Fermions out of dipolar bosons in the lowest Landau level

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(Received 7 January 2008; published 4 April 2008)

In the limit of very fast rotation atomic Bose-Einstein condensates may entirely reside in the lowest two-dimensional Landau level (LLL). For small enough filling factor of the LLL, one may have formation of fractional quantum Hall states. We investigate the case of bosons with dipolar interactions as may be realized with chromium-52 atoms. We show that at a filling factor equal to unity the ground state is a Moore-Read (or Pfaffian) paired state as is the case of bosons with purely *s*-wave scattering interactions. This Pfaffian state is destabilized when the interaction in the *s*-wave channel is small enough and the ground state is a stripe phase with unidimensional density modulation. For filling factor 1/3, we show that there is formation of a Fermi sea of "composite fermions." These composites are made of one boson bound with three vortices. This phase has a wide range of stability and the effective mass of the fermions essentially depends only on the scattering amplitude in momentum channels larger or equal to 2. The formation of such a Fermi sea opens up a new possible route to detection of the quantum Hall correlations.

I. INTRODUCTION

The quantum behavior of a Bose-Einstein condensate is readily apparent in the response to rotation. Instead of solid-body rotation as in a viscous fluid, there is formation of lines of quantized vorticity. These vortices then form a regular lattice eventually distorted by the container, i.e., the trapping potential. In the case of structureless bosons interacting by purely *s*-wave scattering, the equilibrium vortex lattice is the celebrated triangular Abrikosov lattice [1,2]. Many experiments have been able to observe and study in detail this vortex lattice. Its structure and dynamics have been a very active field with fruitful interplay between theory and experiments [3]. The angular momentum of the gas in this regime is of the order of the number of particles times the number of vortices. From theoretical grounds, it is expected that a physically different regime is reached when the angular momentum is raised up to the order of the square of the number of particles. In this case, various quantum phases at zero temperature may appear [4–6], related to the fractional quantum Hall liquids already known from the physics of two-dimensional electron systems. These phases exist when the gas enters a bidimensional regime, either forced by an external confining potential along the rotation axis or induced by the centrifugal force. The mere existence of these phases is due to the formation of Landau levels in which kinetic energy is quenched and all the physics is determined by the interaction potential between atoms. When the centrifugal force compensates the harmonic trapping potential, one is left with the Coriolis force, which is formally equal to a fictitious magnetic field acting upon the neutral atoms. In a two-dimensional situation, the fictitious field leads to one-body energy levels that are degenerate, the Landau levels (LLs). The systematics of the apparition of the phases is ruled by the so-called filling factor ν, i.e., the number of atoms divided by the number of states in the lowest Landau level (LLL). The quantum Hall phases are expected at filling factors of order unity. Even when all bosons entirely reside in the LLL, there will be a vortex lattice when the filling

factor is large enough [7]. This lattice may melt [6] and be replaced by quantum Hall liquids in the region $\nu \approx 1$. The case of bosons with pure *s*-wave scattering is special. Indeed one may model the interaction in appropriate circumstances by a pure delta function interaction whose coefficient is proportional to the *s*-wave scattering length. Right at $\nu = 1/2$ it is the celebrated Laughlin wave function that is the exact ground state of the system of interacting bosons. In this case for filling factors smaller than $\nu = 1/2$ interactions play absolutely no role in the ground state. Indeed the ground state becomes increasingly degenerate with increasing angular momentum, being given by the product of an arbitrary symmetric polynomial times the Laughlin wave function. In fact the gas is noninteracting in this regime since all pairs of atoms have a relative angular momentum that is at least 2.

It has been recently possible to create condensates of chromium atoms that have dipolar interactions [8–13]. This opens up the possibility to study the change of interaction potential in many-body problems. The problematic of response to rotation can be again investigated with this new interaction potential. For example, the vortex lattice has been shown to display a much richer phase structure [15–17]. Beyond triangular and square arrangements of vortices, there is also evidence [16,18,19] of phases with unidimensional density modulation pattern and regular two-dimensional arrays of bubbles containing more than one atom. The physics of atoms in the LLL is also affected by the dipolar interactions. The most interesting situation is realized by perpendicularly aligning the dipolar moments to the plane where the atoms are confined. The interactions are then purely repulsive within the plane. It is likely that the "conventional" quantum Hall states as described by the composite fermion picture [20–24] will survive with dipolar interactions. However, the dipolar interaction induces interactions between particles with all even relative angular momenta in the LLL. As a consequence, the degeneracy when the filling factor is less than 1/2 is lifted and new physics takes place. Experience with electron systems has shown that one expects the formation of composite particles in which there is binding of vortices to the bosonic atoms. It is known [25–27] that there is

formation of such composite objects with one vortex per atom leading to the appearance of a series of incompressible liquids for filling factors $\nu = p/(p+1)$ with p an integer. Similarly for smaller filling factors one may have formation of composite fermions (CFs) made of three vortices plus one boson giving rise to fractions $\nu = p/(3p \pm 1)$ (p integer). Indeed from some numerical studies there is limited evidence for such phases although they have not been investigated in detail.

A fascinating property of the quantum Hall phases for bosons is that there are also states that do not belong to the composite-fermion family. This is the case for complete filling of the LLL by spinless bosons ($\nu = 1$). Here the ground state is described by the Moore-Read (or Pfaffian) state, which is a paired state of composite fermions. This very special state of matter [28,29] supports excitations with non-Abelian statistics and has been the object of much attention for many years. It has been suggested in the context of electronic systems as a plausible explanation of a quantum Hall state observed for filling factor $\nu = 5/2$ of two-dimensional electron gases. In the case of bosons with purely s-wave scattering, this state has been observed from exact diagonalizations in the planar geometry and then confirmed by finding the signature of its paired character in studies on the spherical geometry. This kind of paired state is in competition with the Fermi sea made of composite fermions [30] and this competition is solely governed by the nature of interactions. In the case of electrons for example at filling factor $\nu = 1/2$ in the LLL, it is known that a Fermi sea of composite fermions is the ground state of the system. The second Landau level at electron filling factor $\nu = 5/2$ may be approximately regarded as a $\nu = 1/2$ electron gas with renormalized Coulomb interactions due to the change of electronic wave functions sitting on top of an inert $\nu = 2$ background (the filled LLL) provided the cyclotron gap is large enough compared to interactions. This effective $\nu = 1/2$ system has a ground state that is gapped and may possibly be described by the Pfaffian state.

The case of dipolar atoms offers an additional parameter to tune the interactions. Indeed there is the s-wave scattering length that can be approximated by a zero-range potential in addition to the long-range dipolar scattering whose strength is governed by the dipole moment of the atoms. The ratio of energies corresponding to these two phenomena can be tuned by Feshbach resonance manipulation, which acts upon the s-wave scattering length. This ratio is a new parameter that can be varied to explore the many-body physics of ultracold atoms [12,13].

In this paper, we explore the competition between the Pfaffian state and the CF Fermi sea for ultracold atoms in the LLL as a function of the ratio of the dipolar scattering and the pure s-wave scattering. We use exact diagonalizations of the many-body problem for a small number of atoms. Indeed the restriction to the LLL implies that the Hilbert space of the interacting particles is finite dimensional for a finite number of vorticity quanta in the system. Thus we are led to a (huge) linear algebra problem but without any additional approximation. This methodology has been very successful in the study of quantum Hall phases. We consider the spherical as well as torus geometry.

At filling factor $\nu = 1$ we find that there is a wide range of stability of the Pfaffian when s-wave scattering dominates dipolar interactions. If we weaken the latter, there is a phase transition toward a stripe phase with unidimensional density modulation. Further weakening leads to collapse of the gas phase as in the more conventional three-dimensional case. A particularly interesting case is $\nu = 1/3$. Here one may have formation of composite fermions with three vortices bound to each atom instead of one vortex in the $\nu \geq 1/2$ region. We find that there is a stable Fermi sea without pairing instability, i.e., no Pfaffian forms. This Fermi sea is identified by striking closed shell effects observed by exact diagonalizations in the spherical geometry. Again, weakening of s-wave scattering leads to collapse but now without any intervening intermediate phase. The occurrence of this Fermi sea is a striking phenomenon of emergence of fermions out of bosons in a two-dimensional regime. This can be exploited to reveal the quantum Hall physics by measuring the occupation number density. When tuned to this filling factor $\nu = 1/3$ it will display the characteristic signature of a Fermi sea. Evidence for such a compressible state can be obtained by taking a snapshot of the particle distribution after free expansion as suggested in the work of Read and Cooper [14].

In Sec. II, we discuss the interactions between atoms in the LLL when they have dipolar interactions and how this changes the so-called pseudopotentials, the discrete set of energies that are the solution of the two-body problem in the LLL. In Sec. III, we present the two geometries we have used for exact diagonalizations: namely, the sphere and the torus. Section IV is devoted to the Pfaffian state realized at $\nu = 1$, we discuss its stability and the appearance of the stripe phase. In Sec. V, we show the appearance of the spectral signature of CFs with one vortex (^{1}CFs) as well as CFs with three vortices (^{3}CFs) when the filling factor is now 1/3. The formation of ^{3}CFs is shown by looking at the low-lying levels of the many-body problem in the spherical geometry. The energy of these low-lying states when fitted against a simple free fermion model allows one to get an estimate of their effective mass, which is solely governed by interactions as it should be in the LLL. Finally Sec. VI contains our conclusions.

II. DIPOLAR BOSONS IN THE LLL

We consider a gas of bosonic atoms of mass M confined in a harmonic trap with cylindrical symmetry along the z axis. The trap frequencies along the axial and transverse directions are denoted $\omega_\parallel$ and $\omega_\perp$, respectively. There are thus associated characteristic lengths $l_{\parallel,\perp} = \sqrt{\hbar/(M\omega_{\parallel,\perp})}$. We consider the case of a single hyperfine species of bosonic atoms. At long distances the interaction potential includes a Van der Waals interaction tail. For ultracold atoms it may be replaced by a simple isotropic short-range δ function potential proportional to the s-wave scattering length a_s

$$V_s(\mathbf{r}) = \frac{4\pi\hbar^2 a_s}{M}\delta^3(\mathbf{r}). \tag{1}$$

Atoms with a permanent dipole moment also have a dipole-dipole interaction in addition to the interaction through V_s. The corresponding potential is

$$V_{dd}(\mathbf{r}) = C_{dd}\frac{1 - 3\cos^2\theta}{r^3}. \qquad (2)$$

Here $C_{dd} = d^2/4\pi\epsilon_0$ where d is the electric dipole moment or $C_{dd} = \mu_0\mu^2/4\pi$ where μ the magnetic moment. The angle θ is between the dipole orientation and the vector joining the two atoms. We can make a dimensionless parameter out of these two kinds of interactions

$$\epsilon_{dd} \equiv \frac{MC_{dd}}{3\hbar^2 a_s}. \qquad (3)$$

In the case of the well-studied ^{87}Rb, dipolar interactions are negligible with $\epsilon_{dd} \approx 0.007$. Experiments have achieved Bose-Einstein condensation of chromium atoms that have a magnetic dipole moment of $6\mu_B$ and an s-wave scattering length of order 100 Bohr radius, leading to a sizeable $\epsilon_{dd} \approx 0.16$. Several Feshbach resonances are known for some hyperfine states of chromium that may allow us to tune a_s to a small value of the order of the Bohr radius leading to a big increase of the parameter ϵ_{dd}. It has been possible to enhance the effect of dipolar interactions up to $\epsilon_{dd} \approx 0.8$ [10].

In this paper we study the special case of dipolar atoms in a rotating two-dimensional (2D) regime for which dipolar moments are perpendicular to the plane [hence $\theta = \pi/2$ in Eq. (2)]. The motion along the z axis is considered to be frozen, i.e., the z dependence of the wave functions is the Gaussian ground state of the harmonic confining potential. The contact potential Eq. (1) effectively becomes two dimensional

$$V_s^{2D}(\mathbf{r}) = \sqrt{8\pi}\frac{\hbar^2}{M}\frac{a_s}{l_\parallel}\delta^2(\mathbf{r}). \qquad (4)$$

The dipolar interaction in this 2D regime is given by the following integral:

$$V_{dd}^{2D}(x_1 - x_2, y_1 - y_2) = \int dz_1 dz_2 V_{dd}(x_1 - x_2, y_1 - y_2, z_1 - z_2)$$

$$\times \frac{1}{\pi l_\parallel^2}e^{-(z_1^2 + z_2^2)/l_\parallel^2}. \qquad (5)$$

We concentrate on the regime where all atoms stay in the LLL. For rotating systems this may happen by tuning the rotation frequency of trap to the frequency of the harmonic confining force. The neutral atoms just experience the Coriolis force that is formally equivalent to a Lorentz force. This correspondence will be used throughout this paper. We will use both languages (rotating and magnetic) to clarify the physics of the rotating system. The LLL is spanned by one-body states of the following form:

$$\phi_m(z) = \frac{1}{\sqrt{2^{m+1}\pi m!}}z^m e^{-|z|^2/4l^2}, \qquad (6)$$

where $z = x + iy$, the length scale is set by the magnetic length $l = \sqrt{\hbar/eB} = \sqrt{\hbar/2M\omega_\perp}$, and m is a non-negative integer,

which is (in units of $\hbar$) the angular momentum along z of the state. The two-body problem can be straightforwardly solved in the LLL when one has an interacting potential that is rotationally invariant. The eigenenergies are given by

$$V_m = \int_0^\infty q\,dq\,\tilde{V}(q)L_m(q^2 l^2)e^{-q^2 l^2}, \qquad (7)$$

where $\tilde{V}(q)$ is the Fourier transform of the 2D potential and L_m is the Laguerre polynomials. These energies are named pseudopotentials and completely parametrize the $\mathcal{N}$-body problem. Indeed the interaction Hamiltonian in the LLL can be written as

$$\mathcal{H} = \sum_{i<j}\sum_m V_m \mathcal{P}_{ij}^{(m)}, \qquad (8)$$

where $\mathcal{P}_{ij}^{(m)}$ projects the state of the pair (ij) onto relative angular momentum m. The local potential V_s^{2D} Eq. (4) only contributes to V_0 since it only scatters particles with zero relative angular momentum.

On the contrary, the dipolar potential V_{dd}^{2D} Eq. (5) contributes to all V_ms. In the limit of strong confinement along the z axis $l_\parallel \to 0$, the dipolar contribution to the V_ms is finite [16] and given by

$$V_m^{dd} = \frac{C_{dd}}{l^3}\frac{\sqrt{\pi}}{8}\frac{(2m-3)!!}{2^{m-1}m!}. \qquad (9)$$

On the contrary, the contribution of the dipolar potential to V_0 is diverging in this limit. In fact V_0 is a local interaction that has contributions from both interactions. We consider the relative ratio to be an adjustable parameter as was realized in the three-dimensional situation by manipulation of Feshbach resonance. We thus fix all pseudopotentials for $m \neq 0$ to their value given by Eq. (9) and consider the ratio $\alpha = V_0/V_2$ as a free parameter. Large values of α will give back the FQHE physics of bosons with contact interactions and novelties emerge at finite α. The ratio α is a function not only of C_{dd} and a_s but is also affected by the lengths l and $l_\parallel$. Its control is then more complex than the case of bulk nonrotating gases where the competition is governed by ϵ_{dd} Eq. (3).

III. GEOMETRIES FOR EXACT DIAGONALIZATIONS

To investigate the ground state properties of interacting dipolar bosons in the LLL, we resort to exact diagonalization techniques. It is convenient to use geometries without boundaries to access the bulk physics. In this paper we use both the sphere and the torus geometry. We first briefly discuss the use of the spherical geometry [31,32]. The Landau problem on the sphere leads to a finite number of one-body states in the LLL proportional to the area of the sphere. The spherical geometry possesses the full rotation symmetry and thus all states can be classified according to their angular momentum referred to as spin in what follows. More precisely if an integer number of flux quanta (vorticity quanta in the rotating language) is piercing the sphere $N_\phi \equiv 2S$, its radius is $R = l\sqrt{S}$ and the LLL is a set of $2S + 1$ states forming a

spin-S multiplet. In spherical coordinates (θ, φ), the (unnormalized) one-body wave functions can be taken as

$$\Phi_M^S = u^{S+M} v^{S-M}, \quad u = \cos(\theta/2) e^{-i\varphi/2}, \quad v = \sin(\theta/2) e^{i\varphi/2},$$
(10)

where $M = -S, \ldots, +S$. The Laughlin wave function [33] in the unbounded plane geometry is given by

$$\Psi^{(m)} = \prod_{i<j} (z_i - z_j)^m e^{-\Sigma_i |z_i|^2/4l^2}.$$
(11)

It can be translated on the sphere by the following formula:

$$\Psi^{(m)} = \prod_{i<j} (u_i v_j - v_i u_j)^m.$$
(12)

This wave function describes a liquid state with filling factor $\nu = 1/m$ smoothly covering the sphere (if m is not very large). Even values of m give candidate ground states for bosonic systems. This state is singlet under rotations and hence non-degenerate on the sphere. It has also a specific relation between the flux and the number of particles, namely, $N_\phi = m(N-1)$ when requiring that all particles are in the LLL. This geometry is convenient for the study of liquid states that possess full rotation symmetry but it is not adapted to the detection of phases with broken translational symmetry similar to one- or two-dimensional analogs of charge density waves. In general a state with filling factor ν will have a flux-number relation of the form $N_\phi = N/\nu - X$ where X is a nontrivial shift depending upon the state under consideration.

The other geometry is that of the torus [34,35]. We use a periodic rectangular geometry of sides a and b. Due to the presence of the magnetic field, standard translation operators do not commute and their implementation is not trivial. Haldane has shown how to construct eigenstates of the many-body problem with two conserved pseudomomenta corresponding to the translations along the two directions. This pseudomomentum **K** lies in a restricted Brillouin zone, which contains only N_0^2 points where N_0 is the GCD of the number of particles and the number of flux quanta. For convenience, we measure K_x in units of $2\pi\hbar/a$ and K_y in units of $2\pi\hbar/b$. Contrary to the case of the sphere there is no special shift in the relation between the flux and the number of particles. There is, however, a center-of-mass degeneracy that is exact for all system sizes and that is given by q at filling fraction p/q. We always factor out this degeneracy in our numerical studies. The efficient coding of the magnetic translation leads to diagonalization in spaces of reduced dimension in addition to the possibility to classify states by their pseudomomentum. In the rectangular case there are also discrete symmetries relating states at $(\pm K_x, \pm K_y)$. The Laughlin wave function can again be translated in the torus geometry [35] with the help of θ functions

$$\Psi^{(m)} = F_{\text{c.m.}}^{(m)} (\sum_i z_i) \prod_{i<j} \theta_1(z_i - z_j | \tau)^m e^{-\Sigma_i y_i^2/2l^2},$$
(13)

where $\tau = ib/a$ and the center-of-mass dependence $F_{\text{c.m.}}^{(m)}$ is itself a θ function with characteristics.

IV. PFAFFIAN vs STRIPE AT $\nu = 1$

Many aspects of the physics of the FQHE states can be understood by use of the CF picture. In this approach one is reasoning with effective CF fermions that are made of the original particle bound with some number (n) of flux quanta depending on circumstances. If the bare particle is a boson then we need an odd number n of flux tubes to convert it into a CF. The effective number of flux quanta for ^{n}CFs is then $N_\phi^* = N_\phi - nN$. Since this flux is reduced with respect to the field acting on the bare particles the ^{n}CFs may occupy p higher-lying LLs and we expect commensurability effects for each integer filling of the effective LLs. Formation of ^{1}CFs leads to a satisfactory explanation of a series of incompressible states numerically observed at filling factor $\nu = p/(p+1)$ for at least $p = 1, 2, 3$. This series of fractions converges toward $\nu = 1$. At this filling factor the ^{1}CFs experience zero flux and one possible candidate for the ground state is a Fermi sea of ^{1}CFs as is realized in electronic systems at $\nu = 1/2$. However, formation of a Fermi sea may be preempted by instability toward pairing of CFs. The possibility of pairing has been proposed in the context of electronic systems and boils down to the consideration of the so-called Pfaffian wave function [28]

$$\Psi_{\text{Pf}} = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j) e^{-\Sigma_i |z_i|^2/4l^2},$$
(14)

where Pf stands for the Pfaffian symbol. This latter object is defined for an arbitrary skew-symmetric $N \times N$ (N even) matrix A

$$\text{Pf}(A) = \sum_\sigma \epsilon_\sigma A_{\sigma(1)\sigma(2)} A_{\sigma(3)\sigma(4)} \cdots A_{\sigma(N-1)\sigma(N)},$$
(15)

where the sum runs over all permutations of the index with N values and ϵ_σ is the signature of the permutation. This state can be formulated in the spherical geometry by use of the substitution $z_i - z_j \rightarrow u_i v_j - u_j v_i$. This leads to a singlet state that has the special flux-number of particle relationship $N_\phi = N-2$. Exact diagonalizations on the disk geometry have given evidence for this state in the case of pure contact interactions between bosons [5]. This has been clarified in the spherical geometry [25].

A. Evidence for Pfaffian state on the torus

The Pfaffian state at $\nu = 1$ has also been observed in the torus geometry [6] for the contact interaction Eq. (4). Its striking signature is then the ground state degeneracy, which is of topological origin. This can be seen in several ways. We first note that in the case of the Laughlin state the Jastrow factor is rendered periodic by simply doing the substitution $z_i - z_j \rightarrow \theta_1(z_i - z_j | \tau)$ (forgetting about the center-of-mass motion). While this is correct for the Jastrow factor in the Pfaffian wave function (14), it cannot be correct for the Pfaffian factor that involves inverse powers of $z_i - z_j$. The correct substitution is, in fact [29], $1/(z_i - z_j) \rightarrow \theta_a(z_i - z_j | \tau)/\theta_1(z_i - z_j | \tau)$ with $a = 2, 3, 4$. This leads to three ground states: this degeneracy has nothing to do with that of the center-of-mass, which is absent for $\nu = 1$ anyway. Such a topological degen-
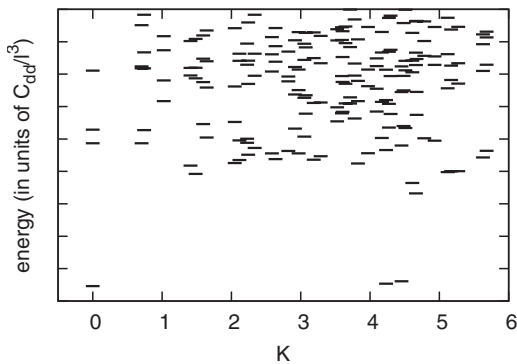
FIG. 1. Low-lying energy levels of 12 bosons at $\nu=1$ with dipolar interactions characterized by $\alpha=V_0/V_2=3$ as a function of the pseudomomentum K. The rectangular cell has aspect ratio $a/b=0.95$.
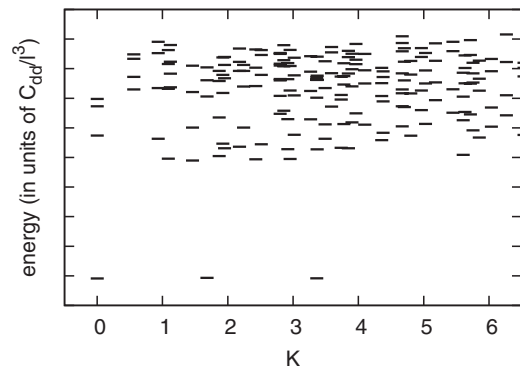

FIG. 2. Low-lying energy levels of 12 bosons at $\nu=1$ with dipolar interactions characterized by $\alpha=V_0/V_2=2$ as a function of the pseudomomentum K. The rectangular cell has aspect ratio $a/b=0.6$.

eracy is expected to be approximate for any finite size system and should become more and more manifest as we converge toward the thermodynamic limit.

For large values of $\alpha=V_0/V_2$ we find that the Pfaffian phase survives the addition of dipolar interactions. This is of course what one can expect for a gapped phase that exists in the $\alpha\rightarrow\infty$ limit. A typical spectrum in this regime is given in Fig. 1 for $\mathcal{N}=12$ bosons. We have used a small departure from square unit cell to more clearly show the degenerate states.

The three ground states have $\mathbf{K}=(0,0)$, $\mathbf{K}=(6,0)$, and $\mathbf{K}=(0,6)$. These quantum numbers are exactly those expected from the bosonic Pfaffian formulated on the torus. These states are separated by a clear gap from the bulk of the spectrum and this situation is robust with respect to the change of the aspect ratio.

B. Evidence for stripe phase

This range of stability of the Pfaffian terminates around $\alpha\approx2.4$. Below this critical value we find that the ground state degeneracy changes and that the whole spectrum looks compressible contrary to the Pfaffian case. In a square unit cell we find ground states at $\mathbf{K}=(0,0),(0,\pm4),(\pm4,0)$. This regime is very sensitive to the aspect ratio of the unit cell contrary to the Pfaffian phase. The degeneracy is lifted in an anisotropic cell toward $\mathbf{K}=(0,0),(0,\pm4)$. These wave vectors are now inside the magnetic Brillouin while the Pfaffian ground states were at half reciprocal lattice vectors. This set of vector form a 1D array, they are separated by a characteristic ordering wave vector $\mathbf{Q}=(0,3)$. This special configuration persists for a wide range of aspect ratios $0.4\lesssim a/b\lesssim0.9$. This is exactly what we expect for the formation of a unidimensional density modulation or stripe phase [36–38]. A typical spectrum in this phase is shown in Fig. 2 for $\mathcal{N}=12$ bosons.

If we now further reduce the parameter α there is a collapse of the system of bosons for $\alpha\approx1$ below which there is a huge structureless degeneracy of the ground state. Although we have not studied this non-FQHE phase in detail, we believe it is the LLL analog of the collapse of three-dimensional nonrotating gases observed when the parameter ϵ_{dd} becomes less than 1 [11–13]. It is worth stressing that we have no evidence for other intermediate phases between the Pfaffian and the collapsed phase. This is contrary to what is known [18] in the case of the Laughlin state at $\nu=1/2$. Here the Laughlin state also has a wide range of stability. For small values of α it is replaced by a striped phases as we find for $\nu=1$ but further reduction leads to so-called bubble phases, i.e., regular 2D arrays of density excess with two or more bosons per site.

V. ³CF FERMI SEA AT $\nu=1/3$

We now turn to the fate of the bosons interacting with dipolar interactions when the filling factor is tuned to 1/3. This is also a very special case. The composite fermion construction certainly suggests that at small Bose filling factors there may be formation of ³CFs. This leads to the possibility of standard incompressible liquids for $\nu=p/(3p\pm1)$ that would be the Bose analogs of the series of ⁴CF states observed for electrons for $\nu<1/3$. There is in fact some evidence reported in literature for such fractions [16,25]. This series also has a termination point at $\nu=/1/3$ at which the ³CFs experience zero effective flux and we are again facing the competition between the Fermi sea of CFs and the Pfaffian state. To construct a Pfaffian candidate [28] at 1/3 it is enough to add two powers to the Jastrow factor in Eq. (14) leading to

$$\Psi_{\mathrm{Pf}} = \mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)\prod_{i<j}(z_i-z_j)^3 e^{-\Sigma_i|z_i|^2/4l^2}. \qquad (16)$$

Translated on the sphere it leads to a series of incompressible states with $N_\phi=3N-4$. On the torus it also has a topological degeneracy. This is not what we observe. In fact we find no evidence for the Pfaffian neither on the sphere nor on the torus for any values of α. Our findings are compatible with the presence of a Fermi sea of ³CFs. The wave function of such a state on the disk geometry is [43]

$$\mathcal{P}_{\text{LLL}}\{\prod_{i<j}(z_i - z_j)^3 \det[\exp(i\mathbf{k}_i \cdot \mathbf{r}_j)]\}. \qquad (17)$$

In this equation the Jastrow factor ensuring filling factor 1/3 is multiplied by the Slater determinant of plane waves that describes the Fermi sea. Since this determinant is not automatically in the LLL one needs to project it explicitly onto the LLL. The choice of the $\mathbf{k}$ vectors in the determinant to find the ground state is dependent upon details of the geometry of the system. The signature of such a state is that the ground-state pseudomomentum in the torus geometry is not as simple as in the case of FQHE states or striped phases we have encountered above. It is thus much simpler to study the properties of the Fermi sea on the spherical geometry.

A. ^{1}CFs versus ^{3}CFs

We first discuss the nature of the correlations between bosons at filling factors smaller than 1/2. The first point is that the ground state becomes highly degenerate when we are dealing with the pure contact interaction Eq. (4). Indeed if we multiply the Laughlin wave function Eq. (11) for $\nu = 1/2$ by any symmetric polynomial of the z_i coordinates it is still trivially a zero-energy eigenstate. Some of these states are edge modes when the degree of the polynomial is small enough but this also extends to states with smaller filling factors, i.e., the $\nu = 1/4$ Laughlin state is itself lost in the manifold of such states. These states do not experience the delta interactions since the relative angular momentum between any two bosons is at least two. However excited states, hence with nonzero energies, experience the effect of the pseudopotential V_0. We thus expect to find above the ground state a set of states corresponding to many-body states in which there is one pair of bosons having relative angular momentum zero: such states will have energy of order V_0. There will then be yet another band with two pairs of bosons interacting with V_0 and energy of order $2V_0$ and so on. These quasidegeneracies are most clearly exhibited in the spherical geometry for which they are not further complicated by the center-of-mass degeneracy that exists on the disk. If we now introduce some dipolar interactions all the pseudopotentials V_m, $m \geq 2$ become nonzero. The ground state manifold is now no longer degenerate and will split into subbands according to the number of pairs of bosons having relative angular momentum 2 at least and so on. This picture of classification by relative angular momentum has been put forward some time ago by Wojs and Quinn [39,40] for electrons interacting by the Coulomb potential. In their case the separation of scales is not tunable while the dipolar system offers an opportunity to observe the build-up of these hierarchical structures. Since the pair angular momentum is not a conserved quantity one should keep in mind that the classification is approximate. The quantum numbers of these various sets of states can be found by using the language of composite fermions [41]. The ^{1}CFs experience an effective flux $N_\phi^* = N_\phi - (N-1)$. At filling factor 1/3 the configuration of these fermions is thus highly degenerate. Let us take the example of $\mathcal{N} = 6$ bosons given in Fig. 3. The ^{1}CF LLL now has 11 states. The values of the total angular momentum for 6 fermions in an 11-fold degenerate shell are exactly those found
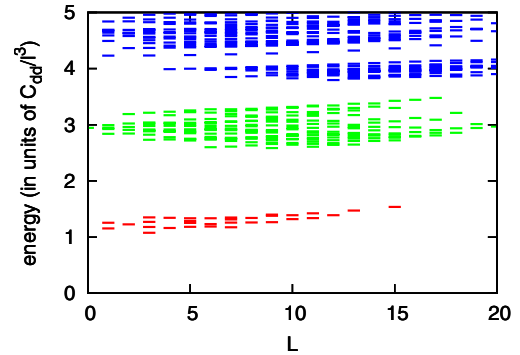

FIG. 3. (Color online) The lower part of the spectrum of 6 bosons on the sphere at $N_\phi = 15$ with dipolar interactions characterized by $\alpha = V_0/V_2 = 10$, $V_0 = 2.73$, and $V_2 = 0.273$ as a function of the total angular momentum L. The lowest band of states at energies close to one unit avoids relative angular momentum zero, the next band of states has on average one pair with zero momentum and hence the energy scale is raised by $\approx V_0$, the counting extends to higher levels. The low-energy manifold is in one-to-one correspondence with the spectrum of six electrons at $N_\phi = 10$, i.e., at $\nu = 1/2$.

in the lowest energy band in Fig. 3. If we promote one ^{1}CF to the $\mathcal{N} = 1$ ^{1}CF LL then the momentum of the states we generate in this configuration are those of $\mathcal{N} = 5$ fermions in an 11-fold degenerate shell and one extra fermion in a 13-fold shell. This is the exact content of the second band of states in Fig. 3. The energy scale of the ^{1}CF LLs is related to V_0 and it is only the remaining pseudopotentials $V_m, m \geq 2$ that will lift the degeneracy. If we look at the band of states predicted by the ^{1}CF reasoning we indeed find that the degeneracy is lifted by the dipolar force. Now the ordering of energy levels in the ground-state manifold of ^{1}CF is predicted by introducing ^{3}CF that correctly predict the smaller splitting due to the dipolar interaction. There is formation of a set of effective ^{3}CFs LL with an energy separation essentially governed by the pseudopotential beyond V_0. We render this reasoning quantitative in the next section by evaluating the effective mass of the ^{3}CF as a function of α.

In this peculiar physical system, the ratio of the energy scales corresponding to ^{1}CF and ^{3}CF correlation is tunable. This is a very interesting situation that is able to shed light on the coexistence of the two species of composite particles ^{1}CF and ^{3}CF since we can in principle separate them. This means that one can separately tune the effective CF Landau levels energy scales for the ^{1}CFs and ^{3}CFs. Similar coexistence effects have been experimentally observed [42] in the context of two-dimensional electron gases at filling factor smaller than 1/3. In this context, these experiments have given evidence for coexistence of ^{2}CF and ^{4}CF excitations. Since the electrons are interacting through the Coulomb interaction, there is not a huge difference between the pseudopotentials V_1 and V_3 corresponding to these two kinds of correlations. The dipolar system renders the corresponding separation manifest.

B. Shell effects on the sphere

The ultimate low-energy behavior of the Bose system is governed by the lowest-lying states described by the ^{3}CFs if

there is formation of a Fermi sea. We have studied the zero-flux line in the N_ϕ-N plane, namely, $N_\phi = 3(N-1)$. Since the ^{3}CF are more or less freely moving on a sphere we expect to observe closed shell effects when N is a square as it should be if the effective kinetic energy of the CFs is of the form L^2 with L the total momentum. This is what happens for electrons [43–46] in the LLL at $N_\phi = 2(N-1)$. When $N = p^2$, p integer, on the zero-flux line the system can be considered as a member of the two families of incompressible states with $\nu = p/(3p \pm 1)$. Indeed the flux-number of particles for these two Jain-like series of states are given by

$$N_\phi = \frac{3p \pm 1}{p} N \mp p - 3. \tag{18}$$

Along the line belonging to the fraction $p/(3p+1)$ the effective flux is negative [47] for $N < p^2$, vanishes at $N = p^2$, and becomes positive beyond that value. Between two successive squares, Rezayi and Read [43] have shown that the signature of a CF Fermi sea state is that the angular momentum of the ground state should be given by the second Hund rule for electrons in zero field. This is exactly what we observe between $\mathcal{N} = 4$ and $\mathcal{N} = 9$ bosons in Figs. 4(a)–4(f). At $\mathcal{N} = 4$ we have filled s and p shells hence a singlet ground state. At $\mathcal{N} = 5$ there is one CF in the d shell, hence $L = 2$. At $\mathcal{N} = 6$, the two CFs in the d shell couple to $L = 3$. The sequence is then reproduced backward by CF particle-hole symmetry up to the next closed shell configuration: $\mathcal{N} = 7$ has also $L = 3$, $\mathcal{N} = 8$ has one hole in the d shell hence $L = 2$ and $\mathcal{N} = 9$ has s,p,d filled shells. The ordering is similar to that of electrons in the LLL at $N_\phi = 2(N-1)$. At a qualitative level it is exactly what is expected from the formation of ^{3}CFs. One can also recognize the excitations involving effective CF LLs. For example, at $\mathcal{N} = 4$, the promotion of one CF from the filled p shell to the empty d shell leads to the branch of states with $L = 2, 3, 4$. Similarly at $\mathcal{N} = 9$, promotion of one CF from the d shell to the next f shell leads to a branch of states terminating at $L = 5$.

While this is readily analyzed, we find that the torus geometry leads to a complex dependence of the ground-state wave vector as a function of the aspect ratio and size. This is coherent with an appearance of a compressible state. In addition we do not find evidence for any ordering wave vector such as that at $\nu = 1$ so we rule out the possibility of a striped phase. Due to the exponential growth of the Hilbert, we are not able to confirm the level pattern up to the next closed shell (f), which should appear for $\mathcal{N} = 16$ bosons (prohibitively large for exact diagonalizations).

C. Effective mass

One can now give a more quantitative treatment of the "free" ^{3}CF picture by evaluating the effective mass entirely due to the interactions. The idea is to reproduce ground-state energies between closed shell configurations by a simple free fermion model where the particles are living on the surface of the sphere. Energies are thus given by
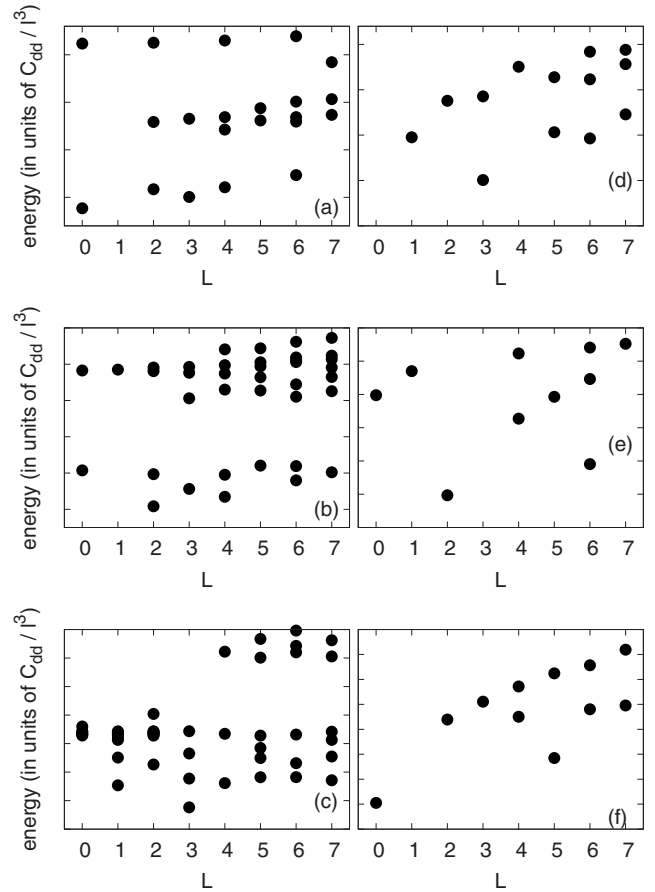

FIG. 4. Shell effects of ^{3}CFs on the sphere for $\alpha = 4$. We follow the zero-flux line $N_\phi = 3(N-1)$. The low-lying states are plotted as a function of total angular momentum. (a) $\mathcal{N} = 4$, closed $s+p$ shells of ^{3}CFs, (b) $\mathcal{N} = 5$: one ^{3}CF in a d shell leads to the $L = 3$ ground state, (c) $\mathcal{N} = 6$: two ^{3}CFs couple to $L = 3$, (d) $\mathcal{N} = 7$: two holes in the d shell of ^{3}CFs lead also to $L = 3$, (e) $\mathcal{N} = 8$: one hole is similar to the $\mathcal{N} = 5$ case, (f) $N = 9$: The three closed shell singlet state is recovered.

$$E(\{l_i\}) = \frac{\hbar^2}{2m^* R^2} \sum_{i=1}^{N} l_i(l_i + 1), \tag{19}$$

where $R = l\sqrt{S}$ is the radius of the sphere, the l_i are the momenta of one-body occupied states and the effective mass is proportional to the interactions for dimensional reasons

$$\frac{\hbar^2}{m^*} = \mu(\alpha) \frac{C_{dd}}{a}, \tag{20}$$

where a is the ion-disk radius, conveniently defined from electronic systems by $\pi a^2 = 1/\text{density}$, and μ is dimensionless. As usual in the spherical geometry there is a difference between the actual density of particles and the density at the thermodynamic limit for a given filling factor. This is due to the nonzero shift in the flux number of particle relation. While negligible for large numbers, it is sizeable for the values of $\mathcal{N}$ at hand. We correct for this effect by renormalizing the magnetic length or ion-disk radius by a factor $\sqrt{\nu N_\phi / N}$.
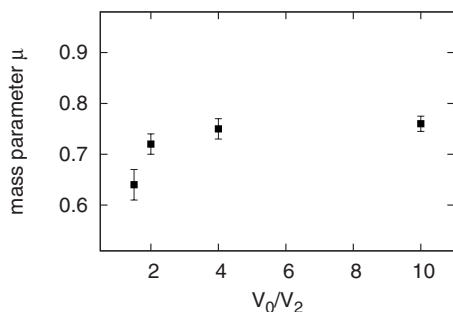
FIG. 5. The inverse effective mass parameter μ of the ^{3}CF at $\nu=1/3$ obtained from the ground-state energies for $4 \leq N \leq 9$ bosons. The error bars are tentative estimates of the systematic error due to the fact that real spectra deviate from the free fermion predictions. As expected from three-flux composite fermions, the effective mass saturates for large α to a value ≈ 0.75 that is independent of the hardcore pseudopotential V_0.

We find that the energies can be reproduced by Eq. (19) as soon as $\alpha \gtrsim 1$. For lower values the system is in the collapsed phase. When increasing the hardcore interaction through V_0 we find that the effective mass becomes constant with $\mu \approx 0.75$: see Fig. 5. This is exactly what we expect from the separation of scales between ^{1}CF and ^{3}CFs. The degeneracy of the ^{3}CF manifold of states at $\nu=/1/3$ is lifted by the interactions beyond the pure hardcore.

We now comment on the possible strategy to characterize such compressible states. Indeed the incompressible liquid states of the FQHE picture are featureless, i.e., no snapshot of the density will reveal their character. One must thus resort to more complex techniques [14,48] to measure correlation functions as a function of the distance to possible indicators. Also the spatial density variation should display a steplike structure [49] with pinning at magic densities. The Fermi sea that we have found is a gapless, compressible state. Since we are in the LLL the signal snapshot measurement of the density after free expansion directly reveals the particle distribution as pointed out by Read and Cooper [14].

One may then observe the density oscillations in space as expected for a Fermi-liquid-like state.

VI. CONCLUSIONS

We have investigated the behavior of bosons in the LLL with dipolar interactions. The special filling factors we chose on general grounds are expected to display quantum states different from the Laughlin-like incompressible states found, for example, at $\nu=1/2$. At $\nu=1$ we find that there is an extended stability region of the Pfaffian state already known for pure contact interactions. If we weaken repulsion enough the hardcore, this state is destroyed and replaced by a unidimensional density modulation, a stripe. In both cases it is the ground state degeneracy that is the main evidence for the occurrence of such phases. We have studied the fraction $\nu =/1/3$, which is nontrivial in the presence of dipolar interactions. Since this state is the termination point of the Jain-like series of fractions at $p/(3p \pm 1)$, it is plausible that there is formation of a CF Fermi sea as is observed for electron systems at the electronic filling factor 1/2. We have given evidence for such a state and evaluated the effective mass of the CFs that forms. We also give a low-energy approximate description of the system. The dipolar system is a very interesting case of adjustable separation between the energy scales involved in the formation of ^{1}CF versus ^{3}CFs. The mass at $\nu=1/3$ is remarkably insensitive to the hardcore repulsion.

Finally, we note that formation of fermions out of bosons is an interesting phenomenon in the realm of the dipolar bosons. While this phenomenon is well documented in the one-dimensional context through the Tonks-Girardeau regime, it remains to be observed in two space dimensions.

ACKNOWLEDGMENTS

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