitude $f(k)$. As a consequence of the unitarity of the S-matrix of scattering theory, it obeys the optical theorem

$$\mathrm{Im}\, f(k) = k|f(k)|^2, \tag{2}$$

which implies

$$f(k) = -\frac{1}{u(k)+ik}, \tag{3}$$

where $u(k)$ is a real function. For cold atoms, the low-energy scattering properties are crucial and we assume that $u(k)$ has the following low-k series expansion:

$$u(k) = \frac{1}{k^2\mathcal{V}_s} + \alpha + O(k^2). \tag{4}$$

The so-called scattering volume $\mathcal{V}_s$ plays a role similar to the scattering length in the s-wave channel: the resonant situation corresponds to the limit $|\mathcal{V}_s|\to\infty$.

Another crucial property of the reduced scattering amplitude is that its analytic continuation to negative energies, that is to imaginary values of k, gives information on possible bound states in the two-body problem, in the form of poles of $f(k)$. More precisely, setting $k=iq$, where $q>0$, the solutions $q_{\mathrm{dim}}>0$ of the equation

$$\frac{1}{f(iq)} = 0, \tag{5}$$

correspond to bound states of the scattering potential, that is here to dimers of rotational quantum number $S=1$, with a binding energy

$$E_{\rm dim} = \hbar^2 q_{\rm dim}^2/m. \tag{6}$$

The wave function of such a dimer, "out" of the potential (again, this has an exact meaning for a compact support potential), is a solution of the free Schrödinger's equation in the p-wave channel, so that we may take it of the form

$$\phi(\mathbf{r}) = \mathcal{N}\left(\frac{3}{4\pi}\right)^{1/2} \frac{r_\gamma}{r}\frac{d}{dr}\left(\frac{e^{-q_{\rm dim}r}}{r}\right), \tag{7}$$

where $\mathcal{N}$ is a normalization factor and r_γ is the component of $\mathbf{r}$ along direction $\gamma=x$, y or z [48].

The knowledge of the dimer wave function "inside" the potential requires a full solution of Schrödinger's equation. However, it is possible to access the normalization factor $\mathcal{N}$ directly from the knowledge of the scattering amplitude. Using the closure relation

$$\int \frac{d^3k}{(2\pi)^3}\psi_{\mathbf{k}}(\mathbf{r})\psi_{\mathbf{k}}^*(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') - \sum_i \phi_i(\mathbf{r})\phi_i^*(\mathbf{r}'), \tag{8}$$

in the limit of large r and r', we obtain

$$|\mathcal{N}|^2 = -\frac{2}{q_{\rm dim}[1 - iu'(iq_{\rm dim})]}, \tag{9}$$

assuming that $u(k)$ has a series expansion with even powers of k only and using contour integration in the complex plane to single out the contribution of the poles of $f(k)$. As we shall see, this relation (9) may be used to put constraints on the parameter α.

B. Constraint on the parameter α close to resonance

Whereas the scattering volume can be adjusted at will by a Feshbach resonance driven by a magnetic field, the value of α on resonance cannot be adjusted the same way so it is important to determine what are its possible values on resonance. We assume that $\alpha_{\rm res} \neq 0$, where $\alpha_{\rm res}$ is the value of α on resonance [49]. In the resonant limit, we see from the low-k expansion of $u(k)$ that there exists a weakly bound dimer on the side $\alpha_{\rm res}\mathcal{V}_s > 0$ of the resonance [50]

$$q_{\rm dim} \sim \frac{1}{\sqrt{\alpha_{\rm res}\mathcal{V}_s}}. \tag{10}$$

From Eqs. (4) and (9) we obtain

$$|\mathcal{N}|^2 \sim \frac{1}{\alpha_{\rm res}}. \tag{11}$$

This imposes $\alpha_{\rm res} > 0$. This is in sharp contrast with the case of s-wave scattering, where the effective range r_e can take any sign on resonance.

For a compact support potential, vanishing outside a sphere of radius b, that is for $r > b$, the normalization of the dimer wave function to unity imposes

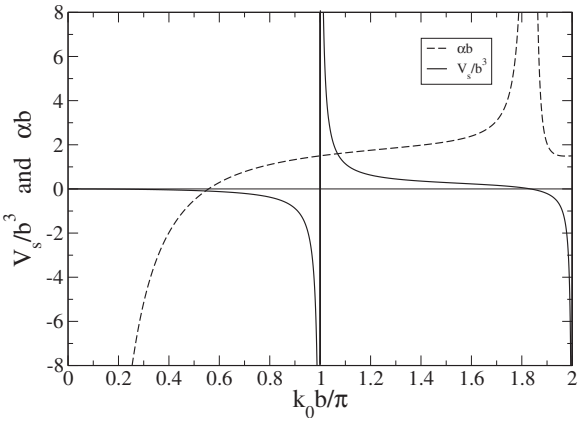

FIG. 1. For a square well interaction potential $V(r) = -\frac{\hbar^2 k_0^2}{m} \times \theta(b-r)$, where θ is the Heaviside function, values of α in units of $1/b$ (dashed line) and $\mathcal{V}_s$ in units of b^3 (solid line) as functions of $k_0 b$. Note the divergence of α when $\mathcal{V}_s = 0$.

$$\int_{r>b} d^3r |\phi(\mathbf{r})|^2 \leq 1. \tag{12}$$

Calculating the resulting integral with the expression Eq. (7) leads to [51]

$$|\mathcal{N}|^2 q_{\rm dim}\left[\frac{1}{2} + \frac{1}{q_{\rm dim}b}\right]e^{-2q_{\rm dim}b} \leq 1. \tag{13}$$

In the limit $|\mathcal{V}_s| \to \infty$ this leads to [52]

$$\alpha_{\rm res} \geq \frac{1}{b}, \tag{14}$$

where, again, $\alpha_{\rm res}$ is the value of α on resonance $|\mathcal{V}_s| = \infty$. In the zero-range limit $b \to 0$, we see that $\alpha_{\rm res}$ cannot tend to zero, but on the contrary has to diverge. This is in sharp contrast with the s-wave case, where one can find models for the interaction potential where $r_e \to 0$ in the zero-range limit $b \to 0$.

To illustrate these properties on a simple example, we give in Fig. 1 the values of α and $\mathcal{V}_s$ for a square well interaction potential, as functions of the well depth. We see on the figure that Eq. (14) is satisfied at resonance, and that α is no longer constrained by this condition away from resonance, and may even vanish and become negative.

III. MODELING OF THE RESONANT p-WAVE INTERACTION

In this section, we introduce the main model used in this paper to describe the p-wave interaction between same spin state fermions close to a resonance. It is simply a two-channel model of a Feshbach resonance, that is a direct generalization of the s-wave two-channel model [53] to the p-wave case, in the spirit of Ref. [30]. It is extended in Sec. VI to include direct interactions among atoms in the open channel.

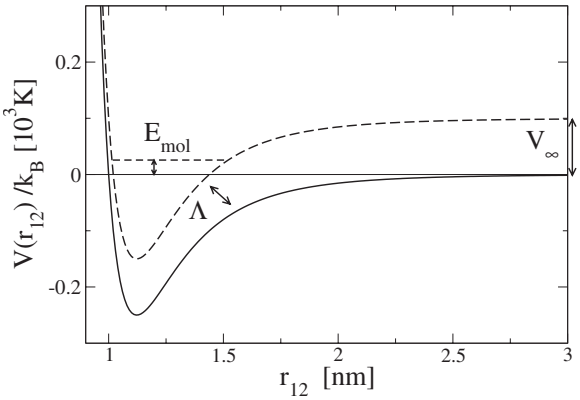

FIG. 2. Schematic view of a Feshbach resonance configuration: the atoms interact via two potential curves, plotted as a function of the interatomic distance. Solid line: open channel potential curve. Dashed line: closed channel potential curve. When one neglects the coupling Λ between the two curves, the closed channel has a molecular state of energy E_{mol} with respect to the dissociation limit of the open channel. Note that the energy dependence of the two curves is purely indicative, and the spacing between the solid curve and the dashed curve was greatly exaggerated for clarity.

A. Model Hamiltonian

As is standard in a two-channel model, the atoms may populate either the open channel, where they are treated explicitly as fermionic particles, or the closed channel, where they exist only under the form of specific tight two-body bound states, here referred to as molecules; these molecules are treated as bosons, and have an internal rotational state of spin $S_{\mathrm{mol}}=1$ since they are p-wave two-body bound states. We assume that the three rotational sublevels of a molecule are degenerate: even if this is not exactly true in practice because of the effect of the dipole-dipole interaction in presence of the magnetic field used to produce the Feshbach resonance [25], this will make our model rotationally invariant and greatly simplify the algebra for the three-body problem. For simplicity, we also assume that there is no direct interaction among the fermionic particles, the resonant p-wave atomic interaction being taken into account through the coupling between fermions and molecules. As already mentioned, this simplifying assumption is removed in Sec. VI.

The situation is represented schematically in Fig. 2. Mathematically, it corresponds to the following free space Hamiltonian written in second quantized form

$$H = \int \frac{d^3k}{(2\pi)^3}\left[\frac{\hbar^2k^2}{2m}a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}} + \left(E_{\mathrm{mol}} + \frac{\hbar^2k^2}{4m}\right)\sum_{\gamma} b_{\gamma,\mathbf{k}}^{\dagger}b_{\gamma,\mathbf{k}}\right] + \Lambda\int \frac{d^3kd^3k'}{(2\pi)^6}\left[\sum_{\gamma}\chi_{\gamma}^{*}\left(\frac{\mathbf{k}-\mathbf{k}'}{2}\right)b_{\gamma,\mathbf{k}+\mathbf{k}'}^{\dagger}a_{\mathbf{k}}a_{\mathbf{k}'} + \mathrm{H.c.}\right]. \tag{15}$$

The annihilation and creation operators for fermions (that is for the atoms in the open channel) in plane waves of wave vectors $\mathbf{k}$ and $\mathbf{k}'$ obey the anticommutation relation

$$\{a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}\} = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}') \tag{16}$$

which corresponds to the convention $\langle \mathbf{r}|\mathbf{k}\rangle = e^{i\mathbf{k}\cdot\mathbf{r}}$ for the plane wave. The operator $b_{\gamma,\mathbf{k}}$ annihilates a molecule (in the closed channel), with a center-of-mass momentum $\hbar\mathbf{k}$, in one of the three degenerate internal states γ in the $S_{\mathrm{mol}}=1$ molecular rotational manifold; we take here for γ one of the directions x, y, or z, which amounts to using the chemistry basis $\{|\gamma\rangle\}$, where $|\gamma\rangle$ is an eigenstate of zero angular momentum along direction γ, rather than the standard basis $\{|m=0,\pm 1\rangle\}$. As we mentioned, molecules are treated as bosons so that the b's obey commutation relations

$$[b_{\gamma,\mathbf{k}}, b_{\gamma',\mathbf{k}'}^{\dagger}] = \delta_{\gamma\gamma'}(2\pi)^3\delta(\mathbf{k}-\mathbf{k}'). \tag{17}$$

Also, the $a, a^{\dagger}$ fermionic operators commute with the bosonic ones b and $b^{\dagger}$. In addition to its center of mass kinetic energy, each molecule has an internal energy E_{mol}, defined in the absence of coupling between the open and the closed channels, and counted with respect to the dissociation energy of the open channel.

Whereas the first contribution in the right-hand side of Eq. (15) simply corresponds to noninteracting gases of atoms and molecules, the second contribution describes the coupling between the two species, that is between the open and closed channels, responsible for the p-wave resonance. This interchannel coupling depends on the relative momentum between two atoms through the functions χ_{γ}; here, we are in the case of a p-wave coupling so we take

$$\boldsymbol{\chi}(\mathbf{k}) = \mathbf{k}e^{-k^2b^2/2}, \tag{18}$$

where b is the range in real space of the interchannel coupling, of the order of the radius of the closed channel molecule. The overall amplitude of the interchannel coupling is measured by the coupling constant Λ, taken here to be real; it has not the dimension of an energy, but rather has the same dimension as $\hbar^2b^{1/2}/m$. As we already mentioned, the model is summarized in Fig. 2. It holds at low kinetic energies, below the dissociation limit V_{∞} of the closed channel.

B. Two-body aspects

Before solving the three-body problem, it is important to understand the two-body aspects of the model, in the form of the reduced scattering amplitude $f(k)$ and the related properties of possible dimers, according to the general discussion of Sec. II. We thus calculate the scattering state of two atoms in the center-of-mass frame, that is, for a zero total momentum. The most general state vector is thus a coherent superposition of two atoms (in the open channel) and one molecule (in the closed channel):

$$|\Psi\rangle = \sum_{\gamma}\beta_{\gamma}b_{\gamma,0}^{\dagger}|0\rangle + \int \frac{d^3k}{(2\pi)^3}A(\mathbf{k})a_{\mathbf{k}}^{\dagger}a_{-\mathbf{k}}^{\dagger}|0\rangle. \tag{19}$$

Since the molecule has a zero total momentum, its state is characterized by the three complex amplitudes β_{γ} in each of the internal rotational states $\gamma=x$, y and z. On the contrary, the two atoms can have opposite but arbitrary momenta $\mathbf{k}$ and $-\mathbf{k}$, hence the *a priori* unknown function $A(\mathbf{k})$.

Injecting this ansatz in Schrödinger's equation $(E-H)|\Psi\rangle=0$, and projecting onto the molecular subspace and the atomic subspace, respectively, one finds that Schrödinger's equation is satisfied when A and $\boldsymbol{\beta}$ satisfy

$$(E-E_{\mathrm{mol}})\boldsymbol{\beta}+2\Lambda\int\frac{d^3k}{(2\pi)^3}A(\mathbf{k})\boldsymbol{\chi}^*(\mathbf{k})=0, \tag{20}$$

$$\left(E-\frac{\hbar^2k^2}{m}\right)A(\mathbf{k})+\Lambda\boldsymbol{\beta}\cdot\boldsymbol{\chi}(\mathbf{k})=0. \tag{21}$$

Equation (21) does not specify $A(\mathbf{k})$ in a unique way, for a positive energy E, since $E-\hbar^2k^2/m$ vanishes for some values of $\mathbf{k}$. To obtain the scattering state of two atoms, one takes a more specific form of the ansatz, corresponding to the superposition in the open channel of an incoming wave of wave vector $\mathbf{k}_0$ and a purely outgoing scattered wave

$$A(\mathbf{k})=(2\pi)^3\delta(\mathbf{k}-\mathbf{k}_0)-\Lambda\frac{\boldsymbol{\beta}\cdot\boldsymbol{\chi}(\mathbf{k})}{E+i0^+-\dfrac{\hbar^2k^2}{m}}. \tag{22}$$

Here $E=\hbar^2k_0^2/m\geq 0$ is the total energy of the scattering state.

The general scattering theory [54] relates the scattering state to the incoming state K_{dim} by $|\Psi\rangle=(1+G_0T)|\Psi_0\rangle$ where T is the T matrix and G_0 the resolvent of the noninteracting Hamiltonian. From this identity it is then apparent that the matrix element of the T matrix in Fourier space is related to the numerator of the last term of Eq. (22):

$$\langle\mathbf{k}|T(E+i0^+)|\mathbf{k}_0\rangle=-\Lambda\boldsymbol{\beta}\cdot\boldsymbol{\chi}(\mathbf{k}). \tag{23}$$

From the known relation between the scattering amplitude and the T matrix [54], we get the reduced scattering amplitude

$$f(k_0)=\frac{-mk_0^2e^{-k_0^2b^2}/(4\pi\hbar^2)}{\dfrac{3(E-E_{\mathrm{mol}})}{2\Lambda^2}-\displaystyle\int\frac{d^3k}{(2\pi)^3}\frac{k^2e^{-k^2b^2}}{E+i0^+-\frac{\hbar^2k^2}{m}}}. \tag{24}$$

The choice of the Gaussian envelope in $\boldsymbol{\chi}(\mathbf{k})$ allows an explicit expression for the scattering amplitude. After complexification of k_0 by analytic continuation, setting $k_0=iq_0$, $q_0>0$, we obtain

$$\frac{1}{f(iq_0)}=\frac{4\pi}{q_0^2e^{q_0^2b^2}}\left[-\frac{3\hbar^4}{2m^2\Lambda^2}(q_0^2+mE_{\mathrm{mol}}/\hbar^2)+\int\frac{d^3k}{(2\pi)^3}\frac{k^2e^{-k^2b^2}}{q_0^2+k^2}\right], \tag{25}$$

$$=e^{-q_0^2b^2}\left[\frac{1+q_0^2b^2}{q_0^2\mathcal{V}_s}-\alpha\right]+q_0\,\mathrm{erfc}(q_0b), \tag{26}$$

where erfc is the complementary error function that tends to unity in zero. With this complexification technique, it is straightforward to identify the parameters $\mathcal{V}_s$ and α appearing in the low-k expansion (4) and to get the explicit expressions

$$\frac{1}{\mathcal{V}_s}=\frac{1}{2\pi^{1/2}b^3}-\frac{6\pi\hbar^2}{m\Lambda^2}E_{\mathrm{mol}}, \tag{27}$$

$$\alpha=\frac{b^2}{\mathcal{V}_s}+\alpha_{\mathrm{res}}, \tag{28}$$

$$\alpha_{\mathrm{res}}=\frac{1}{\pi^{1/2}b}+\frac{6\pi\hbar^4}{m^2\Lambda^2}. \tag{29}$$

This illustrates the fact that one can tune $\mathcal{V}_s$ to $-\infty$ or $+\infty$ by shifting the molecular energy E_{mol} (in practice with a magnetic field B) around the value E_{mol}^0 such that the right-hand side of Eq. (27) vanishes, $E_{\mathrm{mol}}-E_{\mathrm{mol}}^0\simeq\mu(B-B_0)$.

We have introduced the convenient quantity α_{res}, which is the value of α exactly on the Feshbach resonance. We see that α_{res} depends on the interchannel coupling Λ, and is bounded from below by the inverse of the potential range, within a numerical factor depending on the details of the model, here $1/\pi^{1/2}$. In principle, α_{res} can take any possible value above this limit, depending on the value of the interchannel coupling Λ; in practice, of course, Λ is not easily tunable so α_{res} is fixed for a given experimental configuration.

By a direct generalization of a well established s-wave terminology, we may classify the p-wave Feshbach resonances as a broad resonance ($\Lambda\gg\hbar^2b^{1/2}/m$): $\alpha_{\mathrm{res}}\sim 1/b$ and a narrow resonance ($\Lambda<\hbar^2b^{1/2}/m$): $\alpha_{\mathrm{res}}\gg 1/b$. We recall that this terminology can be motivated as follows: If one assumes that E_{mol} is an affine function of the magnetic field B with a slope μ, and that $\mathcal{V}_s=\mathcal{V}_s^{\mathrm{bg}}[1-\Delta B/(B-B_0)]$ in a more complete theory including the fact that $\mathcal{V}_s$ takes a finite value $\mathcal{V}_s^{\mathrm{bg}}$ far from the resonance (due to the direct interaction in the open channel, neglected here) and presumably of the order of b^3, one finds a resonance width

$$\mu\Delta B=\frac{m\Lambda^2}{6\pi\hbar^2\mathcal{V}_s^{\mathrm{bg}}}. \tag{30}$$

It remains to compare this resonance width to the "natural" energy scale $\hbar^2/mb^2$ to obtain the abovementioned terminology.

The last point to discuss for the two-body problem is the existence or not of a two-body bound state in the open channel. We shall refer to such a bound state as a dimer, in order not to confuse it with the molecular state in the closed channel. Mathematically, such a dimer is a zero of $1/f(iq_0)$ with $q_0>0$. The expression in between square brackets in the right-hand side Eq. (25) is a decreasing function of q_0 that tends to $-\infty$ for $q_0\to+\infty$. Hence there exists at most one dimer in our model Hamiltonian. There exists one if and only if the expression between square brackets is positive in $q_0=0$, that is if and only if $\mathcal{V}_s>0$ [55].

When a dimer is present, one can express analytically its wave function $\phi(\mathbf{r})$ in the open channel, in terms of exponential and erfc functions, and one can calculate the occupation probability of the closed channel, $p_{\mathrm{closed}}=|\boldsymbol{\beta}|^2$ after proper normalization of $|\Psi\rangle$ in the center-of-mass frame [56]

$$|\boldsymbol{\beta}|^2+2\int\frac{d^3k}{(2\pi)^3}|A(\mathbf{k})|^2=1. \tag{31}$$

An equivalent way to obtain $|\boldsymbol{\beta}|^2$ is to calculate the large r behavior of $\phi(\mathbf{r})$, which is proportional to β and which is

related to the normalization factor $\mathcal{N}$ in Eq. (7), and then to use the general relation Eq. (9) [57]. Both ways lead to the expression

$$\frac{1}{p_{\text{closed}}} = \frac{m^2\Lambda^2}{6\pi\hbar^4}\frac{e^{q_{\text{dim}}^2 b^2}}{|\mathcal{N}|^2}. \tag{32}$$

The value of p_{closed} for an infinite scattering volume can be cast in the very simple forms

$$p_{\text{closed}}^{\text{res}} = \frac{6\pi\hbar^4}{m^2\Lambda^2}\alpha_{\text{res}}^{-1}, \tag{33}$$

$$= 1 - \frac{1}{\pi^{1/2}\alpha_{\text{res}} b}. \tag{34}$$

The expression (33) is quite remarkable since it is "universal:" It does not involve the interaction range b and, as we have checked, it is not specific to the choice of a Gaussian cutoff function in $\boldsymbol{\chi}(\mathbf{k})$. It was already derived in Ref. [30], see the unnumbered equation following Eq. (9) of that reference. In the vicinity of the resonance, we see on the expression (34) that, in the dimer wave function, the closed channel is strongly occupied for a narrow resonance and is weakly occupied for a broad resonance; p_{closed} tends to zero in the broad resonance limit.

To conclude this review of the two-body aspects, we point out a striking property of the dimer, very different from the usual s-wave case: In the limit $\mathcal{V}_s/b^3 \to +\infty$, we find that the dimer wave function $\phi(\mathbf{r})$ has a well defined, nonzero limit, tending to zero as $O(1/r^2)$ at large r. This can be directly seen in momentum space: for $q_{\text{dim}}=0^+$, the function $A(\mathbf{k})$ is $O(1/k)$ at low k, which is indeed square integrable around the origin $\mathbf{k}=0$. In other words, at the threshold for the formation of the dimer, the dimer wave function is a well-defined nonzero and square integrable function.

IV. SOLUTION OF THE THREE-BODY PROBLEM

This is the central section, where we solve the three-body problem within the two-channel model close to a p-wave resonance. The mathematical structure of the model, with a single molecular state occupied in the closed channel and no interaction potential in the open channel, is such that the three-body problem is amenable to an integral equation for a one-body "wave function." This integral equation becomes easily solvable numerically if one further uses the rotational symmetry of the Hamiltonian. We then obtain predictions for three physical situations, (i) the existence of three-body bound states, that is, of trimers, (ii) the scattering of an atom on a dimer, and (iii) the scattering of three atoms, leading to recombination processes, that is to the formation of a weakly bound dimer and a free atom.

A. Derivation of an integral equation

We start with the most general ansatz for the three-body problem in the center-of-mass frame, that is, for a zero total momentum. Because of the conversion of pairs of atoms into molecules and vice versa, the three-body ansatz is a coherent superposition of three fermions (all three atoms in the open channel) and of one molecule plus one fermion (one atom in the open channel and two atoms tightly bound in a molecule in the closed channel):

$$|\Psi\rangle = \int \frac{d^3K}{(2\pi)^3}\sum_\gamma \beta_\gamma(\mathbf{K}) b^\dagger_{\gamma,\mathbf{K}} a^\dagger_{-\mathbf{K}}|0\rangle + \int \frac{d^3k d^3K}{(2\pi)^6} A(\mathbf{K},\mathbf{k}) a^\dagger_{1/2\mathbf{K}+\mathbf{k}} a^\dagger_{1/2\mathbf{K}-\mathbf{k}} a^\dagger_{-\mathbf{K}}|0\rangle. \tag{35}$$

The one molecule plus one fermion part is parametrized by three one-body wave functions β_γ, here in Fourier space; we shall derive an integral equation for them. The three fermion part A can be parametrized by two Jacobi-like coordinates in momentum space since the total momentum is zero. For pure convenience, we impose that $A(\mathbf{K},\mathbf{k})$ is an odd function of $\mathbf{k}$, to reduce the number of terms involving A in the integral equation for $\boldsymbol{\beta}$.

We inject the general ansatz for $|\Psi\rangle$ in Schrödinger's equation $(E-H)|\Psi\rangle=0$, where the total energy E is at this stage of arbitrary sign. Projecting Schrödinger's equation on the subspace with one molecule and one fermion gives an equation for $\boldsymbol{\beta}$ with the function A appearing in a source term:

$$\left[E - E_{\text{mol}} - \frac{3\hbar^2K^2}{4m}\right]\boldsymbol{\beta}(\mathbf{K}) + 2\Lambda\int\frac{d^3k}{(2\pi)^3}\boldsymbol{\chi}^*(\mathbf{k})\left[A(\mathbf{K},\mathbf{k}) + 2A\left(-\frac{1}{2}\mathbf{K}+\mathbf{k}, -\frac{3}{4}\mathbf{K}-\frac{1}{2}\mathbf{k}\right)\right] = 0. \tag{36}$$

Projecting Schrödinger's equation on the subspace with three fermions leads to

$$\int\frac{d^3Kd^3k}{(2\pi)^6}\left\{\left[E - \frac{\hbar^2}{m}\left(\frac{3}{4}K^2+k^2\right)\right]A(\mathbf{K},\mathbf{k}) + \Lambda\boldsymbol{\beta}(\mathbf{K})\cdot\boldsymbol{\chi}(\mathbf{k})\right\}a^\dagger_{1/2\mathbf{K}+\mathbf{k}}a^\dagger_{1/2\mathbf{K}-\mathbf{k}}a^\dagger_{-\mathbf{K}}|0\rangle = 0. \tag{37}$$

This equation is satisfied for the choice

$$A(\mathbf{K},\mathbf{k}) = A_0(\mathbf{K},\mathbf{k}) - \frac{\Lambda\boldsymbol{\beta}(\mathbf{K})\cdot\boldsymbol{\chi}(\mathbf{k})}{E + i0^+ - \frac{\hbar^2}{m}\left(\frac{3}{4}K^2+k^2\right)}. \tag{38}$$

For a positive total energy $E>0$: A_0 represents a possible incoming wave of three free atoms, and it is an eigenstate of the kinetic energy operator in the center-of-mass frame with energy E; in presence of such an incoming free wave, the second term in A represents the scattered wave in the open channel, which is guaranteed to be outgoing by the standard substitution $E\to E+i0^+$. As we have imposed the convention that $A(\mathbf{K},\mathbf{k})$ should be an odd function of $\mathbf{k}$, one has to apply the same convention to $A_0(\mathbf{K},\mathbf{k})$; note that the last term of Eq. (38) is automatically an odd function of $\mathbf{k}$, since $\chi(\mathbf{k})$ is. For a negative total energy $E<0$, the expression between square brackets in Eq. (37) cannot vanish, $A_0\equiv 0$ and the $+i0^+$ in the denominator can be omitted.

Injecting Eq. (38) in Eq. (36), we obtain an integral equation for $\boldsymbol{\beta}$. The term $A(\mathbf{K},\mathbf{k})$ of Eq. (36) gives a contribution

simply proportional to $\boldsymbol{\beta}(\mathbf{K})$, with a K-dependent factor; collecting it with the factor in between square brackets in the first term of Eq. (36) gives a K-dependent expression that can be recognized as being proportional (with a K-dependent factor) to the inverse of the scattering amplitude of two atoms at the energy

$$E_{\mathrm{rel}} = E - \frac{3\hbar^2 K^2}{4m} \equiv \frac{\hbar^2 k_{\mathrm{rel}}^2}{m}, \tag{39}$$

with the determination $k_{\mathrm{rel}} \geq 0$ for $E_{\mathrm{rel}} \geq 0$ and $k_{\mathrm{rel}}/i > 0$ for $E_{\mathrm{rel}} < 0$. This relation can be seen as a consequence of the Jacobi-like parametrization of the momenta of the three fermions that we have used in Eq. (35): if three free fermions of total energy E have momenta $\pm\mathbf{k}_{\mathrm{rel}} + \mathbf{K}/2$ and $-\mathbf{K}$, then the modulus k_{rel} will obey Eq. (39).

We finally obtain the general integral equation for $\beta_\gamma(\mathbf{K})$:

$$\begin{aligned}
&\frac{k_{\mathrm{rel}}^2 e^{-k_{\mathrm{rel}}^2 b^2}}{3f(k_{\mathrm{rel}})}\boldsymbol{\beta}(\mathbf{K}) + 8\pi \int \frac{d^3k}{(2\pi)^3}\boldsymbol{\chi}^*\left(\frac{1}{2}\mathbf{K} + \mathbf{k}\right) \\
&\quad \times \frac{\boldsymbol{\beta}(\mathbf{k}) \cdot \boldsymbol{\chi}\left(\mathbf{K} + \frac{1}{2}\mathbf{k}\right)}{K^2 + k^2 + \mathbf{K}\cdot\mathbf{k} - m(E + i0^+)/\hbar^2} \\
&= \frac{4\pi\hbar^2}{m\Lambda} \int \frac{d^3k}{(2\pi)^3}\boldsymbol{\chi}^*(\mathbf{k}) \\
&\quad \times \left[A_0(\mathbf{K},\mathbf{k}) + 2A_0\left(-\frac{1}{2}\mathbf{K} + \mathbf{k}, -\frac{3}{4}\mathbf{K} - \frac{1}{2}\mathbf{k}\right)\right].
\end{aligned} \tag{40}$$

In what follows we shall solve this integral equation for various physical situations. (i) In the search for trimers, one assumes an energy E below zero and below the dimer energy (if there exits a dimer); then $A_0 \equiv 0$ and $\boldsymbol{\beta}(\mathbf{K})$ is not subjected to any specific boundary condition. (ii) In the low-energy scattering of an atom on a dimer, the energy is above the dimer energy but still negative; then $A_0 \equiv 0$ and one has to introduce a specific ansatz for $\boldsymbol{\beta}(\mathbf{K})$ to enforce the boundary conditions corresponding to such a scattering experiment. (iii) In the scattering of three incoming atoms, the total energy is now non-negative so that $A_0 \neq 0$; we shall assume that this scattering experiment is performed for $\mathcal{V}_s > 0$ so that there exists a dimer in the two-body problem, which can be formed by a recombination event in the three-body scattering; then one introduces an ansatz for $\boldsymbol{\beta}(\mathbf{K})$ describing the presence of a purely outgoing wave of such a dimer (with an opposite momentum atom).

B. Symmetry sectors from rotational and parity invariance

Formally Eq. (40) is an equation for a spinor $\boldsymbol{\beta}(\mathbf{K})$, with an internal spin $S_{\mathrm{mol}} = 1$; here this internal spin corresponds to the rotational degrees of freedom of the molecule (in the closed channel); the orbital variable $\mathbf{K}$ here corresponds to the relative atom-molecule momentum. The homogeneous part of Eq. (40) is invariant by a simultaneous rotation of the spin and orbital variables of the spinor. The total momentum J, obtained by addition of the spin S_{mol} and the orbital angular momentum L, is therefore a good quantum number. In this paper, we shall restrict to the manifold $J=1$, which can be obtained from $L=0$ plus $S_{\mathrm{mol}}=1$, or $L=1$ plus $S_{\mathrm{mol}}=1$, or $L=2$ plus $S_{\mathrm{mol}}=1$. In addition, the homogeneous part of Eq. (40) is invariant by parity (combining the parity on the spin variables and on the orbital variables). This decouples the $J=1$ manifold in two sectors, the even sector $L=1$ plus $S_{\mathrm{mol}}=1$ and the odd sector $L=0$ plus $S_{\mathrm{mol}}=1$ and $L=2$ plus $S_{\mathrm{mol}}=1$

Applying the standard algebra of addition of angular momenta, we obtain the following ansatz in the odd sector:

$$\boldsymbol{\beta}(\mathbf{K}) = B_{L=0}(K)\mathbf{e}_z - B_{L=2}(K)\frac{\mathbf{K}\cdot\mathbf{e}_z}{K^2}\mathbf{K}, \tag{41}$$

where $\mathbf{e}_z$ is the unit vector along z axis. This ansatz corresponds to a total angular momentum $J=1$ with vanishing angular momentum component along z, $m_J = 0$. Considering the other components $m_J = \pm 1$, or equivalently the states with vanishing angular momentum component along x and along y, respectively, would lead to equivalent results, as guaranteed by the rotational invariance of the Hamiltonian.

Similarly, we take as ansatz in the even sector

$$\boldsymbol{\beta}(\mathbf{K}) = \frac{B_{L=1}(K)}{K}\mathbf{K} \wedge \mathbf{e}_x = \frac{B_{L=1}(K)}{K}[(\mathbf{K}\cdot\mathbf{e}_z)\mathbf{e}_y - (\mathbf{K}\cdot\mathbf{e}_y)\mathbf{e}_z], \tag{42}$$

which corresponds to the even state with $J=1$ and vanishing angular momentum component along x axis. After some calculations Eq. (40) can be turned into an integral equation for $B_{L=1}$ (in the even sector) or into coupled integral equations for $B_{L=0}$ and $B_{L=2}$ (in the odd sector), as detailed in Appendix A. The remaining unknown functions depend on a single real variable K so that a numerical solution is reasonable.

C. Existence of weakly bound trimers

We investigate here the existence of three-body bound states, that is of trimers, in our model Hamiltonian. These trimers have, of course, a negative total energy E. If one is on the $\mathcal{V}_s > 0$ side of the resonance, where a dimer of energy $-E_{\mathrm{dim}}$ exists, one further has $E < -E_{\mathrm{dim}}$ to have stability of the trimers with respect to dissociation into an atom and a dimer; if this condition was not satisfied, the trimers would not exist as true stationary states but would rather be resonances in the atom-dimer scattering process.

These constraints on the energy have the following mathematical consequences. Since $E<0$, the source term A_0 in Eq. (38) is identically zero, so that Eq. (40) becomes homogeneous. Since $E < -E_{\mathrm{dim}}$, on the side $\mathcal{V}_s > 0$ of the resonance, the scattering amplitude $f(k_{\mathrm{rel}})$ in the denominator of the first term of Eq. (40) is nonzero for all $\mathbf{K}$ and the linear operator $L(E)$ representing the integral equation has a smooth action over the class of regular $\boldsymbol{\beta}(\mathbf{K})$ functions. Numerically, one can then discretize the variable K with no particular care, and approximate $L(E)$ by a matrix. The existence of a trimer corresponds to a non-zero-dimensional kernel of the operator $L(E)$; in practice, we look for the values of E such that the approximating matrix has a vanishing eigenvalue. The explicit form of $L(E)$ for the ansatz in the

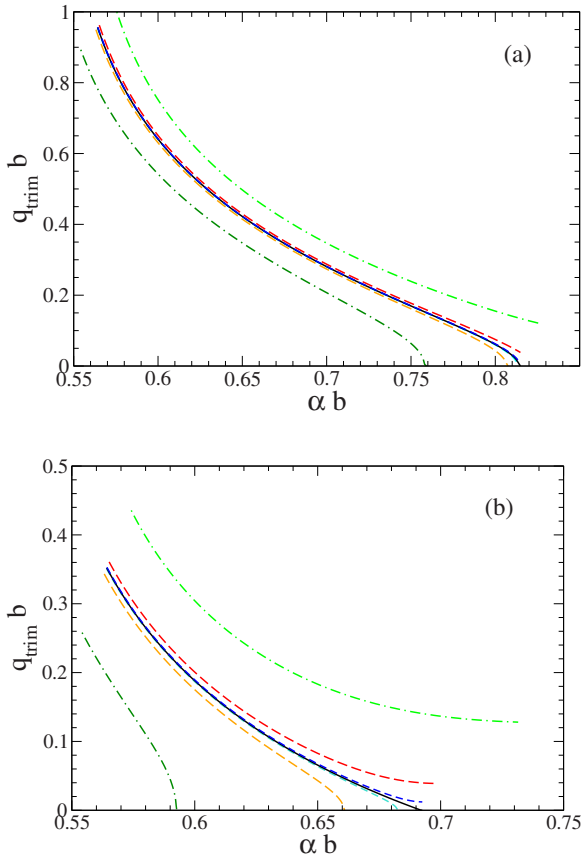

FIG. 3. (Color online) For fixed values of the scattering volume $\mathcal{V}_s$, parameter q_{trim} of the trimer (when it exists) as a function of αb; q_{trim} is related to the negative energy $-E_{\mathrm{trim}}$ of the trimer by $E_{\mathrm{trim}}=\hbar^2 q_{\mathrm{trim}}^2/m$. (a) Even sector, (b) odd sector, as defined in Sec. III B. Solid line (black): $|\mathcal{V}_s|/b^3=\infty$. Above the solid line, positive values of $\mathcal{V}_s$: short dashed line (blue): $\mathcal{V}_s=10^4 b^3$; dashed line (red): $\mathcal{V}_s=10^3 b^3$; dashed-dotted (green): $\mathcal{V}_s=100 b^3$. Below the solid line, negative values of $\mathcal{V}_s$: short dashed line (light blue): $\mathcal{V}_s=-10^4 b^3$; dashed line (orange): $\mathcal{V}_s=-10^3 b^3$; dashed-dotted (dark green): $\mathcal{V}_s=-100 b^3$. At the threshold for the existence of the trimer as a true bound state, on the $\mathcal{V}_s>0$ side of the resonance, where a dimer exists, the trimer binding energy vanishes, so that the energy of the trimer coincides with the one of the dimer, and $q_{\mathrm{trim}}=q_{\mathrm{dim}}$ (see text).

even and odd sector can be deduced from Appendix A. In the same appendix, it is also explained how to normalize the state vector of the trimer.

For values of $|\mathcal{V}_s|\gg b^3$, we have found either zero or one trimer in each symmetry sector (with threefold rotational degeneracy when the trimer exists). The energy of the trimer is written as $-\hbar^2 q_{\mathrm{trim}}^2/m$. Then q_{trim} as a function of αb is given in Fig. 3, for the even and the odd sectors. We found no evidence of Efimov effect: in each symmetry sector, we found at most one trimer, and there is no oscillation of the $\boldsymbol{\beta}(\mathbf{K})$ with K as a function of K, see Fig. 4.

We note that, in real experiments with atoms, these trimers may acquire a finite lifetime, due to the formation of deeply bound dimers by three-body collisions. This process is not contained in our Hamiltonian, since H does not support deeply bound dimers for $|\mathcal{V}_s|\gg b^3$; its rate is estimated by a simple recipe in Sec. V A.

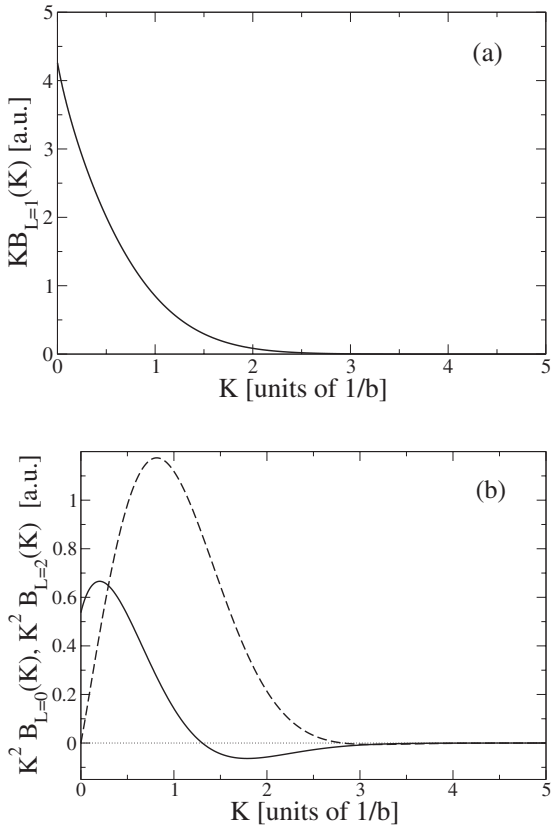

FIG. 4. For $\mathcal{V}_s=\infty$, and $\alpha_{\mathrm{res}}=\alpha_{\mathrm{th}}$ [right on the thresholds for the existence of a trimer, see Eqs. (54) and (92)], K dependence of the functions (a) $B_{L=1}$ (even sector), (b) $B_{L=0}$ (solid line), $B_{L=2}$ (dashed line) (odd sector), for the trimers. To avoid diverging functions, these functions were multiplied by K in (a) and by K^2 in (b). The normalization is arbitrary.

D. Atom-dimer scattering

We consider here the scattering problem of an atom on a dimer, which corresponds to the positive $\mathcal{V}_s$ side of the resonance and to a total energy $E\geq -E_{\mathrm{dim}}$. For simplicity, we restrict to the low-energy limit of this scattering, with a relative kinetic energy of the incoming atom and the dimer much smaller than the binding energy of the dimer

$$E+E_{\mathrm{dim}}\ll E_{\mathrm{dim}}. \tag{43}$$

As a consequence, the total energy is negative, so that energy conservation prevents the dimer from being dissociated by the interaction with the incoming atom and the scattering is elastic. Furthermore, a multipolar expansion can be performed in terms of the atom-dimer relative orbital momentum. In the mathematical limit of a vanishing kinetic energy, the atom-dimer incoming wave is a s wave and the scattering is characterized by the atom-dimer scattering length a_{ad} that we shall calculate. To next order of the multipolar expansion the atom-dimer incoming wave is a p wave and we shall calculate a corresponding atom-dimer scattering volume $\mathcal{V}_s^{\mathrm{ad}}$.

The property of elastic scattering at $E<0$ rigorously holds for the model Hamiltonian (15), since we have shown that it admits at most one dimer state (with rotational degeneracy). Reality with atoms goes beyond this model Hamiltonian:

there exist in general deeply bound dimers, which can make the atom-dimer scattering inelastic even at arbitrarily low relative kinetic energy. The corresponding three-body loss rate is estimated in Sec. V B.

S-wave atom-dimer scattering. Since we have assumed a negative total energy $E<0$, there cannot be a free incoming three-atom state so that $A_0 \equiv 0$ in Eq. (38). In the center-of-mass frame, the incoming state is an atom impinging on a dimer with vanishing kinetic energy; in the subspace with one atom and one molecule in the closed channel, this corresponds to a relative orbital angular momentum $L=0$, that is to a total momentum $J=1$ since the molecule is of spin unity. According to Sec. IV B, the incoming state is in the odd sector. Mathematically, this scattering experiment corresponds to the following splitting for $\boldsymbol{\beta}$, into the sum of an incoming wave of zero momentum (a δ distribution in $\mathbf{K}$ space) and a scattered wave which is a regular function of K:

$$\boldsymbol{\beta}(\mathbf{K}) = (2\pi)^3 \delta(\mathbf{K}) p_{\text{closed}}^{1/2} \mathbf{e}_z + \boldsymbol{\beta}^{\text{out}}(\mathbf{K}). \tag{44}$$

This is of the form Eq. (41), the δ being in the $L=0$ sector. Note that the incoming dimer has a probability p_{closed} to be in the closed channel, so that the amplitude of the incoming wave for $\boldsymbol{\beta}$, that is in the subspace of one atom and one molecule, includes a factor $p_{\text{closed}}^{1/2}$.

In practice, one injects the form of Eq. (44) into Eq. (40). The $\delta(\mathbf{K})$ term gives a zero contribution in the first term of the left-hand side, since $\mathbf{K}=0$ and $E=-E_{\text{dim}}$ leads to $k_{\text{rel}} = iq_{\text{dim}}$ and $1/f(k_{\text{rel}})=0$. The $\delta(\mathbf{k})$ inserted in the second term of the left-hand side of Eq. (40), that is the integral term, produces a smooth source term in the left-hand side

$$\mathbf{T}(\mathbf{K}) = 4\pi p_{\text{closed}}^{1/2} (\mathbf{e}_z \cdot \mathbf{K}) \mathbf{K} \frac{e^{-5b^2K^2/8}}{K^2 + mE_{\text{dim}}/\hbar^2}. \tag{45}$$

One is left with a linear and inhomogeneous system for the vectorial function $\boldsymbol{\beta}^{\text{out}}(\mathbf{K})$, which is then taken of the form (41), with coefficients $B_{L=0}^{\text{out}}(K)$ and $B_{L=2}^{\text{out}}(K)$. The explicit form of the resulting system is derived in Appendix A, and we obtain

$$\begin{aligned} D(K) \begin{pmatrix} B_{L=0}^{\text{out}}(K) \\ B_{L=2}^{\text{out}}(K) \end{pmatrix} + \frac{4}{\pi} \int_0^{+\infty} dk k^2 e^{-5(K^2+k^2)b^2/8} \\ \times M(K,k) \begin{pmatrix} B_{L=0}^{\text{out}}(k) \\ B_{L=2}^{\text{out}}(k) \end{pmatrix} \\ = 4\pi p_{\text{closed}}^{1/2} \frac{K^2 e^{-5b^2K^2/8}}{K^2 + q_{\text{dim}}^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{46}$$

where we have introduced the diagonal part

$$D(K) = \frac{k_{\text{rel}}^2 e^{-k_{\text{rel}}^2 b^2}}{3f(k_{\text{rel}})} \tag{47}$$

and the two by two matrix $M(K,k)$ is given in Appendix A.

Let us start with an intuitive presentation of the results. We expect that, at low K, the scattered wave in the $L=0$ channel diverges as $1/K^2$, so that we set

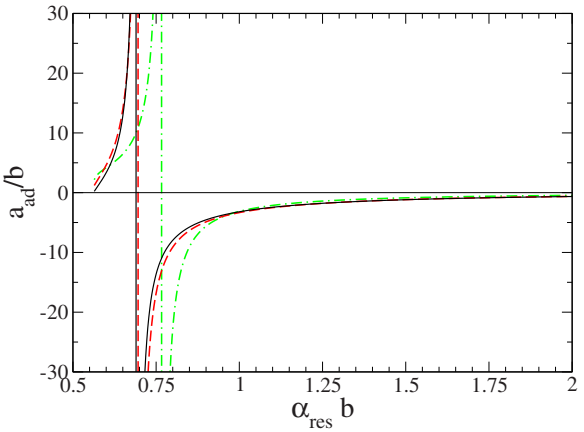

FIG. 5. (Color online) Atom-dimer scattering length a_{ad} as a function of α_{res} for fixed values of the scattering volume $\mathcal{V}_s$, $\mathcal{V}_s = 10^6 b^3$ (black solid line), $\mathcal{V}_s = 10^3 b^3$ (red dashed line), $\mathcal{V}_s = 10 b^3$ (green dotted-dashed line). The divergence of a_{ad} coincides with the threshold of existence of a trimer in the odd sector. In the limit of a broad Feshbach resonance $\alpha_{\text{res}} b \to 1/\pi^{1/2}$, a_{ad} tends to $\approx 0.2b$.

$$B_{L=0}^{\text{out}}(K) \sim -p_{\text{closed}}^{1/2} \frac{4\pi}{K^2} a_{\text{ad}}. \tag{48}$$

In position space this indeed corresponds to the large r behavior $1 - a_{\text{ad}}/r$, where r is the distance between the molecule and the atom, so that a_{ad} is indeed the atom-dimer scattering length. In the channel $L=2$, the outgoing wave is expected to scale as $1/r^3$ at large r, because of the centrifugal barrier; this corresponds to $B_{L=2}(K)$ having a finite limit in $K=0$.

What typical values of a_{ad} can we expect? For $\mathcal{V}_s > 0$ and much larger than b^3, the scattering amplitude of two atoms has a modulus $\leq 1/\alpha_{\text{res}}$, which is a small value at most of the order of b. For $k \simeq q_{\text{dim}} \ll \alpha_{\text{res}}$, one finds that $|f(k)| \simeq 1/\alpha_{\text{res}}$. One may then expect intuitively that a_{ad} weakly depends on $\mathcal{V}_s$, and is at most of the order of $1/\alpha_{\text{res}}$, that is at most $\simeq b$. This expectation is correct, see Fig. 5, except close to the threshold for the existence of a trimer in the odd sector, where a_{ad} diverges.

We now turn to a more rigorous analysis of the integral (46). The key ingredient is the low-K behavior of the various coefficients for $q_{\text{dim}} > 0$. Consider the diagonal term $D(K)$. As we have already mentioned, $D(K)$ vanishes in $K=0$; since here $k_{\text{rel}} = i(q_{\text{dim}}^2 + 3K^2/4)^{1/2}$, we see that k_{rel} is an expandable function of K which varies to second order in K. The same conclusion holds for $D(K)$, which therefore vanishes quadratically in $K=0$; in the limit $q_{\text{dim}} b \ll 1$ we find the simple result

$$\lim_{K \to 0} \frac{D(K)}{K^2} \simeq \frac{\alpha_{\text{res}}}{4}. \tag{49}$$

Consider next the coefficients of the matrix $M(K,k)$. From the explicit expressions given in Appendix A, we obtain for a fixed k:

$$\lim_{K \to 0} M(K,k) = \frac{k^2/6}{k^2 + q_{\text{dim}}^2} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}. \tag{50}$$

Assuming that the functions $k^2 B_L^{\text{out}}(k)$ are bounded, we find that $D(K)B_{L=0}^{\text{out}}(K)$ has a finite limit in $K=0$, obeying the exact relation

$$\lim_{K\to 0} D(K)B_{L=0}^{\text{out}}(K) = -\frac{2}{3\pi}\int_0^{+\infty} dk \frac{k^4 e^{-5/8b^2k^2}}{k^2+q_{\text{dim}}^2}[B_{L=0}^{\text{out}} - B_{L=2}^{\text{out}}](k). \tag{51}$$

On the contrary, we find that the second line of the matrix $M(K,k)$ vanishes quadratically for $K\to 0$, and the source term also vanishes quadratically in K, so that $B_{L=2}^{\text{out}}(K)$ has indeed a finite limit in $K=0$.

The existence of a well defined limit for a_{ad} in the large scattering volume can also be argued in simple terms. All the coefficients in the integral equation (46) have a well-defined limit for $q_{\text{dim}}\to 0$. In particular, the diagonal term in this limit assumes the simple form

$$\lim_{\mathcal{V}_s/b^3\to+\infty} D(K) = \frac{K^2}{4}[\alpha_{\text{res}} - h(K)], \tag{52}$$

where $h(K)=qe^{q^2b^2}\,\text{erfc}(qb)$, with $q=\sqrt{3}K/2$, varies monotonically from zero to $1/(\pi^{1/2}b)$; since $\alpha_{\text{res}} > 1/(\pi^{1/2}b)$, the expression in between square brackets cannot vanish. Taking as new functions $G_0(K)=K^2B_{L=0}^{\text{out}}(K)$ and $G_2(K)=K^2B_{L=2}^{\text{out}}(K)$, one faces for $\mathcal{V}_s\to+\infty$ an integral equation of the form

$$\frac{\alpha_{\text{res}}}{4}\mathbf{G}(K) - \mathbf{O}[\mathbf{G}] = \mathbf{S}(K), \tag{53}$$

where the source term is the infinite $\mathcal{V}_s$ limit of the right hand side of Eq. (46), and $\mathbf{O}$ is a bounded operator depending on b but not on α_{res}. The value $G_0(0)$ is finite for $\mathcal{V}_s=+\infty$, so is the atom-dimer scattering length. As shown in Appendix A, a simple transformation can make the operator $\mathbf{O}$ Hermitian; numerically, one finds that the positive part of the spectrum of $\mathbf{O}$ consists of a continuum extending from zero to $1/(4\pi^{1/2}b)$, and of a discrete state of energy above the continuum. We see that $\alpha_{\text{res}}/4$ cannot match an eigenvalue of the continuum, but can indeed match the discrete eigenvalue, for

$$\alpha_{\text{th}}^{\text{odd}} \simeq 0.69208/b. \tag{54}$$

This particular value of α_{res} corresponds to the threshold for the formation of an odd trimer at $\mathcal{V}_s=\infty$, and the corresponding eigenvector was plotted in Fig. 4(a). For α_{res} close to the threshold value, the solution of Eq. (53) acquires a diverging component on this eigenvector; since the eigenvector has a value $G_0(0)\neq 0$ in $K=0$, this leads to an atom-dimer scattering length a_{ad} diverging as $1/(\alpha_{\text{res}}-\alpha_{\text{th}}^{\text{odd}})$.

The writing of Eq. (53) also makes it clear that asymptotic expressions can be obtained in the narrow resonance limit $\alpha_{\text{res}}b \gg 1$: in this limit, the term proportional to α_{res} dominates over the bounded operator $\mathbf{O}$, which can thus be treated as a perturbation. To leading order, $(G_0(K),G_2(K)) = 4\mathbf{S}(K)/\alpha_{\text{res}}$, which, injected into Eq. (51), gives the asymptotic equivalent

$$a_{\text{ad}} \overset{\alpha_{\text{res}}b\gg 1}{\sim} -\frac{32}{3(5\pi)^{1/2}\alpha_{\text{res}}^2 b} \tag{55}$$

valid in the limit of large $\mathcal{V}_s/b^3$ and large $\alpha_{\text{res}}b$. We have checked that this relation is obeyed by the numerical results. It is important physically to point out that, as we shall see in Sec. VI, this asymptotic result no longer holds in the presence of direct interaction between atoms in the open channel. Anyway, it clearly shows that a_{ad} depends not only on the effective range parameter α_{res} but also on the range b, which is sensitive to the microscopic details of the model interaction. In this sense, the large scattering volume limit of a_{ad} is not a "universal" quantity.

This differs from the bosonic case on a narrow Feshbach resonance, where the atom-dimer scattering length is a function of the scattering length a and the effective range r_e only, as soon as a greatly exceeds the range of the potential; furthermore, this function is not bounded in the large a limit, but rather exhibits, on top of an overall linear growth with a, a series of divergences for values of a/r_e corresponding to a threshold for the formation of an Efimov trimer [58].

P-wave atom-dimer scattering. We now assume that the incoming atom-dimer relative wave is a p wave, that is it has a unit orbital momentum $L=1$. In the subspace with one atom and one closed-channel molecule, the corresponding orbital wave function is obtained in momentum space from the low-K_0 expansion of the Dirac distribution corresponding to a molecule of wave vector $\mathbf{K}_0$ impinging on an atom of wave vector $-\mathbf{K}_0$:

$$(2\pi)^3\delta(\mathbf{K}-\mathbf{K}_0) = (2\pi)^3[\delta(\mathbf{K}) - \mathbf{K}_0\cdot(\text{grad}\,\delta)(\mathbf{K}) + \cdots], \tag{56}$$

and one may take exactly $E=-E_{\text{dim}}$ at this order. Since the molecule has a spin $S_{\text{mol}}=1$ this may correspond to a total spin $J=0$, 1, or 2. The present work is restricted to a total spin $J=1$, and the corresponding ansatz turns out to be in the even sector

$$\boldsymbol{\beta}(\mathbf{K}) = (2\pi)^3(-K_0)p_{\text{closed}}^{1/2}\{[\mathbf{e}_z\cdot(\text{grad}\,\delta)(\mathbf{K})]\mathbf{e}_y - [\mathbf{e}_y\cdot(\text{grad}\,\delta)(\mathbf{K})]\mathbf{e}_z\} + \frac{B_{L=1}^{\text{out}}(K)}{K} \times[(\mathbf{K}\cdot\mathbf{e}_z)\mathbf{e}_y - (\mathbf{K}\cdot\mathbf{e}_y)\mathbf{e}_z]. \tag{57}$$

We insert this ansatz in the integral equation (40), keeping in mind that here $A_0\equiv 0$. The part of the ansatz involving the gradient of the Dirac distribution gives a vanishing contribution in the diagonal term of the equation [since $1/f(k_{\text{rel}})$ vanishes quadratically in $K=0$ for the total energy $E=-E_{\text{dim}}$], but gives a nonzero, smooth contribution in the integral term, serving as a source term for the scattered wave $B_{L=1}^{\text{out}}$. Performing the angular average as detailed in Appendix A we obtain

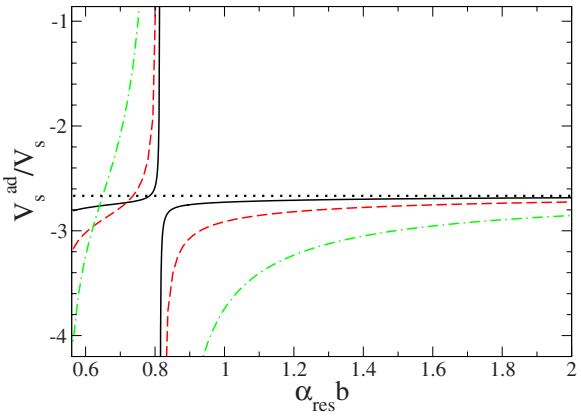

FIG. 6. (Color online) Atom-dimer scattering volume $\mathcal{V}_s^{\mathrm{ad}}$ for a total spin $J=1$ (see text) as a function of $\alpha_{\mathrm{res}}b$, for a fixed value of the atom-atom scattering volume $\mathcal{V}_s/b^3=100$ (dashed-dotted green line), $\mathcal{V}_s/b^3=10^3$ (dashed red line), $\mathcal{V}_s/b^3=10^4$ (solid black line). To reveal the scaling of $\mathcal{V}_s^{\mathrm{ad}}$ with $\mathcal{V}_s$ close to the Feshbach resonance, $\mathcal{V}_s^{\mathrm{ad}}$ is expressed in units of $\mathcal{V}_s$. Dotted horizontal line: analytical prediction (66) in the limit $\mathcal{V}_s\to+\infty$.

$$D(K)B_{L=1}^{\mathrm{out}}(K)-\frac{2}{\pi}\int_0^{+\infty}dkk^2[C_0(K,k)-C_2(K,k)]\times B_{L=1}^{\mathrm{out}}(k)e^{-5(K^2+k^2)b^2/8}=(-K_0)p_{\mathrm{closed}}^{1/2}\frac{(-8\pi K)}{K^2+q_{\mathrm{dim}}^2},\tag{58}$$

where the functions C_0 and C_2 are defined in Appendix A taking $q=q_{\mathrm{dim}}$, and $D(K)$ is given by Eq. (47).

The analysis performed for the atom-dimer s-wave scattering is readily extended to the present p-wave scattering. Since the inhomogeneous term in the right-hand side of Eq. (58) vanishes linearly in $K=0$ and the diagonal part $D(K)$ vanishes quadratically, $B_{L=1}^{\mathrm{out}}(K)$ diverges as $1/K$. Such a low-K behavior was expected: From Eq. (1) expanded to first order in the incoming wave vector, here called $\mathbf{K}_0$ rather than $\mathbf{k}$, one obtains for the wave function at large distances

$$\psi_{\mathbf{K}_0}(\mathbf{r})\simeq i\mathbf{K}_0\cdot\mathbf{r}\left[1-\frac{3\mathcal{V}_s^{\mathrm{ad}}}{r^3}\right].\tag{59}$$

Taking the Fourier transform with respect to the relative atom-molecule coordinates $\mathbf{r}=\mathbf{r}_{\mathrm{mol}}-\mathbf{r}_{\mathrm{at}}$ leads to the low-K behavior

$$\tilde{\psi}_{\mathbf{K}_0}(\mathbf{K})\simeq-(2\pi)^3(\mathbf{K}_0\cdot\mathrm{grad})\delta(\mathbf{K})-12\pi\mathcal{V}_s^{\mathrm{ad}}\frac{\mathbf{K}\cdot\mathbf{K}_0}{K^2}.\tag{60}$$

So we conclude that

$$B_{L=1}^{\mathrm{out}}\sim p_{\mathrm{closed}}^{1/2}(-K_0)\frac{12\pi\mathcal{V}_s^{\mathrm{ad}}}{K}.\tag{61}$$

From the numerical solution of Eq. (58), the atom-dimer scattering volume $\mathcal{V}_s^{\mathrm{ad}}$ seems to scale as the atom-atom scattering volume $\mathcal{V}_s$ itself close to the Feshbach resonance. So we plot in Fig. 6 the ratio $\mathcal{V}_s^{\mathrm{ad}}/\mathcal{V}_s$ as a function of α_{res} for increasing values of $\mathcal{V}_s$. Another interesting feature is the divergence of $\mathcal{V}_s^{\mathrm{ad}}$ at the threshold for a trimer formation in the even sector.

The same analytical techniques as in the case of s-wave atom-dimer scattering may be used to predict the scaling of $\mathcal{V}_s^{\mathrm{ad}}$ with $\mathcal{V}_s$. First we divide Eq. (58) by K and we take the limit $K\to0$. As discussed in the s-wave atom-dimer scattering case, $D(K)/K^2$ has a finite limit, so does $D(K)B_{L=1}^{\mathrm{out}}(K)/K$. Furthermore one can show from Eq. (A9) (with $q=q_{\mathrm{dim}}$) that

$$\lim_{K\to0}\frac{C_0(K,k)-C_2(K,k)}{K}=\frac{2}{3}\frac{k}{k^2+q_{\mathrm{dim}}^2}.\tag{62}$$

We thus obtain the exact relation

$$\lim_{K\to0}\frac{D(K)}{K^2}3\mathcal{V}_s^{\mathrm{ad}}-\frac{4}{3\pi}\int_0^{+\infty}dk\frac{k^3e^{-5k^2b^2/8}}{k^2+q_{\mathrm{dim}}^2}\frac{B_{L=1}^{\mathrm{out}}(k)}{4\pi(-K_0)p_{\mathrm{closed}}^{1/2}}=-\frac{2}{q_{\mathrm{dim}}^2}.\tag{63}$$

Next we take the limit of an infinite scattering volume in Eq. (58), that is, we take $q_{\mathrm{dim}}\to0$. The source term now diverges as $1/K$ in $K=0$; since $D(K)$ vanishes as K^2, we expect that the function

$$B_\infty(K)\equiv\lim_{\mathcal{V}_s\to+\infty}B_{L=1}^{\mathrm{out}}(K)\tag{64}$$

diverges as $1/K^3$ in $K=0$. To check the existence of B_∞ as a limit, one thus has to check that the integral in Eq. (58) does not have a divergence in $k=0$ for such a $1/k^3$ behavior of the $B(k)$ function: the factor k^2 of three-dimensional integration and the fact that $C_0(K,k)-C_2(K,k)$ vanishes linearly with k indeed bring an overall k^3 factor that compensates the divergence. As a consequence it is reasonable to assume that there exists a constant C such that

$$|B_{L=1}^{\mathrm{out}}(K)|\le\frac{C}{K^3}\tag{65}$$

uniformly in K and $\mathcal{V}_s$. This allows to show that the integral term in Eq. (63) is $O(1/q_{\mathrm{dim}})$ and is thus negligible as compared to $1/q_{\mathrm{dim}}^2$. Using Eq. (49) and $q_{\mathrm{dim}}\sim1/(\alpha_{\mathrm{res}}\mathcal{V}_s)^{1/2}$ we obtain

$$\mathcal{V}_s^{\mathrm{ad}}\sim-\frac{8}{3}\mathcal{V}_s.\tag{66}$$

This result corresponds to the dotted line in Fig. 6. Strictly speaking, it asymptotically holds for all values of $\alpha_{\mathrm{res}}b$ except right on the threshold for the even trimer formation, for reasons that are explained in Sec. IV E. Away from this threshold we thus reach the important conclusion that, very close to the Feshbach resonance, the atom-dimer scattering volume for a total angular momentum $J=1$ is a "universal" quantity in the sense that it does not depend on the range b of the interaction, but only on the atom-atom scattering volume.

E. Scattering of three atoms: recombination rate

In this subsection, we consider the case of three incoming atoms, in the form of plane waves of wave vectors $\mathbf{k}_1^0$, $\mathbf{k}_2^0$,

and $\mathbf{k}_3^0$. Without loss of generality we move to the center-of-mass frame and assume a vanishing total momentum. We consider the case of a positive scattering volume $\mathcal{V}_s>0$, so that there exists a dimer state in the two-body problem, that may be populated by the collision of three atoms. The goal here is to determine the rate with which such a dimer state is formed, the so-called recombination rate.

For this physical situation, the total energy is positive so A_0 in Eq. (38) does not vanish, but rather defines the state of the three incoming fermions

$$|\Psi_0\rangle = \int \frac{d^3k d^3K}{(2\pi)^6} A_0(\mathbf{K},\mathbf{k}) a^\dagger_{1/2\mathbf{K}+\mathbf{k}} a^\dagger_{1/2\mathbf{K}-\mathbf{k}} a^\dagger_{-\mathbf{K}} |0\rangle. \quad (67)$$

Setting $\mathbf{K}_0=\mathbf{k}_1^0+\mathbf{k}_2^0$ and $\mathbf{k}_0=(\mathbf{k}_1^0-\mathbf{k}_2^0)/2$, one has

$$A_0(\mathbf{K},\mathbf{k}) = (2\pi)^6 \delta(\mathbf{K}-\mathbf{K}_0)\frac{1}{2}[\delta(\mathbf{k}-\mathbf{k}_0)-\delta(\mathbf{k}+\mathbf{k}_0)]. \quad (68)$$

To derive a simplified expression in the low incoming kinetic energy limit

$$K_0, k_0 \ll q_{\mathrm{dim}}, \frac{1}{b}, \quad (69)$$

which implies $E \ll E_{\mathrm{dim}}$, one expands A_0 in powers of k_0 and K_0. The expression between square brackets gives k_0 times a gradient of $\delta+O(k_0^3)$. The expansion in powers of K_0 gives

$$A_0(\mathbf{K},\mathbf{k}) = (2\pi)^6[\delta(\mathbf{K}) - \mathbf{K}_0\cdot(\mathrm{grad}\,\delta)(\mathbf{K}) + O(K_0^2)] \times[-\mathbf{k}_0\cdot(\mathrm{grad}\,\delta)(\mathbf{k}) + O(k_0^3)]. \quad (70)$$

One has to keep the leading order in $k_0 \sim K_0$ giving a nonzero value for the incoming state $|\Psi_0\rangle$. Keeping the first term in the first factor gives a vanishing contribution so that one has to keep the second term

$$A_0(\mathbf{K},\mathbf{k}) \simeq (2\pi)^6[\mathbf{K}_0\cdot(\mathrm{grad}\,\delta)(\mathbf{K})][\mathbf{k}_0\cdot(\mathrm{grad}\,\delta)(\mathbf{k})]. \quad (71)$$

This choice for A_0 corresponds to the limit of a vanishing total energy, so that we now take $E=0$.

This expression for A_0, when inserted in Eq. (40), gives in the right-hand side the source term

$$-\frac{6\pi\hbar^2}{m\Lambda}(2\pi)^3\{[\mathbf{K}_0\cdot(\mathrm{grad}\,\delta)(\mathbf{K})]\mathbf{k}_0 - [\mathbf{k}_0\cdot(\mathrm{grad}\,\delta)(\mathbf{K})]\mathbf{K}_0\}. \quad (72)$$

Since the gradient of δ can be seen as the product of $\mathbf{K}$ with an isotropic distribution, one finds that this source term is in the even sector, of the form (42), where $\mathbf{e}_z$ is taken along the direction of $\mathbf{K}_0$ and $\mathbf{e}_y$ is taken along the direction of $\mathbf{k}_0^\perp$, the component of $\mathbf{k}_0$ in the plane orthogonal to $\mathbf{K}_0$. We take for $\boldsymbol{\beta}$ the even ansatz with a specific form adapted to the present physical situation

$$\boldsymbol{\beta}(\mathbf{K}) = \mathcal{G}(K)\{[\mathbf{K}_0\cdot(\mathrm{grad}\,\delta)(\mathbf{K})]\mathbf{k}_0 - [\mathbf{k}_0\cdot(\mathrm{grad}\,\delta)(\mathbf{K})]\mathbf{K}_0\} + \frac{4\pi g(K)}{K^2-K_{\mathrm{dim}}^2-i0^+}\frac{1}{K}[(\mathbf{K}\cdot\mathbf{e}_z)\mathbf{e}_y - (\mathbf{K}\cdot\mathbf{e}_y)\mathbf{e}_z]. \quad (73)$$

The first term in the right-hand side is motivated by the fact that the source term contains a gradient of δ, so that $\boldsymbol{\beta}$ has also to contain a gradient of δ. In the second term, we have pulled out explicitly a singularity with a pole at $K=K_{\mathrm{dim}}+i0^+$, where

$$K_{\mathrm{dim}} = \frac{2}{\sqrt{3}} q_{\mathrm{dim}} \quad (74)$$

which is the value of K given by Eq. (39) when $k_{\mathrm{rel}}=iq_{\mathrm{dim}}$, keeping in mind that the total energy is here $E\simeq 0$. Physically K_{dim} is the value K_{out} of K corresponding to the motion in opposite directions of a flying atom and a flying dimer formed by the three-atom collision, and the term $i0^+$ in the denominator of the ansatz ensures that this relative motion is a purely outgoing wave. The conservation of energy indeed imposes $E=3\hbar^2K_{\mathrm{out}}^2/4m-E_{\mathrm{dim}}$, that is, $K_{\mathrm{out}}\simeq K_{\mathrm{dim}}$ since we assumed $E\ll E_{\mathrm{dim}}$.

We now inject the ansatz Eq. (73) in the integral equation Eq. (40). The bit in gradient of δ in the ansatz, when injected in the diagonal term of Eq. (40), gives a contribution which is a distribution of the same structure as the source term (72) created by A_0; the function $\mathcal{G}(K)$ is adjusted to have an exact cancellation:

$$\mathcal{G}(K) = -(2\pi)^3\frac{6\pi\hbar^2}{m\Lambda}\frac{3f(k_{\mathrm{rel}})}{k_{\mathrm{rel}}^2 e^{-k_{\mathrm{rel}}^2 b^2}}, \quad (75)$$

where k_{rel} is defined in Eq. (39) and is equal here to $i(\sqrt{3}/2)K$. When injected in the integral on the left hand side of Eq. (40), the bit in gradient of δ in the ansatz gives rise to a smooth function of $\mathbf{K}$ (not a distribution). After lengthy calculations and angular averages detailed in Appendix A, one finds an inhomogeneous integral equation for $g(K)$:

$$-\frac{K^2 e^{3K^2b^2/4}}{4f\left(i\frac{\sqrt{3}}{2}K\right)}\frac{g(K)}{K^2-K_{\mathrm{dim}}^2-i0^+} - \frac{2}{\pi}\int_0^{+\infty} dK' K'^2[C_0(K,K') - C_2(K,K')]\frac{g(K')}{K'^2-K_{\mathrm{dim}}^2-i0^+}e^{-5b^2(K^2+K'^2)/8} = -\frac{36\pi\hbar^2}{m\Lambda}K_0 k_0^\perp \mathcal{V}_s \frac{e^{-5K^2b^2/8}}{K}, \quad (76)$$

where C_0 and C_2 are given by Eqs. (A10) and (A12) with $k=K'$ and $q=0$. It remains to solve this integral equation; one notes that there is no δ distribution arising in the first term of this equation, since $1/f(k_{\mathrm{rel}})=1/f(iq_{\mathrm{dim}})=0$ for $K=K_{\mathrm{dim}}$, so that $i0^+$ may be omitted in the denominator of this first term [59].

To obtain the recombination rate from the solution $g(K)$ of Eq. (76), we proceed in two steps. First, we calculate the rate of dimer formation, that is the recombination rate, in terms of $g(K_{\mathrm{dim}})$, after having enclosed the three atoms in a fictitious cubic box of size L. Second, we construct an op-

erator $\hat{O}$ such that its expectation value in the unperturbed incoming state (67) of the three atoms gives the recombination rate; calculating the expectation value of this operator for a Fermi sea in the thermodynamic limit then gives the recombination rate for a macroscopic gas.

Recombination rate for three atoms. Enclosing the three atoms in a arbitrarily large cubic box with periodic boundary conditions introduces the following normalization factor in the state vector

$$|\Psi^{\text{box}}\rangle \simeq \frac{1}{L^{9/2}}|\Psi\rangle. \tag{77}$$

This is directly seen on the incoming state vector (67): each of the three atoms is in a plane wave, with a wave function in the box differing from the free space one by the normalization factor $1/L^{3/2}$.

To calculate the probability flux of dimer formation, the most convenient is to perform the reasoning in the subspace of Eq. (35) with one atom and one closed-channel molecule, where the formation of a dimer manifests itself by an outgoing wave of the molecule of momentum K_{dim} and an outgoing wave of the atom with the same momentum in the opposite direction. In momentum space, this outgoing wave results from the existence of a pole of $\boldsymbol{\beta}(\mathbf{K})$ in $K=K_{\text{dim}}$, as was made apparent in the ansatz (73). In position space, taking the Fourier transform of $\boldsymbol{\beta}(\mathbf{K})$ and writing $g(K)=g(K_{\text{dim}})+[g(K)-g(K_{\text{dim}})]$, we isolate the outgoing wave, and we obtain in the limit of a large atom-molecule separation

$$[\boldsymbol{\Psi}^{\text{box}}_{\text{mol}}]_{\text{out}}(\mathbf{r}_{\text{mol}};\mathbf{r}_{\text{at}}) \simeq \frac{g(K_{\text{dim}})}{L^{9/2}}\frac{e^{iK_{\text{dim}}r}}{r}\mathbf{e}_r \wedge \mathbf{e}_x \tag{78}$$

with $\mathbf{r}=r\mathbf{e}_r \equiv \mathbf{r}_{\text{mol}}-\mathbf{r}_{\text{at}}$ is the position of the relative particle. The associated probability current for the relative particle of reduced mass $2m/3$ is then

$$\mathbf{j}_{\text{out}} = \sum_{\gamma}\frac{\hbar}{2m/3}\text{Im}[\Psi^{*}_{\gamma,\text{out}}\partial_{\mathbf{r}}\Psi_{\gamma,\text{out}}]$$

$$\simeq \frac{3\hbar K_{\text{dim}}}{2mr^2}|g(K_{\text{dim}})|^2[(\mathbf{e}_r\cdot\mathbf{e}_z)^2+(\mathbf{e}_r\cdot\mathbf{e}_y)^2]\mathbf{e}_r. \tag{79}$$

One then calculates the total flux of the current through the 4π Steradian and one integrates over the center of mass position. Since the flying dimer has a probability amplitude $p^{1/2}_{\text{closed}}$ to be in the form of a molecule in the closed channel, it remains to divide the total flux by p_{closed} to get the rate of dimer formation for three atoms in the box

$$\frac{d}{dt}N^{\text{box}}_{\text{dim}} = 4\pi\frac{\hbar K_{\text{dim}}}{m}\frac{|g(K_{\text{dim}})|^2}{p_{\text{closed}}}\frac{1}{L^6}. \tag{80}$$

Recombination rate for a macroscopic gas. To extend Eq. (80) to a macroscopic number of atoms, we heuristically generalize to fermions an operatorial expression derived in Ref. [60] for bosons. In the bosonic case, the recombination rate in a macroscopic gas is expressed in terms of $\langle[\hat{\psi}^{\dagger}(\mathbf{R})]^3[\hat{\psi}(\mathbf{R})]^3\rangle_0$, where $\hat{\psi}$ is the bosonic field operator and the expectation value $\langle\cdots\rangle_0$ is taken in a mean-field state for the bosons not including the short-range microscopic correlations induced by the interaction potential [61]. Such a local formula results from the assumption that the size of a produced dimer is much smaller than the macroscopic correlation lengths of the gas, such as the healing length and the thermal de Broglie wavelength [60].

In the case of fermions, one has to rederive the formula since $\hat{\psi}^3=0$. This is done in Appendix B and leads to the following prescription for the recombination rate:

$$\frac{d}{dt}N_{\text{dim}} = \mathcal{K}_{\text{rec}}\int d^3R \sum_{(\alpha,\beta)\in\{(x,y),(x,z),(y,z)\}}$$

$$\times\langle(\partial_{R_\beta}\hat{\psi})^{\dagger}(\partial_{R_\alpha}\hat{\psi})^{\dagger}\hat{\psi}^{\dagger}\hat{\psi}(\partial_{R_\alpha}\hat{\psi})(\partial_{R_\beta}\hat{\psi})\rangle_0, \tag{81}$$

where the field operator and its derivatives are all evaluated in $\mathbf{R}$. This expression involves as a factor the recombination constant $\mathcal{K}_{\text{rec}}$, not to be confused with the recombination rate. In the considered limit of a fermionic kinetic energy smaller than the dimer binding energy (69) we indeed expect $\mathcal{K}_{\text{rec}}$ to be a constant, that is not to depend on the fermionic kinetic energy. On the contrary, the recombination rate dN_{dim}/dt will involve a factor proportional to the square of the kinetic energy of the fermions, as predicted in Ref. [42] with a different approach.

To illustrate this point, let us consider the case of a spatially homogeneous weakly interacting zero temperature Fermi gas. The condition of low kinetic energy is then that the Fermi energy $\hbar^2k_F^2/(2m)$ is smaller than the dimer binding energy E_{dim}. It remains to calculate the expectation value $\langle\cdots\rangle_0$ of Eq. (81) in the Fermi sea of the ideal Fermi gas of density $n=k_F^3/(6\pi^2)$, to get

$$\frac{d}{dt}N_{\text{dim}} = \frac{3k_F^4}{25}\mathcal{K}_{\text{rec}}Nn^2. \tag{82}$$

The factor k_F^4 reveals the expected kinetic energy dependence of the recombination rate. The recombination rate will weakly depend on temperature as long as the gas remains strongly degenerate, $k_BT \ll \hbar^2k_F^2/(2m)$.

Value of the recombination constant. We obtain $\mathcal{K}_{\text{rec}}$ by applying Eq. (81) to our solution of the three-body problem. In this case, the uncorrelated state $|\Psi_0\rangle$ over which to average in the expectation value $\langle\cdots\rangle_0$ is a Slater determinant with three atoms in plane waves of wave vectors $\mathbf{k}_1^0=\mathbf{k}_0+\mathbf{K}_0/2$, $\mathbf{k}_2^0=-\mathbf{k}_0+\mathbf{K}_0/2$, and $\mathbf{k}_3^0=-\mathbf{K}_0$, respectively. Using Wick's theorem [62], we obtain

$$\frac{d}{dt}N^{\text{box}}_{\text{dim}} = 9\mathcal{K}_{\text{rec}}\frac{(\mathbf{k}_0\wedge\mathbf{K}_0)^2}{L^6}. \tag{83}$$

Equating this expression to Eq. (80) we obtain

$$\mathcal{K}_{\text{rec}} = \frac{4\pi}{9}\frac{\hbar K_{\text{dim}}}{m}\frac{|g(K_{\text{dim}})|^2}{(\mathbf{K}_0\times\mathbf{k}_0)^2p_{\text{closed}}}. \tag{84}$$

We find that, as expected, this recombination constant does not depend on the incoming energy, that is on the norms K_0 and k_0, in the present limit of vanishing incoming energy: $g(K)$ is indeed proportional to $K_0k_0^{\perp}=\|\mathbf{K}_0\times\mathbf{k}_0\|$, as the source term in the linear equation (76) is.

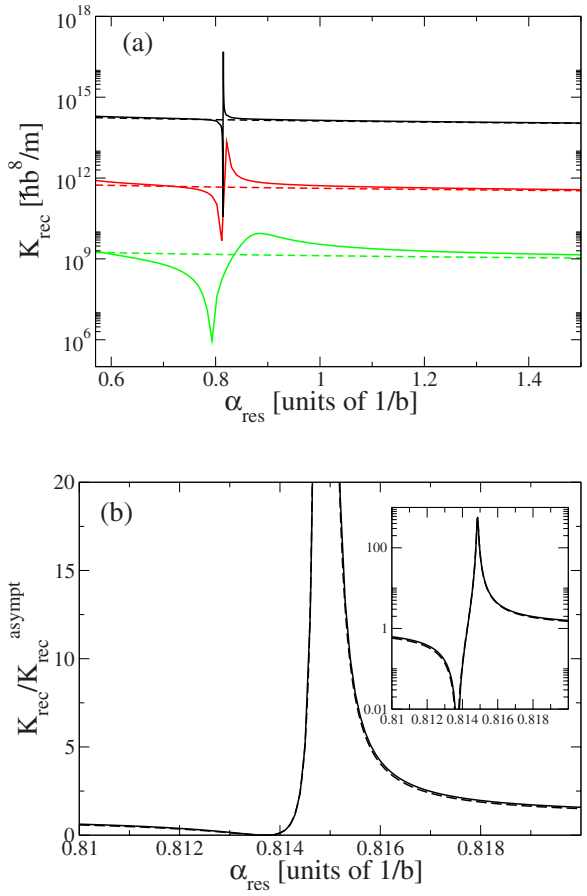

FIG. 7. (Color online) Recombination constant $\mathcal{K}_{rec}$ appearing in the expression (81) giving the rate of formation of weakly bound dimers when three low energy atoms are colliding, as a function of α_{res} for a fixed value of the scattering volume. (a) $\mathcal{K}_{rec}$ in units of $\hbar b^8/m$. The scattering volume is, from bottom to top, $\mathcal{V}_s=100b^3$ (green line), $\mathcal{V}_s=1000b^3$ (red line), and $\mathcal{V}_s=10^4b^3$ (black line). Solid lines: numerical solution. Dashed lines: asymptotic formula (90). (b) Ratio of $\mathcal{K}_{rec}$ to the asymptotic formula (90), for $\mathcal{V}_s=10^4b^3$. Solid line: numerical solution. Dashed line: analytically predicted Fano profile (93). The inset is exactly the same figure but with a log scale on the vertical axis.

One solves the integral equation (76) numerically, to access $g(K_{dim})$. The corresponding values of the recombination constant are given as functions of α_{res} in Fig. 7(a), for three values of the scattering volume. As expected, a rapid rise of the recombination constant is observed when one gets closer to the Feshbach resonance, that is for increasing values of $\mathcal{V}_s$. For a fixed $\mathcal{V}_s$, one observes a smooth dependence of $\mathcal{K}_{rec}$ with α_{res}, except in the vicinity of $\alpha_{res}b=0.8$: both a dip and a peak in $\mathcal{K}_{rec}$ are observed; this singular structure becomes extremely narrow in the large $\mathcal{V}_s$ limit, both the distance between the dip and the peak, and the width of the peak, apparently tending to zero. These features can be obtained analytically as follows, by investigating the large $\mathcal{V}_s$ limit of Eq. (76).

Let us examine first the diagonal term in the left hand side of the Eq. (76). At low values of K, much below $1/b$, one can approximate the inverse scattering amplitude as $1/f(iq) \simeq -1/(q^2\mathcal{V}_s)+\alpha \simeq \alpha(q^2-q_{dim}^2)/q^2$, where we used Eq. (10) since we are close to resonance. Setting $q=\sqrt{3}K/2$, we then see that this diagonal term at low energy is close to $\alpha g(K)/4$, so it is very smooth. At high values of K, of the order of $1/b$ or larger, one can directly set $\mathcal{V}_s=+\infty$, and one sees that the factor of $g(K)$ in this diagonal term decreases smoothly from $\alpha_{res}/4$ to the positive quantity $[\alpha_{res}-1/(\pi^{1/2}b)]/4$ when K increases to infinity.

Let us now turn to the integral term. The value of K' is cut to values at most of the order of $1/b$ by the Gaussian factor. For low values of K, below $1/b$, an approximate expression of the kernel can be obtained [63],

$$C_0(K,K') - C_2(K,K') \simeq \frac{2KK'}{3(K^2+K'^2)}. \tag{85}$$

This shows that the kernel of the integral part is smooth and bounded, even in the low K and K' limit. Neglecting K_{dim} in the denominator of the integral term gives a diverging factor $1/K'^2$ which is, however, exactly compensated by the K'^2 Jacobian term of three-dimensional integration.

The only source of singularity in the solution $g(K)$ may thus be the source term, in the right hand side of Eq. (76). The presence of a factor $\mathcal{V}_s$ will cause $g(K)$ to diverge at high $\mathcal{V}_s$, by linearity of the equation, and the $1/K$ divergence of the source will lead to a singular behavior of $g(K)$ in $K=0$. These two problems can be eliminated by taking as an unknown function

$$F(K) = \frac{Kg(K)}{\mathcal{V}_s}. \tag{86}$$

We multiply Eq. (76) by $K/\mathcal{V}_s$. The kernel of the integral term for $F(K)$ taken in the limit $K_{dim}=0$ (thus neglecting K_{dim}^2 in the denominator) now behaves at low momenta as $(K/K')[C_0(K,K)-C_2(K,K')] \simeq (2K^2/3)/(K^2+K'^2)$, which remains a bounded quantity. Having eliminated the singularity in the source term, we can suppose that

$$F_\infty(K) = \lim_{\mathcal{V}_s\to+\infty} F(K) \tag{87}$$

is a regular and bounded function $|F_\infty(K)| \le |F|_{max}$. Then, if one uses the approximate expression (85), one easily sees that the integral term in the equation for $F_\infty(K)$ is bounded by

$$\frac{2}{\pi}\int_0^{+\infty} dK' \frac{2K^2}{3(K^2+K'^2)}|F|_{max} = \frac{2}{3}K|F|_{max}, \tag{88}$$

so that it tends to zero in $K=0$. Since the source term for F_∞ is nonzero for $K=0$ we conclude that

$$F_\infty(0) = -\frac{144\pi\hbar^2}{m\Lambda\alpha_{res}}K_0 k_0^\perp. \tag{89}$$

Using the useful expression (33) for the probability to find the dimer in the closed channel, we thus obtain analytically the asymptotic value of the recombination constant for large $\mathcal{V}_s/b^3$ [64]

$$\mathcal{K}_{\rm rec} \sim \mathcal{K}_{\rm rec}^{\rm asympt} = \frac{\hbar}{m}(48\pi)^2\left(\frac{\mathcal{V}_s^5}{3\alpha_{\rm res}}\right)^{1/2}. \quad (90)$$

We first note that this result, contrarily to the atom-dimer scattering length, is "universal," that is it does not depend on the potential range b but only on the parameters $\mathcal{V}_s$ and α entering in the low-k expansion of the two-body scattering amplitude. In particular, Eq. (90) is not specific to our choice of a Gaussian cutoff function in $\boldsymbol{\chi}(\mathbf{k})$, as we have checked for a general cutoff function that is derivable with respect to $\mathbf{k}$. Second, the exponent governing the dependence in $\mathcal{V}_s$ is the same for a broad or a narrow Feshbach resonance. It may thus make sense to compare this prediction to the earlier work of Ref. [42], where a numerical calculation was performed for a resonant interaction in a single channel model: the recombination rate was found to increase as a power law in $\mathcal{V}_s$, with an exponent argued to be equal to 8/3. Since 8/3 and 5/2 differ by about 6% only, it seems difficult to see this difference from the numerical results of Ref. [42].

To see how our numerical results approach the asymptotic prediction (90), we have plotted in dashed lines in Fig. 7(a) the asymptotic value $\mathcal{K}_{\rm rec}^{\rm asympt}$, as a function of $\alpha_{\rm res}$, for the considered values of the scattering volume. For increasing values of $\mathcal{V}_s$, we indeed observe convergence of the ratio $\mathcal{K}_{\rm rec}/\mathcal{K}_{\rm rec}^{\rm asympt}$ to unity, but this convergence is not uniform in $\alpha_{\rm res}$: the singular structure already apparent in Fig. 7(a) becomes narrower and narrower for increasing $\mathcal{V}_s$ but, e.g., the peak in this singular structure leads to increasing deviation from unity of the ratio $\mathcal{K}_{\rm rec}/\mathcal{K}_{\rm rec}^{\rm asympt}$.

The existence of this singular structure and the dependence of the recombination rate on $\alpha_{\rm res}$ within this structure can be obtained analytically as follows. First, we formally write the integral equation obtained for $F_\infty(K)$ in the limit $\mathcal{V}_s \to +\infty$ for a fixed value of $\alpha_{\rm res}$:

$$\frac{\alpha_{\rm res}}{4}F_\infty(K) - I_0[F_\infty](K) = S(K), \quad (91)$$

where the source term $S(K)$ is obtained by multiplication of the right-hand side of Eq. (76) by $K/\mathcal{V}_s$, and I_0 is a linear operator, given explicitly in Appendix C. We find numerically that the spectrum of I_0 consists of a continuum extending from 0 to $1/(4\pi^{1/2}b)$, and of one discrete eigenvalue above the continuum. If one remembers that, from Eq. (29), $\alpha_{\rm res} > 1/(\pi^{1/2}b)$, it becomes clear that the homogeneous equation obtained by replacing S with zero will admit a nonzero solution $u_0(K)$ only for $\alpha_{\rm res} = \alpha_{\rm th}^{\rm even}$, where mathematically $\alpha_{\rm th}^{\rm even}/4$ is the discrete eigenvalue of I_0, and physically $\alpha_{\rm th}^{\rm even}$ is the threshold value of $\alpha_{\rm res}$ for the existence on resonance of an even trimer. The operator appearing in Eq. (91) is indeed the infinite scattering volume limit and the zero energy limit of the operator $L(E)$ of Sec. IV C on trimers, restricted to the even sector. Numerically we find

$$\alpha_{\rm th}^{\rm even} \simeq 0.81408/b. \quad (92)$$

In presence of the source term S, and for a value of $\alpha_{\rm res}$ slightly deviating from $\alpha_{\rm th}^{\rm even}$, one realizes that a component of $F_\infty(K)$, proportional to $u_0(K)$, may diverge as $1/(\alpha_{\rm res} - \alpha_{\rm th}^{\rm even})$. The appearance of such a small denominator implies that $F_\infty(K)$ is not uniformly bounded in $\alpha_{\rm res}$ in the vicinity of $\alpha_{\rm th}^{\rm even}$, so that the asymptotic law (90) may not hold uniformly in $\alpha_{\rm res}$.

This very simply reveals that the singular structure in the recombination coefficient is a consequence of the existence of a weakly bound trimer. Quantitatively, as shown in Appendix C, by going beyond the $\mathcal{V}_s = \infty$ approximation, one can calculate analytically the contribution to $F(K)$ which becomes large for $\alpha_{\rm res}$ close to $\alpha_{\rm th}^{\rm even}$. This leads to a Fano profile [65]

$$\mathcal{K}_{\rm rec} \simeq \mathcal{K}_{\rm rec}^{\rm asympt}\frac{(\alpha_{\rm res} - \alpha_0)^2}{(\alpha_{\rm res} - \alpha_1)^2 + \Delta\alpha^2} \quad (93)$$

with the following low-$q_{\rm dim}$ expansions

$$\alpha_0 \simeq \alpha_{\rm th}^{\rm even} - 3.2q_{\rm dim}^2 b, \quad (94)$$

$$\alpha_1 \simeq \alpha_{\rm th}^{\rm even} + 6.294q_{\rm dim}^2 b, \quad (95)$$

$$\Delta\alpha \simeq 35.89q_{\rm dim}^3 b^2, \quad (96)$$

$$\frac{\Delta\alpha}{\alpha_1 - \alpha_0} \simeq \frac{16}{3\sqrt{3}}\frac{q_{\rm dim}}{\alpha_{\rm th}^{\rm even}}, \quad (97)$$

where $q_{\rm dim} \simeq 1/\sqrt{\alpha_{\rm th}^{\rm even}\mathcal{V}_s}$. This is in agreement with the numerical results at finite but large $\mathcal{V}_s$, see Fig. 7(b).

Note that such a Fano profile in the recombination constant as a function of the width of the Feshbach resonance does not occur in the case of bosons in the large scattering length limit: In the bosonic case, when the scattering length a becomes much larger than the range and the effective range of the interaction potential, the recombination constant, apart from an asymptotic a^4 factor, has only a bounded oscillatory behavior as a function of a or of the potential range [66,67]. This is a consequence of the fact that trimers of bosons always exist for large enough a, whatever the width of the Feshbach resonance, and they exist in arbitrarily large numbers for a arbitrarily large, the so-called Efimov effect; the oscillatory behavior in the recombination constant then results from the successive entrances of new Efimov trimers, as a grows.

Finally, turning back to the fermionic case, we note that, in reality, there may exist deeply bound dimers even in the limit $|\mathcal{V}_s| \gg b^3$, which may be formed by the collision of three incoming atoms, in competition with the weakly bound dimer. This effect, beyond our model Hamiltonian, is discussed in Sec. V C, where the corresponding recombination constant towards deeply bound dimers is estimated.

V. ESTIMATION OF LOSSES DUE TO DEEPLY BOUND DIMERS

Very close to the Feshbach resonance, the model Hamiltonian that we have considered in this work can support only a weakly bound dimer in the two-body problem, that is with an energy much smaller than $\hbar^2/mb^2$. In real experiments, with alkali atoms, the van der Waals interaction potential is very deep and support several deeply bound dimers in the

two-body problem. These deeply bound dimers can be formed by three-body collisions, and liberation of a huge binding energy, leading to particle losses. As a consequence, the trimers will acquire a finite lifetime, the atom-dimer scattering will not be purely elastic, and the scattering of three atoms will lead to recombination not only to the weakly bound dimer, but also to the deeply bound dimers.

A first possibility to estimate these inelastic contributions is to modify the Hamiltonian, so as to have deeply bound dimers in the model, e.g., by including a separable interaction potential in the open channel [68,69], or by considering a true potential and introducing the adiabatic potential curves in hyperspherical coordinates [70,71]. This, however, modifies the mathematical structure of the problem and is beyond the scope of the present paper.

Fortunately one may easily estimate the loss rate, that is the rate of formation of deeply bound dimers, by the following recipe, expected to be accurate within an unknown approximately constant factor [39,72]

$$\Gamma_{\rm loss} = \frac{\hbar}{mb^2} P_{<b}, \tag{98}$$

where $P_{<b}$ is essentially the probability that the three atoms be all within a volume of the order of b^3. More precisely, since we are using a two-channel model, this probability can be split in two contributions, one coming from the purely atomic component (all three particles in the open channel)

$$P_{<b}^{\rm at} = \int_{\rho<b} d^3r_1 d^3r_2 d^3r_3 |\Psi_{\rm at}^{\rm norm}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)|^2, \tag{99}$$

where ρ is the hyper-radius (B2), and the other contribution coming from the molecular component (with one open channel atom and one closed channel molecule),

$$P_{<b}^{\rm mol} = \int_{|\mathbf{r}_{\rm mol}-\mathbf{r}_{\rm at}|<b} d^3r_{\rm mol} d^3r_{\rm at} |\mathbf{\Psi}_{\rm mol}^{\rm norm}(\mathbf{r}_{\rm mol};\mathbf{r}_{\rm at})|^2. \tag{100}$$

The wave functions are here normalized, hence the apex "norm," as we shall explain case by case. It remains to apply a Fourier transform to $\boldsymbol{\beta}$ and to an antisymmetrized version of A in Eq. (35) to calculate numerically the corresponding wave functions $\Psi_{\rm mol}^{\rm norm}$ and $\Psi_{\rm at}^{\rm norm}$. One can also have analytic estimates, by approximating the wave functions by their small-radius expansions $|\mathbf{r}_i-\mathbf{r}_j| \ll b$, as we shall see.

A. Lifetime of the trimers

In the case of trimers, the state vector can be normalized in the center-of-mass frame, as detailed in Appendix A. One may then calculate the probabilities in Eqs. (99) and (100) numerically; the corresponding rate $\Gamma_{\rm loss}$ then represents the inverse lifetime of the trimer due to spontaneous decay into a deeply bound dimer and a free atom. We recall that Eq. (98) contains an unknown numerical factor that depends on the microscopic details of the interaction, so the values of the lifetimes that we shall obtain are only indicative.

As a consequence, it seems more interesting physically to obtain the scaling laws of the trimer lifetime close to the trimer formation threshold, that is when the trimer binding energy is $\ll \hbar^2/(mb^2)$: Does the trimer decay rate tend to zero on the threshold? Even if this is the case, one cannot immediately conclude that the weakly bound trimers are long lived, because their binding energies also tend to zero on the threshold. One rather has to see if the decay rate tends to zero faster or not than the binding energy $E_{\rm trim}^{\rm bind}$ of the trimer. To this end, we form what we call the quality factor of the trimer

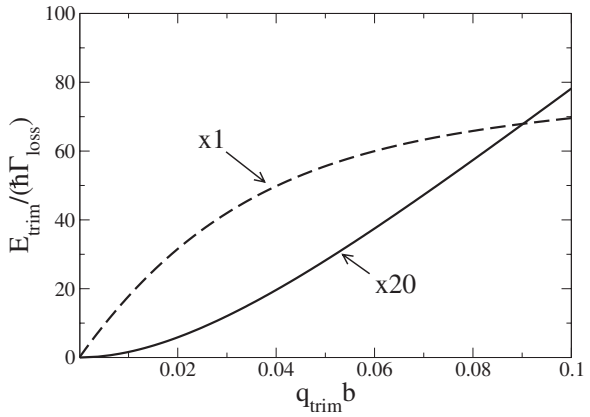

FIG. 8. For an infinite scattering volume, quality factor of the trimers, as a function of $q_{\rm trim}b$, for the even sector (solid line) and for the odd sector (dashed line). For clarity, the quality factor in the even sector was multiplied by 20. The quality factor is defined as the ratio of the trimer binding energy, here equal to $E_{\rm trim} = \hbar^2 q_{\rm trim}^2/m$ since $\mathcal{V}_s = \infty$, and of $\hbar$ times the spontaneous decay rate of the trimer due to the formation of deeply bound dimer and a free atom as estimated by the simple recipe (98).

$$Q = \frac{E_{\rm trim}^{\rm bind}}{\hbar\Gamma_{\rm loss}}. \tag{101}$$

This quality factor is shown as a function of $q_{\rm trim}$ in Fig. 8 for an infinite scattering volume, that is, in practice for $1/|\alpha_{\rm res}\mathcal{V}_s| \ll q_{\rm trim}^2$. One sees that the quality factor Q tends to zero at the threshold for trimer formation, which is not a positive result. The odd sector is, however, much more favorable (keeping in mind that the quality factor in the even sector was multiplied by a factor 20 for clarity in the figure): Values of Q much larger than unity are obtained already for moderately small values of $q_{\rm trim}b$. This is due to the fact that Q vanishes more slowly in the odd than in the even sector: on the figure, $Q_{\rm odd}$ seems to vanish linearly whereas $Q_{\rm even}$ seems to vanish quadratically. The scaling of the quality factor with $q_{\rm trim}$ close to the trimer formation threshold can be obtained analytically from the low-K dependence of $\boldsymbol{\beta}(K)$, considering again an infinite scattering volume.

In the even sector, we have seen in Fig. 4(a) that the function $B_{L=1}(K)$ right on the threshold diverges at low K as $1/K$ only. This can be shown, as done in Appendix C, using the fact that the kernel $(C_0-C_2)(K,k)$ vanishes linearly in K, whereas the diagonal part of the integral equation vanishes quadratically as $K^2\alpha_{\rm res}/4$ for $q_{\rm trim}=0$. As a consequence, the function $\mathcal{B}(K)=KB_{L=1}(K)$ is bounded, and the un-normalized state vector $|\Psi\rangle$ of the trimer is square integrable in the center-of-mass frame, as is apparent on Eqs. (A25) and

(A26). We thus face the same phenomenon as in the two-body case: at threshold, the normalized trimer wave function is nonzero. As a consequence, the probability of finding the particles at relative distances less than b tends to a nonzero value at threshold, $\Gamma_{\rm loss}$ does not vanish,

$$\lim_{q_{\rm trim}b\to 0} \Gamma_{\rm loss}^{\rm even} > 0, \tag{102}$$

and the quality factor $Q_{\rm even}$ tends to zero as $(q_{\rm trim}b)^2$.

In the odd sector, the situation is more favorable. As we have seen in Fig. 4(b), for the arbitrary normalization chosen in that figure [73], the function $B_{L=0}(K)$ right on threshold diverges as $1/K^2$ [whereas the $B_{L=2}(K)$ is $O(1/K)$]. As a consequence, the functions $\boldsymbol{\beta}(\mathbf{K})$ and thus $\Psi_{\rm mol}$ are not square integrable, and $\langle\Psi_{\rm at}|\Psi_{\rm at}\rangle$ is also infinite, see Eqs. (A27) and (A28). This means that the normalization factor $\mathcal{N}_t$ linking the correctly normalized state vector to the unnormalized one $|\Psi\rangle$,

$$|\Psi^{\rm norm}\rangle = \mathcal{N}_t|\Psi\rangle, \tag{103}$$

vanishes for $q_{\rm trim}b$ tending to zero. As a consequence, $P_{<b}$ and $\Gamma_{\rm loss}$ also vanish.

We now determine the corresponding scaling law. The functions $B_{L=0}$ and $B_{L=2}$ solve a homogeneous integral equation, corresponding to the homogeneous part of Eq. (46) [that is, with the source term set to zero in the right-hand side] written for an energy $E=-\hbar^2 q_{\rm trim}^2/m$. One can then recycle the reasoning performed below Eq. (46). At $K\ll 1/b$, the diagonal term

$$D(K) \simeq \alpha_{\rm res}(K^2 + K_{\rm trim}^2)/4, \tag{104}$$

where $K_{\rm trim}=2q_{\rm trim}/\sqrt{3}$. On the other hand, the first line of the matrix $M(K,k)$ has a nonzero limit for $K\to 0$, see Eq. (50). For a choice of normalization of the B's such that the functions $k^2|B_{L=0,2}|(k)$ are uniformly bounded for $K_{\rm trim}$ tending to zero, we reach the form

$$B_{L=0}(K) = \frac{\mathcal{E}(K)}{K^2 + K_{\rm trim}^2}, \tag{105}$$

where the envelope function $\mathcal{E}(K)$ has a finite but nonzero limit in $K=0$, and is uniformly bounded as a function of K and $K_{\rm trim}$ [74]. Since the second line of the matrix $M(K,k)$ vanishes for $K\to 0$, we find that $B_{L=2}$ is dominated by $B_{L=0}$ at low $K, K_{\rm trim}$ and can be neglected [75]. Inserting the form (105) in the normalization integrals (A27) and (A28) we obtain the asymptotic results in the $q_{\rm trim}\to 0$ limit

$$\langle\Psi_{\rm mol}|\Psi_{\rm mol}\rangle \sim \frac{\sqrt{3}}{16\pi}\frac{\mathcal{E}^2(0)}{q_{\rm trim}}, \tag{106}$$

$$\langle\Psi_{\rm at}|\Psi_{\rm at}\rangle \sim \frac{1-p_{\rm closed}^{\rm res}}{p_{\rm closed}^{\rm res}}\langle\Psi_{\rm mol}|\Psi_{\rm mol}\rangle, \tag{107}$$

where $p_{\rm closed}^{\rm res}=0.185\cdots$ here at the odd trimer formation threshold. This shows that $|\mathcal{N}_t|^2$ scales as $q_{\rm trim}b$. In calculating the probability $P_{<b}$ to have the particles "inside" the interaction potential for a nonzero but small $q_{\rm trim}$, we can take directly the un-normalized wave functions Ψ for $q_{\rm trim}=0$, so that $P_{<b}$ scales as $|\mathcal{N}_t|^2$ and

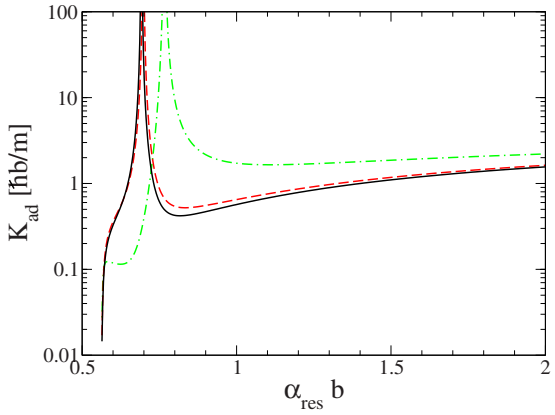

FIG. 9. (Color online) Loss constant $K_{\rm ad}$ due to formation of deeply bound dimers in the atom-dimer collision, as estimated by the recipe Eq. (98) and the relation Eq. (113), as a function of $\alpha_{\rm res}$ for $\mathcal{V}_s=10^6b^3$ (black solid line), $\mathcal{V}_s=10^3b^3$ (red dashed line), $\mathcal{V}_s=10b^3$ (green dotted-dashed line). $K_{\rm ad}$ is in units of $\hbar b/m$. A divergence of $K_{\rm ad}$ occurs at the threshold of odd trimer formation.

$$\Gamma_{\rm loss}^{\rm odd} \propto \frac{\hbar q_{\rm trim}}{mb}. \tag{108}$$

The quality factor $Q_{\rm odd}$ thus vanishes linearly in $q_{\rm trim}b$.

B. Losses in atom-dimer scattering

We now estimate the loss rate in the collision of an atom with a weakly bound dimer. We enclose the atom and the dimer in a fictitious cubic box of volume L^3, so that the state vector in the box can be normalized to unity. The box has a size $L\gg b, |a_{\rm ad}|$ so that this state vector differs from the free space one by a normalization factor only,

$$|\Psi^{\rm norm}\rangle \simeq \frac{1}{L^3}|\Psi\rangle. \tag{109}$$

One has then indeed, in the subspace with one atom and one molecule at large distances

$$\boldsymbol{\Psi}_{\rm mol}^{\rm norm}(\mathbf{r}_{\rm mol};\mathbf{r}_{\rm at}) \simeq \frac{p_{\rm closed}^{1/2}}{L^3}\left[1-\frac{a_{\rm ad}}{|\mathbf{r}_{\rm mol}-\mathbf{r}_{\rm at}|}\right]\mathbf{e}_z, \tag{110}$$

and in the subspace with three atoms, for the position of the third atom going to infinity for fixed positions of atoms one and two [56,76],

$$\Psi_{\rm at}^{\rm norm}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) \simeq \frac{1}{\sqrt{3}L^3}\phi(\mathbf{r}_1-\mathbf{r}_2)\left[1-\frac{a_{\rm ad}}{|\mathbf{r}_3-(\mathbf{r}_1+\mathbf{r}_2)/2|}\right], \tag{111}$$

so that, after spatial integration of the modulus square of these two wave functions in the box, using $\int d^3r|\phi(\mathbf{r})|^2=1-p_{\rm closed}$, one finds that $|\Psi^{\rm norm}\rangle$ is normalized to unity in the cubic box, to zeroth order in $a_{\rm ad}/L$ and b/L.

To link this calculation with an experimentally relevant quantity, we consider a low density mixture of $N_{\rm at}$ atoms and $N_{\rm dim}$ dimers in a volume L^3. The loss rate will be

$$\frac{d}{dt}N_{\rm at} = \frac{d}{dt}N_{\rm dim} = -K_{\rm ad}\frac{N_{\rm at}N_{\rm dim}}{L^3}. \tag{112}$$

The loss constant $K_{\rm ad}$ is related to Eq. (98) by setting $N_{\rm at}=N_{\rm dim}=1$ in the above equation,

$$K_{\rm ad} = L^3\Gamma_{\rm loss}, \tag{113}$$

which can be checked to be independent of L^3. The resulting atom-dimer loss constant is plotted in Fig. 9 as a function of $\alpha_{\rm res}$. We see that, for $\mathcal{V}_s \gg b^3$,

$$K_{\rm ad} \propto \frac{\hbar b}{m}, \tag{114}$$

except close to the trimer formation threshold where, within the simple recipe, $K_{\rm ad}$ diverges. We also see a drop of $K_{\rm ad}$ to a smaller but nonzero value in the limit of broad Feshbach resonances, equal to $\hbar b/m$ within a numerical factor: this drop is due to the fact that $P^{\rm mol}_{<b}$ tends to zero in this limit, so that $P_{<b}$ reduces to the atomic contribution $P^{\rm at}_{<b}$, which is elsewhere dominated by $P^{\rm mol}_{<b}$.

The property (114) can be understood analytically. For example, to estimate $P^{\rm mol}_{<b}$, one can approximate $\Psi^{\rm norm}_{\rm mol}$ in Eq. (100) by its value in $\mathbf{r}_{\rm mol}=\mathbf{r}_{\rm at}$, which is generically nonzero for the odd ansatz. Taking $\mathcal{V}_s=\infty$ gives a finite value for the wave function, since $B^{\rm out}_{L=0}(K)$ diverges as $1/K^2$, see Eq. (48), and this is integrable on a vicinity of $\mathbf{K}=0$ in three dimensions. This explains the weak $\mathcal{V}_s$ dependence of $K_{\rm ad}$ for large scattering volumes. For $\alpha_{\rm res}b$ of the order of unity, Eq. (114) then holds from dimensional analysis, apart from the divergence close to the trimer formation threshold. At large $\alpha_{\rm res}b$, we see from Eq. (53) that the scattered wave $B^{\rm out}_{L=0,2}(K)$ is $O(1/\alpha_{\rm res})$ and is dominated by the contribution of the incoming wave $\propto\delta(\mathbf{K})$. We thus find again Eq. (114), with a dominant contribution $P^{\rm mol}_{<b}$ from the molecular sector and an atomic sector contribution which is about $1-p^{\rm res}_{\rm closed}=O[1/(\alpha_{\rm res}b)]$ times smaller.

To make the discussion more complete, we also estimate the rate of formation of deeply bound dimers when the atom and the dimer scatter in the p wave, each with a momentum of modulus K_0. In this case the wave function for distances between the particles $\ll 1/K_0$ is obtained from the ansatz (57), containing the overall factor K_0. On the contrary, if one takes a box size $L\gg 1/K_0$, the normalization factor linking $|\Psi^{\rm norm}\rangle$ to $|\Psi\rangle$ remains $\simeq 1/L^3$. As a consequence, the p-wave loss constant $K^p_{\rm ad}=L^3\Gamma_{\rm loss}$ will be proportional to K_0^2. Furthermore, to estimate $P^{\rm mol}_{<b}$, we can expand the wave function $\mathbf{\Psi}_{\rm mol}(\mathbf{r}_{\rm mol};\mathbf{r}_{\rm at})$, which is now in the even sector, to leading order in $\mathbf{r}=\mathbf{r}_{\rm mol}-\mathbf{r}_{\rm at}$, that is, to first order, which amounts in the Fourier transform of $\boldsymbol{\beta}(\mathbf{K})$ to replacing $\exp(i\mathbf{K}\cdot\mathbf{r})$ with $i\mathbf{K}\cdot\mathbf{r}$. Taking directly the limit of an infinite scattering volume, we see from Eq. (65) that this first order estimate remains finite, because $K/K^3=1/K^2$ is integrable around $K=0$ in 3D. As a consequence, we expect $K^p_{\rm ad}$ to be of the order of $\hbar K_0^2 b^3/m$, except close to the even trimer formation threshold, where it diverges within the present formalism. This expectation is confirmed by the numerical calculation (not shown). Physically, this means that, away from the even trimer formation threshold, $K^p_{\rm ad}$ will be smaller than $K_{\rm ad}$ by a factor of the order of $(K_0b)^2$, that is, at least by a factor $b^3/\mathcal{V}_s$ since we assume here $K_0 \ll q_{\rm dim}$.

C. Recombination to deeply bound dimers

To complete this section, we now evaluate the rate of formation of deeply bound dimers in the collision of three asymptotically free atoms. We again enclose the atoms in a fictitious cubic box of size L with periodic boundary conditions, so that the normalized state vector in the box is related to the free space one in the large L limit by Eq. (77). Reproducing the reasoning of Appendix B leading to the prescription (81), we can now define a recombination constant towards deeply bound dimers, that we call $\mathcal{K}^{\rm deep}_{\rm rec}$. Applying the resulting prescription to three atoms in the box, we obtain the equivalent of Eq. (83) for the rate of formation of deeply bound dimers, with $\mathcal{K}_{\rm rec}$ replaced with $\mathcal{K}^{\rm deep}_{\rm rec}$. On the other hand, the rate of formation of deeply bound dimers is $\Gamma_{\rm loss}$ defined by the recipe (98). Equating the two expressions of this rate leads to

$$\mathcal{K}^{\rm deep}_{\rm rec} = \frac{L^6\Gamma_{\rm loss}}{9(\mathbf{K}_0\wedge\mathbf{k}_0)^2}. \tag{115}$$

We consider first the molecular sector. One has to perform the Fourier transform of $\boldsymbol{\beta}(\mathbf{K})$ in Eq. (73) to obtain the atom-molecule wave function. The Fourier transform of the first term of the ansatz can be calculated exactly: Since it is composed of gradients of δ distribution in momentum space, it gives a wave function varying linearly with the coordinates of $\mathbf{r}=\mathbf{r}_{\rm mol}-\mathbf{r}_{\rm at}$. It is found that this wave function is proportional to $\mathcal{V}_s$. In the second term of the ansatz, we take the large scattering volume limit, away from the trimer formation threshold, so that we neglect $K^2_{\rm dim}$ in the denominator and we take $g(K)\simeq\mathcal{V}_sF_\infty(K)/K$, as discussed around Eq. (87). This gives again a contribution proportional to $\mathcal{V}_s$. The Fourier transform cannot be calculated analytically, but we only need the wave function for $r\lesssim b$ so that we can restrict to a small-r expansion of the atom-molecule wave function. The linear order in $\mathbf{r}$ is the first nonzero one, and we obtain the contribution from the molecular sector

$$(\mathcal{K}^{\rm deep}_{\rm rec})_{\rm mol} \propto \frac{\hbar}{m}b^3\mathcal{V}_s^2\alpha_{\rm res}p^{\rm res}_{\rm closed} \times\left[1+\frac{16}{3\pi\alpha_{\rm res}}\int_0^{+\infty}dKF_\infty(K)/F_\infty(0)\right]^2. \tag{116}$$

The expression in between square brackets is not a slowly varying function of $\alpha_{\rm res}b$, because it diverges in the vicinity of the trimer formation threshold [an artifact of the approximations performed here on the second term of the ansatz (73)]. We have checked numerically that the expression can be well approximated by the fitting formula

$$[\cdots] \simeq \frac{1+\dfrac{1.239}{\alpha_{\rm res} b - 0.538}}{1-\alpha_{\rm th}^{\rm even}/\alpha_{\rm res}}. \tag{117}$$

The same procedure can be applied in the atomic sector. To estimate the purely atomic wave function, we expand it to leading order in the interatomic distances. The zeroth and first order vanish, and we get to second order

$$\Psi_{\rm at}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) \simeq \frac{\sqrt{6}}{2}\int \frac{d^3k d^3K}{(2\pi)^6} A(\mathbf{K},\mathbf{k})(\mathbf{x}\wedge\mathbf{y})\cdot(\mathbf{k}\wedge\mathbf{K}), \tag{118}$$

where we have introduced the Jacobi-like coordinates $\mathbf{x}=\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{y}=\mathbf{r}_3-(\mathbf{r}_1+\mathbf{r}_2)/2$. This leads to

$$(\mathcal{K}_{\rm rec}^{\rm deep})_{\rm atom} \simeq (\mathcal{K}_{\rm rec}^{\rm deep})_{\rm mol}\frac{1-p_{\rm closed}^{\rm res}}{p_{\rm closed}^{\rm res}} f_{\rm slow}, \tag{119}$$

where the factor $f_{\rm slow}$ depends only on $\alpha_{\rm res} b$, it is of the order of 0.005 for a broad Feshbach resonance and it increases by a factor $\simeq 5$ from broad to narrow Feshbach resonances.

As a consequence, our estimate of the recombination constant towards deeply bound dimers, away from the trimer formation threshold, scales as

$$\mathcal{K}_{\rm rec}^{\rm deep} \propto \frac{\hbar}{m} b^3 \mathcal{V}_s^2 \alpha_{\rm res}, \tag{120}$$

for a given $\alpha_{\rm res}$, in the large scattering volume limit. We thus see from Eq. (90) that the formation of weakly bound dimers wins over the deeply bound ones in this limit. Note that the estimate of $\mathcal{K}_{\rm rec}^{\rm deep}$ also holds on the negative scattering volume side of the resonance, still restricting to the low relative incoming atomic momenta $k \ll 1/(\alpha_{\rm res}|\mathcal{V}_s|)^{1/2}$.

VI. EFFECT OF A NONRESONANT INTERACTION IN THE OPEN CHANNEL

In real life there exists an attractive van der Waals interaction between atoms in the open channel, responsible for a residual interaction in the p wave even very far from the Feshbach resonance. This residual interaction may be characterized by the so-called background scattering volume $\mathcal{V}_s^{\rm bg}$. Usually it is assumed that this residual interaction is weak $|\mathcal{V}_s^{\rm bg}| \approx b^3$, where the interaction range b is of the order of the van der Waals length, so that it is neglected in the vicinity of the Feshbach resonance as compared to the effect of the coupling to the closed channel [30,36]. However, with the pure closed channel coupling Hamiltonian (15) used in this paper, we found that several quantities were depending not only on the low-k scattering properties parametrized by $\mathcal{V}_s$ and α, but also on the range b of the potential, such as the threshold for trimer formation, which raises the issue of their dependence with the microscopic details of the model. Furthermore, we found that the atom-dimer scattering length assumes values smaller than b for broad Feshbach resonances, so that it is not evident that the residual interaction is really negligible.

To address these questions, we model the residual interaction by a separable potential of coupling constant g_0 with the same cutoff function $\boldsymbol{\chi}(\mathbf{k})$ as in the closed channel coupling. This amounts to adding to the Hamiltonian (15) the open channel interaction

$$V_{\rm open} = \frac{g_0}{2}\int \frac{d^3K d^3k d^3k'}{(2\pi)^9} \boldsymbol{\chi}(\mathbf{k}')\cdot\boldsymbol{\chi}^*(\mathbf{k}) \times a^\dagger_{1/2\mathbf{K}-\mathbf{k}'} a^\dagger_{1/2\mathbf{K}+\mathbf{k}'} a_{1/2\mathbf{K}+\mathbf{k}} a_{1/2\mathbf{K}-\mathbf{k}}. \tag{121}$$

We determine g_0 by relating it to $\mathcal{V}_s^{\rm bg}$ from the solution of the two-body problem. Then we solve the three-body problem again with the simultaneous inclusion of the closed channel coupling and the open channel interaction.

A. Modification of the two-body problem

The calculations proceed along the lines of Sec. III B. The same ansatz (19) for the two-body state vector applies; the new term emerging from the action of $V_{\rm open}$ is simply

$$V_{\rm open}|\Psi\rangle = g_0 \boldsymbol{\gamma}\cdot\int \frac{d^3k}{(2\pi)^3}\boldsymbol{\chi}(\mathbf{k}) a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}|0\rangle, \tag{122}$$

where we have set

$$\boldsymbol{\gamma} = \int \frac{d^3k}{(2\pi)^3} A(\mathbf{k})\boldsymbol{\chi}^*(\mathbf{k}). \tag{123}$$

From Schrödinger's equation at energy E, we find the remarkable property

$$(E-E_{\rm mol})\frac{\boldsymbol{\beta}}{2\Lambda} + \boldsymbol{\gamma} = \mathbf{0}, \tag{124}$$

so that $\boldsymbol{\gamma}$ can be expressed in terms of $\boldsymbol{\beta}$ and the new reduced scattering amplitude has a form very similar to the previous one (24),

$$f(k_0) = \frac{-mk_0^2 e^{-k_0^2 b^2}/(4\pi\hbar^2)}{\dfrac{3(E-E_{\rm mol})}{2\Lambda^2+g_0(E-E_{\rm mol})} - \displaystyle\int \frac{d^3k}{(2\pi)^3}\frac{k^2 e^{-k^2b^2}}{E+i0^+-\frac{\hbar^2k^2}{m}}}, \tag{125}$$

where $E=\hbar^2k_0^2/m$ is the energy of the two-body scattering state. This leads to a modified expression for the scattering volume

$$\frac{1}{\mathcal{V}_s} = \frac{1}{2\pi^{1/2}b^3} - \frac{6\pi\hbar^2}{m\Lambda^2}\frac{E_{\rm mol}}{1-g_0 E_{\rm mol}/(2\Lambda^2)}. \tag{126}$$

We first analyze the result very far from the p-wave Feshbach resonance. Taking the limit $E_{\rm mol}\to\infty$ in the above expression gives the background scattering volume as a function of the open channel coupling constant

$$\frac{1}{\mathcal{V}_s^{\rm bg}} = \frac{1}{2\pi^{1/2}b^3} + \frac{12\pi\hbar^2}{mg_0}. \tag{127}$$

Since the van der Waals interaction is attractive, we take $g_0<0$ in all what follows. Then one sees that the background scattering volume has a dependence with $|g_0|$ similar to the left part of the Fig. 1 calculated for a square well potential:

For increasing values of $m|g_0|/\hbar^2$ starting from zero, $\mathcal{V}_s^{\mathrm{bg}}$ decreases from zero to $-\infty$, it diverges on the critical value

$$\frac{m|g_0^c|}{\hbar^2} = 24\pi^{3/2}b^3, \tag{128}$$

then it decreases from $+\infty$ down to $2\pi^{1/2}b^3$. The divergence is due to the formation of a dimer in the open channel, and this dimer is deeply bound when g_0 is away from the critical value g_0^c. The existence of a deeply bound dimer would deeply change the physical nature of the three-body problem with respect to our previous analysis: e.g., it would open a decay channel to the trimer, which would not exist as a true stationary state anymore but at most as a resonance. We thus take from now on $|g_0|<|g_0^c|$ so that $\mathcal{V}_s^{\mathrm{bg}}<0$. We shall keep in mind that

$$\mathcal{V}_s^{\mathrm{bg}} \approx -b^3 \tag{129}$$

for a nonresonant interaction in the open channel. To be complete, we have also calculated the value of the parameter α in the presence of open-channel interactions only

$$\alpha_{\mathrm{bg}} = \frac{1}{\pi^{1/2}b} + \frac{b^2}{\mathcal{V}_s^{\mathrm{bg}}}, \tag{130}$$

which can have any sign since the open-channel interaction is not resonant.

We now come back to the vicinity of the Feshbach resonance, where the closed channel coupling is no longer negligible. First we can prove that, under the condition $\mathcal{V}_s^{\mathrm{bg}}<0$, there is no bound state in the two-body problem for $\mathcal{V}_s<0$ and there is one for $\mathcal{V}_s>0$, as expected; the proof was obtained using the argument of a monotonic variation of an appropriate function as done in the paragraph below Eq. (30)

We restrict for simplicity to the exactly resonant case $\mathcal{V}_s=\infty$. The corresponding values of the closed-channel detuning E_{mol} and of α are then

$$\frac{g_0 E_{\mathrm{mol}}^{\mathrm{res}}}{2\Lambda^2} = \frac{\mathcal{V}_s^{\mathrm{bg}}}{2\pi^{1/2}b^3}, \tag{131}$$

$$\alpha_{\mathrm{res}} = \frac{1}{\pi^{1/2}b} + \frac{6\pi\hbar^4}{m^2\Lambda^2\left(1-\dfrac{\mathcal{V}_s^{\mathrm{bg}}}{2\pi^{1/2}b^3}\right)^2}. \tag{132}$$

We see that the range of variation of α_{res} remains the same as in our previous model. The scattering amplitude, analytically continued to negative energies $E=-\hbar^2q^2/m$, $q>0$, can be put in the simple form

$$\frac{e^{q^2b^2}}{f(iq)} \overset{\mathcal{V}_s\to\infty}{=} qe^{q^2b^2}\,\mathrm{erfc}(qb) - \frac{1}{b\pi^{1/2}} - \frac{\alpha_{\mathrm{res}} - 1/(b\pi^{1/2})}{1+q^2/q_{\mathrm{open}}^2}, \tag{133}$$

making it apparent that the main effect of the open channel interaction is to introduce a new scale q_{open} for the wave vectors, such that

$$q_{\mathrm{open}}^2 = \frac{g_0E_{\mathrm{mol}}^{\mathrm{res}} - 2\Lambda^2}{g_0\hbar^2/m} = \frac{-1/\mathcal{V}_s^{\mathrm{bg}}}{\alpha_{\mathrm{res}} - 1/(\pi^{1/2}b)}. \tag{134}$$

This allows us to reach first conclusions on the effect of the open channel interaction on the properties of the original model (15).

For a broad Feshbach resonance $\Lambda \gg \hbar^2b^{1/2}/m$, we find that $q_{\mathrm{open}}b>1$, so the open channel interaction should have a weak effect. In particular, this suggests that the trimer states in the regime of rather broad resonances should be weakly affected. Furthermore, if $\mathcal{V}_s^{\mathrm{bg}}$ is weak enough to have $q_{\mathrm{open}}b>1$ at the threshold for the formation of the trimers in our previous model,

$$|\mathcal{V}_s^{\mathrm{bg}}| < \frac{b^3}{\alpha_{\mathrm{th}}b - \dfrac{1}{\pi^{1/2}}}, \tag{135}$$

then the threshold itself should be weakly affected by the open channel interaction. For a narrow Feshbach resonance, one has $q_{\mathrm{open}}b<1$ (except for $|\mathcal{V}_s^{\mathrm{bg}}| \ll b^3$); in this case the effect of the open channel coupling is more difficult to guess: It may depend on the considered quantity and a more detailed analysis is required.

If q_{open} was smaller than the estimate $q_{\mathrm{dim}} \approx 1/(\alpha_{\mathrm{res}}\mathcal{V}_s)^{1/2}$ for the wave vector associated to the dimer binding energy close to the resonance, then the open channel interaction would have a dramatic effect. However, one finds $q_{\mathrm{dim}}^2/q_{\mathrm{open}}^2 < |\mathcal{V}_s^{\mathrm{bg}}/\mathcal{V}_s|$ so that $q_{\mathrm{dim}} \ll q_{\mathrm{open}}$ in the resonant regime.

A last relevant quantity is the probability to find the dimer in the closed channel. In the limit $\mathcal{V}_s \to +\infty$, after some calculation, we find

$$p_{\mathrm{closed}}^{\mathrm{res}} = \frac{6\pi\hbar^4}{m^2\Lambda^2}\alpha_{\mathrm{res}}^{-1}\left(1 - \frac{g_0E_{\mathrm{mol}}^{\mathrm{res}}}{2\Lambda^2}\right)^{-2}, \tag{136}$$

$$= 1 - \frac{1}{\pi^{1/2}\alpha_{\mathrm{res}}b}. \tag{137}$$

The relation (33) is therefore affected by the open channel interaction, whereas Eq. (34) is not.

B. Modification of the three-body problem

We now solve the three-body problem in presence of both the closed channel coupling and the open channel interaction. The previous ansatz (35) applies, but a new term arises in Schrödinger's equation of eigenenergy E,

$$V_{\mathrm{open}}|\Psi\rangle = g_0\int \frac{d^3Kd^3k}{(2\pi)^6}\boldsymbol{\chi}(\mathbf{k})\cdot\boldsymbol{\gamma}(\mathbf{K})a_{1/2\mathbf{K}+\mathbf{k}}^\dagger a_{1/2\mathbf{K}-\mathbf{k}}^\dagger a_{-\mathbf{K}}^\dagger|0\rangle, \tag{138}$$

where we have set

$$\boldsymbol{\gamma}(\mathbf{K}) = \int \frac{d^3k}{(2\pi)^3}\left[A(\mathbf{K},\mathbf{k}) + 2A\left(-\frac{1}{2}\mathbf{K}+\mathbf{k}, -\frac{3}{4}\mathbf{K}-\frac{1}{2}\mathbf{k}\right)\right] \times \boldsymbol{\chi}^*(\mathbf{k}). \tag{139}$$

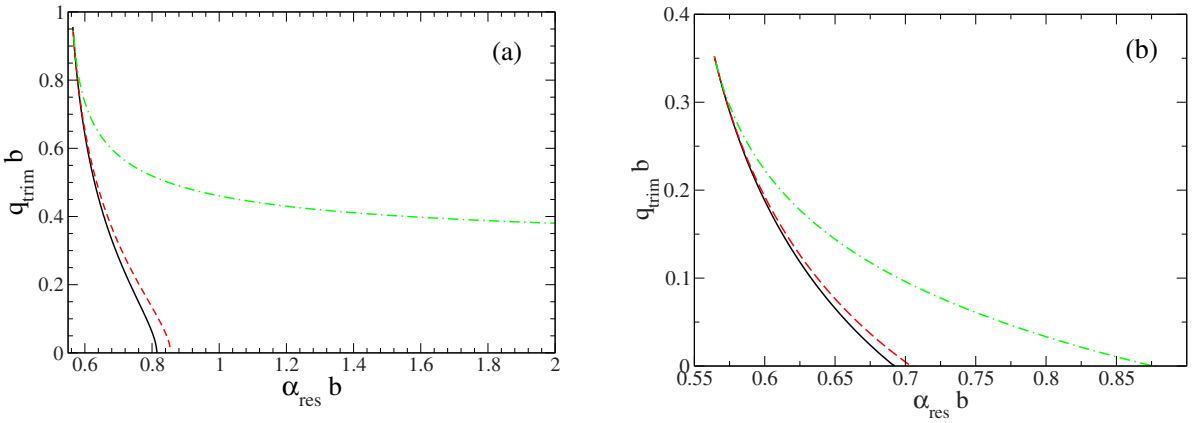

FIG. 10. (Color online) In presence of open channel interactions, characterized by a fixed value of the background scattering volume $\mathcal{V}_s^{bg}$, parameter q_{trim} of the trimer (when it exists) (a) in the even sector and (b) in the odd sector, as a function of $\alpha_{res}b$. Here the scattering volume $\mathcal{V}_s$ is infinite and α_{res} is the corresponding value of the parameter α for the modified scattering amplitude, see Eq. (132). Solid line (black): $\mathcal{V}_s^{bg}=0$. Dashed line (red): $\mathcal{V}_s^{bg}=-b^3$. Dashed-dotted line (green): $\mathcal{V}_s^{bg}=-10b^3$.

Schrödinger's equation projected onto the molecular subspace is unaffected by the open channel interaction, since there is only one atom in that subspace. So Eq. (36) still holds exactly. Then one immediately sees that a simple relation relates $\boldsymbol{\gamma}$ to $\boldsymbol{\beta}$:

$$\left[E - E_{\rm mol} - \frac{3\hbar^2K^2}{4m}\right]\frac{\boldsymbol{\beta}(\mathbf{K})}{2\Lambda} + \boldsymbol{\gamma}(\mathbf{K}) = 0. \tag{140}$$

One may then take as unknown any convenient combination of $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$.

In Schrödinger's equation projected onto the atomic subspace, a new term appears, but of the same structure as the term involving $\boldsymbol{\beta}$ in Eq. (37). Thus the modification to Eq. (38) is minor,

$$A(\mathbf{K},\mathbf{k}) = A_0(\mathbf{K},\mathbf{k}) + \frac{[g_0\boldsymbol{\gamma}(\mathbf{K}) - \Lambda\boldsymbol{\beta}(\mathbf{K})]\cdot\boldsymbol{\chi}(\mathbf{k})}{E + i0^+ - \frac{\hbar^2}{m}\left(\frac{3}{4}K^2 + k^2\right)}. \tag{141}$$

This results immediately suggests which combination of $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ is convenient: We introduce

$$\boldsymbol{\beta}_{\rm eff}(\mathbf{K}) = \boldsymbol{\beta}(\mathbf{K}) - \frac{g_0}{\Lambda}\boldsymbol{\gamma}(\mathbf{K}), \tag{142}$$

the overall factor being such that $\boldsymbol{\beta}_{\rm eff}$ reduces to $\boldsymbol{\beta}$ in the absence of open channel interaction. As a consequence, our unknown field is now

$$\boldsymbol{\beta}_{\rm eff}(\mathbf{K}) = \left[1 + \frac{g_0}{2\Lambda^2}\left(E - E_{\rm mol} - \frac{3\hbar^2K^2}{4m}\right)\right]\boldsymbol{\beta}(\mathbf{K}). \tag{143}$$

Eliminating $\boldsymbol{\beta}$ and A in terms of this unknown field in Eq. (36), we find after some calculations and using Eq. (125) that $\boldsymbol{\beta}_{\rm eff}(\mathbf{K})$ solves the same equation (40) as $\boldsymbol{\beta}$ in our previous model, provided that the modified scattering amplitude (125) is used. As a consequence, all the numerical and most of the analytical techniques developed for the previous model may be reused for the new model.

Existence of weakly bound trimers. We reproduce the numerical calculations of Sec. IV C with the scattering amplitude modified by the open channel interaction. We restrict for simplicity to an infinite scattering volume: See Fig. 10 giving $q_{\rm trim}b$ as a function of $\alpha_{\rm res}b$, where the energy of the trimer is $-\hbar^2q_{\rm trim}^2/m$, for (a) the even sector and (b) the odd one. Then we see that the trimer state still exists for low $\alpha_{\rm res}b$. For a small background scattering volume $\mathcal{V}_s^{\rm bg}=-b^3$, its energy dependence with $\alpha_{\rm res}b$ is only weakly affected by the open channel interaction, as expected from the qualitative condition (135), and the threshold is only slightly shifted. For a much more negative background scattering volume $\mathcal{V}_s^{\rm bg}=-10b^3$, there is simply a larger shift in the odd sector, but the conclusion is radically changed in the even sector, see Fig. 10(a): the trimer seems to exist now for all values of $\alpha_{\rm res}b$. This may be understood as follows: in the large $\alpha_{\rm res}b$ limit, the coupling Λ to the closed channel tends to zero, so does $E_{\rm mol}^{\rm res}$, see Eq. (131), and for a fixed nonzero q, the scattering amplitude (125) converges to the one of a single channel model with a scattering volume $\mathcal{V}_s^{\rm bg}$. If $|\mathcal{V}_s^{\rm bg}|/b^3$ is large enough, then this single channel model can indeed support a trimer [77]. On the other hand, the fact that the threshold for trimer formation survives in the odd sector up to higher values of $|\mathcal{V}_s^{\rm bg}|/b^3$ than in the even sector may be understood from the qualitative argument (135).

S-wave atom-dimer scattering. We consider for $\mathcal{V}_s>0$ the scattering of an atom on a dimer in the limit of a vanishing relative kinetic energy, so that $E=-E_{\rm dim}$, where $E_{\rm dim}$ is the dimer binding energy for the new model, and the corresponding s-wave scattering is characterized by the atom-dimer scattering length $a_{\rm ad}$. Then the field $\boldsymbol{\beta}(\mathbf{K})$ is given by the ansatz (44). As a consequence, the effective field (143) will have the same structure; the delta distribution $\delta(\mathbf{K})$ will simply be multiplied by the factor $1-g_0(E_{\rm dim}+E_{\rm mol})/(2\Lambda^2)$. Also the part of $\boldsymbol{\beta}_{\rm eff}$ diverging as $1/K^2$ will be related to the part of $\boldsymbol{\beta}(\mathbf{K})$ diverging as $1/K^2$ by exactly the same factor. So that one may read the value of the atom-dimer scattering length directly from the effective field. As a consequence,

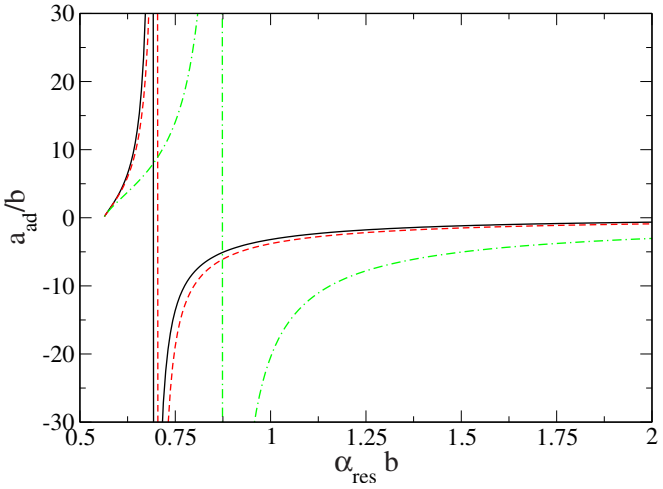

FIG. 11. (Color online) Atom-dimer scattering length a_{ad} as a function of $\alpha_{\mathrm{res}}b$ for the model including the open channel interaction, for an infinite scattering volume $\mathcal{V}_s$. Solid line (black): $\mathcal{V}_s^{\mathrm{bg}}=0$. Dashed line (red): $\mathcal{V}_s^{\mathrm{bg}}=-b^3$. Dashed-dotted line (green): $\mathcal{V}_s^{\mathrm{bg}}=-10b^3$.

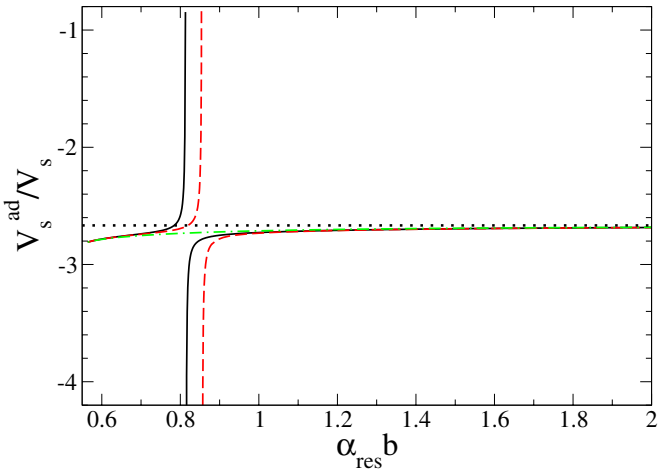

FIG. 12. (Color online) In the model including an open channel interaction, atom-dimer scattering volume $\mathcal{V}_s^{\mathrm{ad}}$ for a total spin $J=1$ (as detailed in Sec. IV D) as a function of $\alpha_{\mathrm{res}}b$, for a fixed value of the atom-atom scattering volume $\mathcal{V}_s=10^4b^3$ (solid black line), but for various background scattering volumes in the open channel: $\mathcal{V}_s^{\mathrm{bg}}=0$ (black solid line), $\mathcal{V}_s^{\mathrm{bg}}=-b^3$ (dashed red line), $\mathcal{V}_s^{\mathrm{bg}}=-10b^3$ (dashed-dotted green line). $\mathcal{V}_s^{\mathrm{ad}}$ is expressed in units of $\mathcal{V}_s$. Dotted horizontal line: analytical prediction Eq. (66) in the limit $\mathcal{V}_s\to+\infty$.

one has to solve the same integral equation (46), just changing the diagonal part $D(K)$ to account for the new scattering amplitude. The corresponding numerical results for a_{ad} are presented in Fig. 11, as functions of α_{res}, for an infinite scattering volume $\mathcal{V}_s$. For the considered values of $|\mathcal{V}_s^{\mathrm{bg}}/b^3|$, the open channel interaction does not qualitatively change the result: There exists a threshold for the formation of a trimer in the odd sector, see Fig. 10(b), and we recover the divergence of a_{ad} at this threshold already observed in our first model.

What happens to the atom-dimer scattering length in the large $\alpha_{\mathrm{res}}b$ limit? Is the analytical prediction (55) obtained in our previous model still valid? In the presence of interactions in the open channel, it seems surprising that a_{ad} can tend to zero at large $\alpha_{\mathrm{res}}b$, since the scattering amplitude for $\Lambda\to0$ for a finite q tends to the one of a single channel model with a scattering volume $\mathcal{V}_s^{\mathrm{bg}}$. This limit, however, is not reached uniformly in q: e.g., for $q\simeq q_{\mathrm{dim}}$, $q\ll q_{\mathrm{open}}$ and the scattering amplitude remains very close to the one of the two channel model with no open channel interaction. This nonuniformity of the $\Lambda\to0$ limit is also revealed at $\mathcal{V}_s=\infty$ from the fact that the parameter α_{res} is very different from Eq. (130) which one would have in the absence of coupling to the closed channel. Mathematically, Eq. (51) still applies, if one considers the effective field $\boldsymbol{\beta}_{\mathrm{eff}}(\mathbf{K})$ rather than $\boldsymbol{\beta}(\mathbf{K})$, but in the infinite scattering volume limit analysis, the function $h(K)$ in Eq. (52) is changed by the open-channel interaction in

$$h(K)=qe^{q^2b^2}\,\mathrm{erfc}(qb)+\left(\alpha_{\mathrm{res}}-\frac{1}{b\pi^{1/2}}\right)\frac{q^2}{q^2+q_{\mathrm{open}}^2},\tag{144}$$

with $q=\sqrt{3}K/2$. Contrarily to our previous model, the function $h(K)$ increases from 0 to α_{res} when K increases from 0 to infinity, in practice to values $\gg1/b$. We can no longer assume $1/(\alpha_{\mathrm{res}}-h(K))\simeq1/\alpha_{\mathrm{res}}$ for all K in the large α_{res} limit, so we have only the weaker result that the right hand side of Eq. (51) is $O(1)$ in this limit. Since $h(K=0)=0$, $D(K)/K^2$ still converges to $\alpha_{\mathrm{res}}/4$ in $K=0$ and we get

$$a_{\mathrm{ad}}\overset{\alpha_{\mathrm{res}}b\gg1}{=}O\left(\frac{1}{\alpha_{\mathrm{res}}}\right).\tag{145}$$

We have successfully compared this analytical prediction to the numerics. For very large values of $\alpha_{\mathrm{res}}b$ (not shown), we numerically find for $\mathcal{V}_s^{\mathrm{bg}}<0$ that a_{ad} tends to zero as C/α_{res}, where the constant C depends on the background scattering volume $C\simeq-0.69$ for $\mathcal{V}_s^{\mathrm{bg}}=-b^3$, and $C\simeq-4.1$ for $\mathcal{V}_s^{\mathrm{bg}}=-10b^3$.

P-wave atom-dimer scattering. Now the incoming atom and dimer have a relative orbital momentum $L=1$, so the vanishing kinetic energy limit of the scattering $E\to-E_{\mathrm{dim}}$ is characterized by an atom-dimer scattering volume $\mathcal{V}_s^{\mathrm{ad}}$, with the ansatz (57) for $\boldsymbol{\beta}(\mathbf{K})$ in the sector of total spin $J=1$ (see Sec. IV D). Since the factor linking $\boldsymbol{\beta}_{\mathrm{eff}}(\mathbf{K})$ to $\boldsymbol{\beta}(\mathbf{K})$ varies only quadratically with K, it may be replaced by its $K=0$ value in front of the gradient of $\delta(\mathbf{K})$, so that $\boldsymbol{\beta}_{\mathrm{eff}}$ and $\boldsymbol{\beta}$ have the same low-K behavior, from which $\mathcal{V}_s^{\mathrm{ad}}$ is readily extracted. We numerically solve Eq. (58) updating the values of the scattering amplitude $f(k)$ and the dimer binding energy E_{dim} to include the open channel interaction. The dependence of $\mathcal{V}_s^{\mathrm{ad}}$ with α_{res} is shown in Fig. 12, for a fixed and large scattering volume $\mathcal{V}_s$ and for various values of the background scattering volume. It is apparent that $\mathcal{V}_s^{\mathrm{ad}}$ is weakly affected by the open channel interaction, apart in the vicinity of the even trimer formation threshold (if it exists). This can be understood analytically, realizing that the reasoning leading to Eq. (66) still applies in presence of the considered open channel interaction. This confirms the "universality" of the asymptotic behavior

$$\mathcal{V}_s^{\mathrm{ad}} \sim -\frac{8}{3}\mathcal{V}_s \tag{146}$$

which was expected in Sec. IV D from the fact that it does not depend on the potential range b.

Recombination rate to weakly bound dimers. Finally, we consider the scattering problem of three atoms in the zero total energy limit, for $\mathcal{V}_s>0$ and in presence of open channel interactions. We take for the effective field $\boldsymbol{\beta}_{\mathrm{eff}}(\mathbf{K})$ the same ansatz as in Eq. (73), putting the subscript "eff" on the various functions of the ansatz. The function $\mathcal{G}_{\mathrm{eff}}(K)$ is still given by Eq. (75) but the scattering amplitude is now changed. As $\mathcal{G}_{\mathrm{eff}}$ is, however, multiplied by the gradient of a delta, only its first derivatives in $\mathbf{K}=0$ matter, so only the scattering volume comes out, and g_{eff} obeys the integral equation (76) with the same source term and the updated scattering amplitude. In the large scattering volume limit, away from the even trimer formation threshold if it exists, we can thus recycle Eq. (89), since it involves only the low-momentum behavior $K \leq K_{\mathrm{dim}} \ll q_{\mathrm{open}}$ of the scattering amplitude and properties of the function C_0-C_2; this leads to

$$g_{\mathrm{eff}}(K_{\mathrm{dim}}) \sim -\frac{144\pi\hbar^2}{m\Lambda\alpha_{\mathrm{res}}}\frac{\mathcal{V}_s}{K_{\mathrm{dim}}}K_0 k_0^{\perp}. \tag{147}$$

On the other hand, the reasoning leading to the recombination constant $\mathcal{K}_{\mathrm{rec}}$ in Eq. (84) still holds, with the function $g(K)$ [and not $g_{\mathrm{eff}}(K)$] and the closed channel probability p_{closed} modified by the open channel interaction. The functions g and g_{eff} differ, because $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_{\mathrm{eff}}$ differ by a factor depending on K. Since the factor in between square brackets in Eq. (143) cannot vanish for our choice $\mathcal{V}_s^{\mathrm{bg}}<0$, $\boldsymbol{\beta}(K)$, and $\boldsymbol{\beta}_{\mathrm{eff}}$ both have a single pole in $K=K_{\mathrm{dim}}$, with

$$g(K_{\mathrm{dim}}) = \frac{g_{\mathrm{eff}}(K_{\mathrm{dim}})}{1-\frac{g_0}{2\Lambda^2}(E_{\mathrm{mol}}+\hbar^2 q_{\mathrm{dim}}^2/m)}. \tag{148}$$

In the large $\mathcal{V}_s/b^3$ limit, one can neglect q_{dim} in the denominator of this expression, which amounts to neglecting q_{dim} with respect to q_{open}. Then we use the expression (136) and remarkably we recover exactly the same asymptotic behavior for $\mathcal{K}_{\mathrm{rec}}$ as in the previous model, under the assumption $\mathcal{V}_s^{\mathrm{bg}}<0$:

$$\mathcal{K}_{\mathrm{rec}} \overset{\mathcal{V}_s \gg b^3}{\sim} \frac{\hbar}{m}(48\pi)^2\left(\frac{\mathcal{V}_s^5}{3\alpha_{\mathrm{res}}}\right)^{1/2}. \tag{149}$$

This indicates some 'universality' of this result, which could be hoped from the fact that it does not depend on the interaction range b.

VII. CONCLUSION

We have solved the free space three-body problem for single spin state fermions resonantly interacting in p-wave via a two-channel Feshbach resonance, in the sector of total angular momentum one. The central model that we used to describe the interaction depends on three parameters, the scattering volume $\mathcal{V}_s$ which diverges on resonance, an effective range parameter α and the spatial range of the interaction b. Whereas b is of the order of the van der Waals length, the parameter α on resonance can range from a strictly positive minimal value of the order of $1/b$ up to plus infinity, the minimal value being model dependent and equal to $1/(b\pi^{1/2})$ for our Gaussian cutoff function. In present experiments one estimates $\alpha b \approx 3$ for ^{40}K [25] and $\alpha b \approx 3$ for ^{6}Li [30] on resonance. The two-body scattering amplitude for $\mathcal{V}_s=\infty$ is $f_k \simeq -1/\alpha$ for low relative momenta $k \ll 1/b$, so that $|f_k|$ right on resonance is at most of the order of b, which is extremely small as compared to the usual s-wave unitary limit. As a consequence, the resonant three-body problem has very different properties from the s-wave one.

First, it does not exhibit the Efimov effect but it admits two trimers, one with even parity and the other with odd parity, for low enough values of αb. Since the considered sector is of angular momentum one, each trimer is threefold degenerate. For $\mathcal{V}_s/b^3$ large and negative, our model Hamiltonian does not have a two-body bound state, so that these trimer states are examples of Borromean states. However, we estimate that the spontaneous decay rate Γ_{loss} of the trimers, due to the formation of deeply bound dimers present in current experiments with real atoms, eventually becomes larger than the binding energy of the trimers (over $\hbar$) if one gets very close to their formation threshold: in the limit of a vanishing trimer binding energy, Γ_{loss} tends to a nonzero limit for the even trimer, and vanishes as the square root of the trimer binding energy for the odd trimer.

Second, the atom-dimer scattering length a_{ad}, characterizing the low-energy (that is s wave) scattering of an atom on a dimer, assumes small values, of the order of b (or even below in absolute value for ultranarrow Feshbach resonances), except close to the odd dimer formation threshold where it diverges. The fact that a_{ad} depends on the interaction range b shows that it is not a "universal" quantity and it is sensitive to the microscopic details of the interaction. Furthermore, the loss constant in the inelastic atom-dimer s-wave scattering (due to the formation of deeply bound dimers) is proportional to $\hbar b/m$, away from the trimer threshold, so that the inelastic rate may dominate over the elastic one. A similar conclusion was reached for the elastic vs inelastic scattering of p wave weakly bound dimers [44]. We have also studied the atom-dimer scattering when the incoming relative wave is a p wave: in the considered sector of total angular momentum one, the corresponding atom-dimer scattering volume $\mathcal{V}_s^{\mathrm{ad}}$ is shown analytically to become proportional to the atom-atom scattering volume $\mathcal{V}_s$ away from the even trimer formation threshold, see Eq. (66). This asymptotic result looks "universal," since it does not involve the interaction range b.

The recombination rate of three atoms into weakly bound dimers, calculated in this work in the limit of low relative atomic wave vectors $k \ll (\alpha\mathcal{V}_s)^{-1/2}$, has properties more similar to the s-wave case. What remains specific to the p-wave case is that the rate is proportional to the square of the mean kinetic energy per particle, see Eq. (81), as already pointed out in Ref. [42]. Apart from that, it includes as a factor the recombination constant $\mathcal{K}_{\mathrm{rec}}$, which is large close to the resonance: it diverges as $\mathcal{V}_s^{5/2}/\alpha^{1/2}$ in the large scattering volume limit, see Eq. (90), an asymptotic expression valid away

from the even trimer formation threshold and which is "universal," since it does not involve the interaction range b and it is not sensitive to the choice of the cutoff function $\boldsymbol{\chi}(\mathbf{k})$ of the two-channel model. In the large scattering volume limit, the recombination constant towards weakly bound dimers dominates over the recombination constant towards deeply bound dimers, which scales as $\mathcal{V}_s^2$ only [still in the limit of low relative atomic momenta with respect to $1/(\alpha|\mathcal{V}_s|)^{1/2}$]. If one applies this last result to a degenerate macroscopic gas, with a Fermi momentum $k_F \approx 1/(\alpha|\mathcal{V}_s|)^{1/2}$, one finds a number of recombination events to deeply bound dimers per unit of time and volume scaling as

$$\gamma_{\rm rec} = O\left(\frac{\hbar}{m} b^4 n^3\right), \tag{150}$$

where we used $1/\alpha = O(b)$ and n is the gas density.

In the last part of the paper, we have made the modelization more realistic by including a fourth parameter, a direct attractive interaction between atoms in the open channel. Physically, this interaction is supposed to be not resonant so that it has a weak background scattering volume $\mathcal{V}_s^{\rm bg}$, of the order of b^3 and much smaller than $\mathcal{V}_s$. To stay in the regime where no deeply bound dimers exist in the Hamiltonian on resonance, one further imposes $\mathcal{V}_s^{\rm bg} < 0$. We then find that the existence of the trimers is preserved. They remain weakly bound in the vicinity of some threshold values of α, provided that $|\mathcal{V}_s^{\rm bg}|$ does not exceed a few b^3; these threshold values for α, however, depend on b and are not "universal." The atom-dimer scattering length $a_{\rm ad}$ is significantly changed by the open channel interaction in the limit of ultranarrow Feshbach resonances, where it now tends to zero for large αb as $1/\alpha$, rather than as $1/(\alpha^2 b)$ in our 3-parameter model; this confirms the nonuniversal character of $a_{\rm ad}$. On the contrary, in presence of open channel interactions, the *same* asymptotic expressions (66) for the atom-dimer scattering volume $\mathcal{V}_s^{\rm ad}$ and Eq. (90) for the recombination constant $\mathcal{K}_{\rm rec}$ to weakly bound dimers are obtained as in our 3-parameter model, in terms of $\mathcal{V}_s$, and in terms of $\mathcal{V}_s$ and α respectively, which confirms the "universal" character of these results.

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APPENDIX A: INTEGRAL EQUATIONS FOR THE *K*-DEPENDENT PART OF THE ANSATZ FOR $\boldsymbol{\beta}$

When injecting the ansatz (41) or (42) in the homogeneous part of the equation (40) for $\boldsymbol{\beta}$, one faces the calculation of the following angular averages over the direction of $\mathbf{k}$,

$$\mathbf{I}_0(\mathbf{K},k) = \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \left(\frac{1}{2}\mathbf{K} + \mathbf{k}\right) \frac{(\mathbf{K}\cdot\mathbf{e}_z) + \frac{1}{2}(\mathbf{k}\cdot\mathbf{e}_z)}{q^2 + K^2 + k^2 + \mathbf{K}\cdot\mathbf{k}} e^{-\mathbf{K}\cdot\mathbf{k}b^2}, \tag{A1}$$

$$\mathbf{I}_1(\mathbf{K},k) = \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \left(\frac{1}{2}\mathbf{K} + \mathbf{k}\right) \times \frac{\mathbf{K}\cdot\left[\left(\frac{\mathbf{k}\cdot\mathbf{e}_z}{k}\right)\mathbf{e}_y - \left(\frac{\mathbf{k}\cdot\mathbf{e}_y}{k}\right)\mathbf{e}_z\right]}{q^2 + K^2 + k^2 + \mathbf{K}\cdot\mathbf{k}} e^{-\mathbf{K}\cdot\mathbf{k}b^2}, \tag{A2}$$

$$\mathbf{I}_2(\mathbf{K},k) = \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \left(\frac{1}{2}\mathbf{K} + \mathbf{k}\right) \times (\mathbf{k}\cdot\mathbf{e}_z) \frac{\frac{1}{2} + \frac{\mathbf{K}\cdot\mathbf{k}}{k^2}}{q^2 + K^2 + k^2 + \mathbf{K}\cdot\mathbf{k}} e^{-\mathbf{K}\cdot\mathbf{k}b^2}. \tag{A3}$$

Note that, for the calculations performed in this paper, one can restrict to the case of a nonpositive total energy E so that we have set $E = -\hbar^2 q^2/m$, $q \geq 0$, and we have omitted $i0^+$.

To perform this angular integration, we use spherical coordinates of polar axis $\mathbf{K}/K$. We need to evaluate first the integral over the azimuthal angle φ,

$$\mathbf{B}_1 = \int_0^{2\pi} \frac{d\varphi}{2\pi}\mathbf{k} = k\cos\theta\frac{\mathbf{K}}{K}, \tag{A4}$$

$$\mathbf{B}_2 = \int_0^{2\pi} \frac{d\varphi}{2\pi}(\mathbf{k}\cdot\mathbf{e}_\alpha)\mathbf{k} = \sum_{i,j} k^2 B_{ij}(\mathbf{E}_i\cdot\mathbf{e}_\alpha)\mathbf{E}_j, \tag{A5}$$

where α stands for y or z, $\{\mathbf{E}_{i=1,2,3}\}$ is an orthonormal basis with $\mathbf{E}_3 = \mathbf{K}/K$ and

$$B_{ij} = \int_0^{2\pi} \frac{d\varphi}{2\pi}(\mathbf{k}\cdot\mathbf{E}_i)(\mathbf{k}\cdot\mathbf{E}_j), \tag{A6}$$

$$= \frac{1}{2}[(1-\cos^2\theta)\delta_{ij} + (3\cos^2\theta - 1)\delta_{i3}\delta_{j3}]. \tag{A7}$$

This leads to

$$\mathbf{B}_2 = \frac{k^2}{2}\left[(1-\cos^2\theta)\mathbf{e}_\alpha + (3\cos^2\theta - 1)\frac{(\mathbf{K}\cdot\mathbf{e}_\alpha)\mathbf{K}}{K^2}\right]. \tag{A8}$$

The integration over the polar angle θ involves basically the integral

$$C_n = \int_{-1}^{1} \frac{dx}{2}\frac{x^n e^{-tx}}{v+x}, \tag{A9}$$

with $v > 1$, $t > 0$ and integer n. The result is

$$C_0 = \frac{e^{vt}}{2}[E_1(vt-t) - E_1(vt+t)], \tag{A10}$$

$$C_1 = -vC_0 + j_0(it), \tag{A11}$$

$$C_2 = -vC_1 + ij_1(it), \tag{A12}$$

$$C_3 = -vC_2 - j_2(it) - \frac{i}{t}j_1(it), \tag{A13}$$

where $E_1(z)$ is the exponential integral

$$E_1(z) = \int_1^{+\infty} ds \frac{e^{-sz}}{s} \tag{A14}$$

and $j_n(z)$ are the usual spherical Bessel functions. A straightforward integration over the θ angle leads to

$$\mathbf{I}_0(\mathbf{K},k) = \frac{u}{4}(C_0 - C_2)\mathbf{e}_z + \left[\frac{u}{4}(3C_2 - C_0) + \frac{5}{4}C_1 + \frac{1}{2u}C_0\right]\frac{(\mathbf{K}\cdot\mathbf{e}_z)\mathbf{K}}{K^2}, \tag{A15}$$

$$\mathbf{I}_1(\mathbf{K},k) = \frac{1}{2}(C_2 - C_0)\left[\frac{\mathbf{K}\cdot\mathbf{e}_z}{K}\mathbf{e}_y - \frac{\mathbf{K}\cdot\mathbf{e}_y}{K}\mathbf{e}_z\right], \tag{A16}$$

$$\mathbf{I}_2(\mathbf{K},k) = \left[\frac{u}{4}(C_0 - C_2) + \frac{1}{2}(C_1 - C_3)\right]\mathbf{e}_z + \left[\frac{u}{4}(3C_2 - C_0) + \frac{1}{2}(3C_3 - C_1) + \frac{1}{4}C_1 + \frac{1}{2u}C_2\right]\frac{(\mathbf{K}\cdot\mathbf{e}_z)\mathbf{K}}{K^2}, \tag{A17}$$

with

$$u = \frac{k}{K}, \quad v = \frac{q^2 + K^2 + k^2}{Kk}, \quad t = b^2 Kk. \tag{A18}$$

For the odd sector, $E<0$ so the source term A_0 vanishes. Projecting Eq. (40) onto the two components $\mathbf{e}_z$ and $(\mathbf{K}\cdot\mathbf{e}_z)\mathbf{K}/K^2$, we obtain an integral system of coupled equations for $B_{L=0}(K)$ and $B_{L=2}(K)$, given in the main text, see Eq. (46), where we have introduced the two by two matrix

$$M(K,k) = \begin{pmatrix} \frac{k}{4K}(C_0 - C_2) & -\frac{k}{4K}(C_0 - C_2) - \frac{1}{2}(C_1 - C_3) \\ -\frac{k}{4K}(3C_2 - C_0) - \frac{5}{4}C_1 - \frac{K}{2k}C_0 & \frac{k}{4K}(3C_2 - C_0) + \frac{3}{2}C_3 - \frac{1}{4}C_1 + \frac{K}{2k}C_2 \end{pmatrix}. \tag{A19}$$

The resulting integral operator can be made Hermitian by the change of variables

$$\begin{pmatrix} B_{L=0}(K) \\ B_{L=2}(K) \end{pmatrix} = P^{-1}\begin{pmatrix} b_0(K)/K \\ b_2(K)/K \end{pmatrix} \tag{A20}$$

with the two by two transformation matrix

$$P = \begin{pmatrix} 2^{1/4} & 0 \\ -2^{-1/4} & 2^{-1/4} \end{pmatrix}. \tag{A21}$$

This results in the Hermitian integral equation

$$0 = D(K)\begin{pmatrix} b_0(K) \\ b_2(K) \end{pmatrix} + \frac{4}{\pi}\int_0^{+\infty} dkKke^{-5(K^2+k^2)b^2/8}N(K,k)\begin{pmatrix} b_0(k) \\ b_2(k) \end{pmatrix}, \tag{A22}$$

with the two by two matrix $N(K,k) = PM(K,k)P^{-1}$ satisfying $N^\dagger(K,k) = N(k,K)$. For the even sector, the integral equation for $B_{L=1}$ is given directly in the main text, see Eq. (76).

To conclude this appendix, we briefly explain how to normalize the state vector of the trimer (when it exists). The normalization can be done directly in momentum space by integration over internal variables, that is after having singled out the total momentum variables $\mathbf{Q}$. In the sector of Eq. (35) with one molecule, using the fact that the parametrization of the molecular and atomic momenta $\mathbf{k}_{\rm mol} = \mathbf{Q}/2 + \mathbf{K}$, $\mathbf{k}_{\rm at} = \mathbf{Q}/2 - \mathbf{K}$, has a unit Jacobian, we find

$$\langle \Psi_{\rm mol} | \Psi_{\rm mol} \rangle = \int \frac{d^3\mathbf{K}}{(2\pi)^3} |\boldsymbol{\beta}(\mathbf{K})|^2. \tag{A23}$$

In the purely atomic sector of Eq. (35), using the fact that the parametrization of the atomic momenta $\mathbf{k}_1 = \mathbf{Q}/3 + \mathbf{K}/2 + \mathbf{k}$, $\mathbf{k}_2 = \mathbf{Q}/3 + \mathbf{K}/2 - \mathbf{k}$, $\mathbf{k}_3 = \mathbf{Q}/3 - \mathbf{K}$, has a unit Jacobian, using Wick's theorem and the fact that $A(\mathbf{K},\mathbf{k})$ is an odd function of $\mathbf{k}$, we obtain

$$\langle \Psi_{\rm at} | \Psi_{\rm at} \rangle = 2\int \frac{d^3\mathbf{K}d^3\mathbf{k}}{(2\pi)^6} A(\mathbf{K},\mathbf{k})\left[A^*(\mathbf{K},\mathbf{k}) - 2A^*\left(\mathbf{k} - \frac{1}{2}\mathbf{K}, \frac{3}{4}\mathbf{K} + \frac{1}{2}\mathbf{k}\right)\right]. \tag{A24}$$

Using the specific form of the ansatz (42) and (41) for $\boldsymbol{\beta}$, with $B_{L=0}(K)$, $B_{L=1}(K)$ and $B_{L=2}(K)$ real, and the link (38) between A and $\boldsymbol{\beta}$ written here for $A_0 \equiv 0$, one can get integrals of lower dimensions:

$$\langle \Psi_{\rm mol}^{\rm even} | \Psi_{\rm mol}^{\rm even} \rangle = \int_0^{+\infty} \frac{dK}{3\pi^2} \mathcal{B}^2(K), \tag{A25}$$

$$\langle \Psi_{\rm at}^{\rm even}|\Psi_{\rm at}^{\rm even}\rangle = \frac{m^2\Lambda^2}{6\pi^4\hbar^4}\int_0^{+\infty} dK \int_0^{+\infty} dk \left[\frac{\frac{2}{3}k^4\mathcal{B}^2(K)e^{-k^2b^2}}{\left(q_{\rm trim}^2 + k^2 + \frac{3}{4}K^2\right)^2} + 2\mathcal{B}(K)\mathcal{B}(k)(D_0 - D_2)(K,k)e^{-5/8(K^2+k^2)b^2} \right], \tag{A26}$$

$$\langle \Psi_{\rm mol}^{\rm odd}|\Psi_{\rm mol}^{\rm odd}\rangle = \int_0^{+\infty} \frac{dK}{3\sqrt{2}\pi^2}\mathbf{b}^2(K), \tag{A27}$$

$$\langle \Psi_{\rm at}^{\rm odd}|\Psi_{\rm at}^{\rm odd}\rangle = \frac{m^2\Lambda^2}{6\sqrt{2}\pi^4\hbar^4}\int_0^{+\infty} dK \int_0^{+\infty} dk \left[\frac{\frac{2}{3}k^4\mathbf{b}^2(K)e^{-k^2b^2}}{\left(q_{\rm trim}^2 + k^2 + \frac{3}{4}K^2\right)^2} - 4\mathbf{b}(K)\cdot N_D(K,k)\mathbf{b}(k)e^{-5/8(K^2+k^2)b^2} \right]. \tag{A28}$$

We have set $E=-\hbar^2 q_{\rm trim}^2/m$, $\mathcal{B}(K)=KB_{L=1}(K)$ and

$$D_n = \int_{-1}^{1} \frac{dx}{2}\frac{x^n e^{-tx}}{(v+x)^2}, \tag{A29}$$

which is minus the derivative of C_n with respect to v for fixed t and obeys the recursive relation $D_{n+1}=C_n - vD_n$. Also, the vector $\mathbf{b}(K)$ has components $b_0(K), b_2(K)$, and the two by two matrix $N_D(K,k)$ is obtained in replacing each C_n by D_n in the matrix $N(K,k)$.

APPENDIX B: PRESCRIPTION FOR THE RECOMBINATION RATE OF FERMIONS

We wish to derive the formula (81) giving the rate of dimer formation in a gas of fermions at low kinetic energy, in terms of a recombination constant depending on the interaction and the expectation value $\langle\cdots\rangle_0$ of some operator in the unperturbed state of the gas. We start with the intuitive idea that a dimer formation can take place by three-body collision when the mutual distances of the atoms are at most of the order of the dimer radius σ, hence the heuristic formula

$$\frac{d}{dt}N_{\rm dim} \propto \int d^3r_1 d^3r_2 d^3r_3 e^{-\rho^2/2\sigma^2} \times \langle \hat\psi^\dagger(\mathbf{r}_1)\hat\psi^\dagger(\mathbf{r}_2)\hat\psi^\dagger(\mathbf{r}_3)\hat\psi(\mathbf{r}_3)\hat\psi(\mathbf{r}_2)\hat\psi(\mathbf{r}_1)\rangle_0. \tag{B1}$$

We have taken for simplicity a Gaussian cutoff function, in terms of the hyper-radius ρ defined as

$$\rho^2 = \sum_{i=1}^{3}(\mathbf{r}_i - \mathbf{R})^2 = \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2)^2 + \frac{2}{3}[\mathbf{r}_3 - (\mathbf{r}_1+\mathbf{r}_2)/2]^2, \tag{B2}$$

where $\mathbf{R}=\Sigma_{i=1}^3 \mathbf{r}_i/3$ is the center-of-mass position. Under the assumption that the typical wave vectors populated in the uncorrelated state of the gas are much smaller than $1/\sigma$, which is the case here since we assume relative momenta $<q_{\rm dim}$, we can expand each field operator $\hat\psi(\mathbf{r}_i)$ around the center-of-mass position $\mathbf{R}$ of the three $\mathbf{r}_i$ in powers of $\mathbf{r}_i - \mathbf{R}$. Since $\hat\psi^2=0$ one has to go to second order in δr:

$$\hat\psi(\mathbf{r}_3)\hat\psi(\mathbf{r}_2)\hat\psi(\mathbf{r}_1) = \sum_{\alpha\neq\beta}\mathcal{A}_{\alpha\beta}\hat\psi(\mathbf{R})[\partial_{R_\alpha}\hat\psi(\mathbf{R})][\partial_{R_\beta}\hat\psi(\mathbf{R})] + O(\delta r^3), \tag{B3}$$

where α and β run over the three directions of space x, y and z, and the matrix $\mathcal{A}$ is given by

$$\mathcal{A}_{\alpha\beta} = \delta r_{2,\alpha}\delta r_{1,\beta} - \delta r_{3,\alpha}\delta r_{1,\beta} + \delta r_{3,\alpha}\delta r_{2,\beta}. \tag{B4}$$

We restrict to this leading order in δr. It remains to integrate the Gaussian weighted products $\mathcal{A}_{\alpha\beta}\mathcal{A}_{\gamma\delta}$ over the internal variables for fixed center of mass position $\mathbf{R}$, by using the Jacobi coordinates. Since $\mathcal{A}_{\alpha\beta}$ for $\alpha\neq\beta$ is odd with respect to the reflection along direction α or along direction β, this integral vanishes if $\{\alpha,\beta\}\neq\{\gamma,\delta\}$. The invariance of the integral by permutation of the x, y, and z axis leads to the final prescription (81).

APPENDIX C: RECOMBINATION CONSTANT CLOSE TO THE TRIMER FORMATION THRESHOLD

The goal is to derive the approximate formula (93) giving the recombination constant $\mathcal{K}_{\rm rec}$ for large scattering volumes and for a value of $\alpha_{\rm res}$ close to the threshold for the even trimer formation. To this end, we rewrite the integral equation (76) as a sum of its $\mathcal{V}_s=\infty$ value and a remainder, then we treat the remainder perturbatively. Taking as unknown $F(K)=Kg(K)/\mathcal{V}_s$, using Eq. (26) and the identity $1/(X - i0^+)=\mathcal{P}\frac{1}{X}+i\pi\delta(X)$, where $\mathcal{P}$ is the Cauchy principal value, we obtain the rewriting

$$\frac{\alpha_{\rm res}}{4}F(K) - I_0[F](K) - I_1[F](K) - iA(K)F(K) = S(K). \tag{C1}$$

We have introduced the two functions

$$A(K) = K(C_0 - C_2)(K,K_{\rm dim})e^{-\frac{5}{8}b^2(K^2+K_{\rm dim}^2)}, \tag{C2}$$

$$S(K) = -\frac{36\pi\hbar^2}{m\Lambda}K_0 k_0^{\perp} e^{-5b^2K^2/8}, \tag{C3}$$

and the two operators

$$I_0[F](K) = \frac{h(K)}{4}F(K) + \frac{2}{\pi}\int_0^{+\infty} dK' \frac{K}{K'}(C_0 - C_2)(K,K') \times e^{-5/8b^2(K^2+K'^2)}F(K'), \tag{C4}$$

$$I_1[F](K) = \frac{K_{\rm dim}^2}{4}\frac{h(K)-h(K_{\rm dim})}{K^2 - K_{\rm dim}^2}F(K) + \frac{2K_{\rm dim}^2}{\pi}\int_0^{+\infty} dK' \frac{K}{K'} \times (C_0 - C_2)(K,K')e^{-5/8b^2(K^2+K'^2)}\mathcal{P}\frac{F(K')}{K'^2 - K_{\rm dim}^2}, \tag{C5}$$

where we have set $h(K)=q\exp(q^2b^2)\mathrm{erfc}(qb)$ with $q=\sqrt{3}\times K/2$.

The operator J such that $J[G](K)=I_0[F](K)/K$ where $F(K)=KG(K)$, is self-adjoint and thus has real eigenvalues; we find numerically that it has one and only one discrete eigenvalue, that we called $\alpha_{\rm th}^{\rm even}/4$, with the corresponding normalized eigenvector $K\to g_0(K)$. As a consequence, the operator I_0 admits, with the eigenvalue $\alpha_{\rm th}^{\rm even}/4$, a discrete eigenvector $u_0(K)=Kg_0(K)$, with the corresponding adjoint (left eigenvector) $v_0(K)=g_0(K)/K$. We note that the kernel in $I_0^\dagger$ behaves as $K'^2/(K^2+K'^2)$ at low momenta, which is bounded, so that $v_0(K)$ is bounded, $g_0(K)$ vanishes linearly with K in $K=0$ and $u_0(K)$ vanishes quadratically [78]. Apart for this discrete eigenvalue, we numerically find that I_0 has a continuous spectrum extending from 0 to $1/(4\sqrt{\pi}b)$.

For $\alpha_{\rm res}$ close to $\alpha_{\rm th}^{\rm even}$, in the large scattering volume limit, a small denominator (of the order of $\delta\equiv\alpha_{\rm res}-\alpha_{\rm th}^{\rm even}$) appears in the direction of u_0 when one solves Eq. (C1). This small denominator is weakly perturbed by I_1 (which shifts the value of $\alpha_{\rm res}$ for which the denominator has a minimal modulus) and by the imaginary part involving the function $A(K)$ (which prevents the denominator from exactly vanishing). These effects can be included systematically by using the ansatz

$$F(K)=F_{\rm bg}(K)+c_0u_0(K). \tag{C6}$$

The background part of the solution $F_{\rm bg}(K)$ and its derivative $F_{\rm bg}(K)$ are supposed to be uniformly bounded in $\mathcal{V}_s$ and $\alpha_{\rm res}$, even in the vicinity of $\alpha_{\rm res}=\alpha_{\rm th}^{\rm even}$. From the low K behavior of $C_0(K,K')-C_2(K,K')$, one finds that the solution $F(K)$ satisfies $F(0)=4S(0)/\alpha_{\rm res}$, this value being reached quadratically in K. Since $|F'|$ is not uniformly bounded in $\mathcal{V}_s$ and $\alpha_{\rm res}$, this does not give information on the value $F(K_{\rm dim})$. One has also $F_{\rm bg}(0)=4S(0)/\alpha_{\rm res}$, but for a $|F'_{\rm bg}|$ bounded by $c_{\rm bg}$, where the constant $c_{\rm bg}$ does not depend on $\mathcal{V}_s$ or $\alpha_{\rm res}$, one now has

$$\left|F_{\rm bg}(K_{\rm dim})-\frac{4S(0)}{\alpha_{\rm res}}\right|\le c_{\rm bg}K_{\rm dim}. \tag{C7}$$

This is the property of $F_{\rm bg}$ that we shall need.

Injecting the ansatz (C6) into Eq. (C1) and projecting onto u_0 by using the adjoint vector v_0, we obtain the exact expression

$$\begin{aligned}&[\delta/4-\langle v_0|I_1|u_0\rangle-iu_0(K_{\rm dim})\langle v_0|A\rangle]c_0\\&\quad=\langle v_0|S\rangle+(\delta/4)\langle v_0|F_{\rm bg}\rangle+\langle v_0|I_1|F_{\rm bg}\rangle+iF_{\rm bg}(K_{\rm dim})\langle v_0|A\rangle,\end{aligned} \tag{C8}$$

where we used $\langle v_0|I_0|F_{\rm bg}\rangle=(\alpha_{\rm th}^{\rm even}/4)\langle v_0|F_{\rm bg}\rangle$ (after justification). Here Dirac's notation means $\langle f|g\rangle=\int_0^{+\infty}dKf^*(K)g(K)$. Replacing c_0 by its expression in Eq. (C6), setting $K=K_{\rm dim}$ and putting all terms on a common denominator, we see that the imaginary contribution $iu_0(K_{\rm dim})\langle v_0|A\rangle$ exactly cancels in the numerator. We finally obtain the still exact expression

$$\begin{aligned}\frac{F(K_{\rm dim})}{F_{\rm bg}(K_{\rm dim})}&=\mathcal{F}[\delta/4-\langle v_0|I_1|u_0\rangle-iu_0(K_{\rm dim})\langle v_0|A\rangle]^{-1}\\&\quad\times\{\delta/4-\langle v_0|I_1|u_0\rangle/\mathcal{F}+u_0(K_{\rm dim})[\langle v_0|S\rangle\\&\quad+\langle v_0|I_1|F_{\rm bg}\rangle]/[F_{\rm bg}(K_{\rm dim})\mathcal{F}]\},\end{aligned} \tag{C9}$$

where $\mathcal{F}\equiv 1-\langle v_0|F_{\rm bg}\rangle u_0(K_{\rm dim})/F_{\rm bg}(K_{\rm dim})$.

The last step is to expand the various terms to leading order in $K_{\rm dim}$. Expanding I_1 to leading order in $K_{\rm dim}$ and using the fact that u_0 is an eigenvector of I_0 to simplify the integral expression appearing in this leading order form of I_1, we obtain

$$\lim_{K_{\rm dim}\to 0}\frac{\langle v_0|I_1|u_0\rangle}{K_{\rm dim}^2}=\frac{\alpha_{\rm th}^{\rm even}}{4}\langle v_0|v_0\rangle>0, \tag{C10}$$

and the quantity $\langle v_0|v_0\rangle$ is readily evaluated numerically. This gives a position of the peak in $\mathcal{K}_{\rm rec}$ shifted to a value of $\alpha_{\rm res}$ larger than $\alpha_{\rm th}^{\rm even}$ by a $O(K_{\rm dim}^2)$, see Eq. (95). Since $u_0(K)$ vanishes quadratically in K, one has $u_0(K_{\rm dim})$ of the order of $K_{\rm dim}^2$; since $A(K)$ vanishes linearly in $K_{\rm dim}$ for a fixed K, one has $\langle v_0|A\rangle$ of the order of $K_{\rm dim}$. A numerical calculation of the corresponding coefficients leads to Eq. (96). Amusingly, using the low-K expansion of $C_0(K,K')-C_2(K,K')$, we find the mathematical equivalence in the zero $K_{\rm dim}$ limit

$$\langle v_0|A\rangle\sim\frac{2}{3}K_{\rm dim}\frac{\langle v_0|S\rangle}{S(0)}, \tag{C11}$$

which leads to Eq. (97). Another result is

$$\langle v_0|I_1|F_{\rm bg}\rangle=O(K_{\rm dim}^2\ln K_{\rm dim}), \tag{C12}$$

so that this contribution, being multiplied by $u_0(K_{\rm dim})$ in Eq. (C9), may be neglected at this order. Finally, we note that $F_{\rm bg}(K_{\rm dim})$ and $\mathcal{F}$ differ from $4S(0)/\alpha_{\rm res}$ and 1, respectively, by terms of order $K_{\rm dim}$ and $K_{\rm dim}^2$, that we neglect to obtain Eq. (93).

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[48] One can equivalently consider a dimer wavefunction of a well defined angular moment component $m=0,\pm 1$ along the quantization axis z, by replacing $(3/4\pi)^{1/2} r_\gamma / r$ in Eq. (7) by $Y_1^m(\theta,\phi)$.

[49] If one assumes $\alpha_{\rm res}=0$, taking $u(k)=1/(k^2\mathcal{V}_s)$ for low k, as done in Ref. [43], one gets a dimer for $\mathcal{V}_s<0$ (contrarily to the realistic case) and one reaches a contradiction in Eq. (9) $|\mathcal{M}|^2<0$.

[50] In the resonant limit $|\mathcal{V}_s|\to+\infty$, on the side $\alpha_{\rm res}\mathcal{V}_s<0$ of the resonance, there is a so-called quasibound state in the two-body problem, which was studied experimentally in Ref. [27]: From Eq. (4) and using the fact that $\alpha_{\rm res}>0$, one finds that the scattering amplitude $f(k)$ has a complex pole with a real part $k_0^0 \simeq 1/(\alpha_{\rm res}|\mathcal{V}_s|)^{1/2}$ much larger than its imaginary part $\simeq -(k_0^0)^2/(2\alpha_{\rm res})$. Physically this is related to the existence of a centrifugal barrier when the two particles approach with nonzero relative momentum, here $l=1$. This quasibound state does not play an important role in the present work: in the exceptional case where we consider a negative rather than positive scattering volume (see end of Sec. V C), the total energy E in the three-body problem is still assumed to be much smaller than $\hbar^2(k_0^0)^2/m$ so that the relative momentum $k_{\rm rel}$ appearing in the three-body problem, as defined in Eq. (39), cannot reach the value k_0^0.

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[55] This reasoning extends to an arbitrary, everywhere finite envelope function $f(k)$ such that $\chi(\mathbf{k})=\mathbf{k}f(k)$.

[56] We used the identity $\langle \mathbf{r}_1,\ldots,\mathbf{r}_N|a^\dagger_{\mathbf{k}_1}\cdots a^\dagger_{\mathbf{k}_N}|0\rangle=(N!)^{-1/2}\times\Sigma_{\sigma\in S_N}\epsilon(\sigma)\exp[i(\mathbf{k}_1\cdot\mathbf{r}_{\sigma(1)}+\cdots+\mathbf{k}_N\cdot\mathbf{r}_{\sigma(N)})]$.

[57] Checking the consistency of these two ways of calculating β is actually useful: contour integration in the space of complex k was used to derive Eq. (9) from Eq. (8), and it is not straightforward that the contribution of the contour integral on an infinite radius half circle is zero, since $|f(iq)|$ actually rapidly diverges as $q\to\infty$ for our two-channel model. This check reduces to proving that $I(\mathbf{r})=\int\frac{d^3k}{(2\pi)^3}e^{-k^2b^2/2}e^{i\mathbf{k}\cdot\mathbf{r}}/(k^2+q_{\mathrm{dim}}^2)\sim e^{-q_{\mathrm{dim}}r}e^{q_{\mathrm{dim}}^2b^2/2}/(4\pi r)$ for $r\to\infty$. This can be done by using the analytic expression for $I(\mathbf{r})$ or by noting that $I(\mathbf{r})$ is the convolution of a Gaussian with a Yukawa function.

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[59] In the second term, an integral of the form $I=\int_0^{K_{\max}}dK'v(K')\mathcal{P}[1/(K'^2-K_{\mathrm{dim}}^2)]$ appears, with $K_{\max}$ being the numerical momentum cut-off, which may lead to numerical difficulties since the integrand diverges in $K=K_{\mathrm{dim}}$. We rewrite it using $v(K')=[v(K')-v(K_{\mathrm{dim}})]+v(K_{\mathrm{dim}})$. Using $\int_0^{K_{\max}}dK'\mathcal{P}[1/(K'^2-K_{\mathrm{dim}}^2)]=-\ln[(K_{\max}+K_{\mathrm{dim}})/(K_{\max}-K_{\mathrm{dim}})]/(2K_{\mathrm{dim}})$, we are left with the numerical evaluation of $\int_0^{K_{\max}}dK'[v(K')-v(K_{\mathrm{dim}})]/(K'^2-K_{\mathrm{dim}}^2)$.

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[62] We get $\langle\partial_{R_\alpha}\hat{\psi}^\dagger\partial_{R_\beta}\hat{\psi}\rangle_0=M_{\alpha\beta}/L^3$, with $M_{\alpha\beta}=\Sigma_{i=1}^3k_{i,\alpha}^0k_{i,\beta}^0$. From Wick's theorem the expectation value $\langle\cdots\rangle_0$ in Eq. (81) is equal to $3L^{-9}(M_{\alpha\alpha}M_{\beta\beta}-M_{\alpha\beta}^2)$. One may then check that $\Sigma_{(\alpha,\beta)}M_{\alpha\alpha}M_{\beta\beta}-M_{\alpha\beta}^2=3(\mathbf{K}_0\times\mathbf{k}_0)^2$, where the sum is taken over the three values (x,y), (x,z), and (y,z) of (α,β).

[63] One sees that the quantity t appearing in the definition of C_0 and C_2 in Appendix A is small and can be neglected; explicit expressions without relying on special functions can be obtained; they can be further simplified by performing a high v expansion, where the parameter v appears in Appendix A, and by keeping the leading term $2/(3v)$. Since the next term in the large v expansion is $O(1/v^3)$, this further approximation of keeping the leading term $\propto 1/v$ only is already good at the minimal possible value of $v,v=2$, where it introduces an extra error of $\sim 5\%$.

[64] One can show the following result, for a fixed value of α_{res}: if there exist constants A and B such that, for all $\mathcal{V}_s$ above some value, one has $|F(K)|<A$ and $|F'(K)|<B$ at all K, then Eq. (90) is exact.

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[73] This arbitrary normalization is $\int_0^{+\infty}dKK^2|\mathbf{b}(K)|^2=1$.

[74] We shall also assume that the first order derivative of $\mathcal{E}(K)$ with respect to K is uniformly bounded as a function of K and K_{trim}.

[75] From the low-K expansion of the matrix elements in the second line of $M(K,k)$, we get $B_{L=2}(K)=O(1/\sqrt{K^2+K_{\mathrm{trim}}^2})$, so that $K|B_{L=2}(K)|$ is uniformly bounded in K and K_{trim}, similarly to what happens in the even sector. The crossed terms in $B_{L=0}B_{L=2}$ give at most a logarithmic divergence in the normalization integrals.

[76] We use here the convention that, in the two-body calculation of the dimer wave function $\phi(\mathbf{r})$, $\boldsymbol{\beta}=p_{\mathrm{closed}}^{1/2}\mathbf{e}_z$.

[77] The infinite $\mathcal{V}_s^{\mathrm{bg}}$ pure single channel model is equivalent to an infinite $\mathcal{V}_s$ two channel model with no open channel interaction and with $\Lambda\to+\infty$, since both models have the same scattering amplitude in these limits.

[78] More precisely, if one assumes that $v_0(K)$ diverges as $1/K^\gamma$ for low K, $\gamma<3$ to ensure convergence in the integral, one sees that the integral in the limit $K\to 0$ either diverges as $1/K^{\gamma-1}$, for $\gamma>1$, or does not diverge, for $\gamma<1$. Since $\alpha_{\mathrm{th}}^{\mathrm{even}}>0$, this leads to a contradiction in the integral equation $\alpha_{\mathrm{th}}^{\mathrm{even}}v_0(K)/4=I_0^\dagger[v_0](K)$, the left-hand side diverging more rapidly than the right-hand side.