Non-Markovian entanglement evolution of two uncoupled qubits

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Two noninteracting qubits, initially prepared in an entangled state, are coupled to their own independent environments and evolve under their influence. The reduced non-Markovian dynamics of two qubits is exact for arbitrary model parameters. Necessary and sufficient conditions for nonvanishing entanglement are formulated for both zero and nonzero temperatures and arbitrary time. It is shown that (i) entanglement dynamics can effectively be controlled by a finite quantum system coupled to one of the qubits and (ii) dynamical symmetry of the controlling quantum system can influence significantly entanglement of the qubits and results in its nonmonotonic decay.

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I. INTRODUCTION

Entanglement of open quantum systems has attracted considerable attention due to its significance for both fundamentals and applications of quantum-information processing [1]. Time evolution of quantum systems can be developed under various and often quite abstract assumptions [2] reflected in the corresponding entanglement dynamics. There are two, qualitatively different, types of behavior. The first one, when the (abstract) time evolution is given by the so-called global transformation [1]. In such a case, the time dependence of entanglement can exhibit both monotonicity and nonmonotonicity. The second class of quantum systems consists of qubits evolving under local transformations possibly accompanied by a classical communication (LOCC), when the entanglement is always a nonincreasing function of time [1].

For open quantum systems, the choice of a model of its reduced dynamics is crucial. One of the most popular guidelines for that choice is the complete positivity provided by a Kossakowski-Lindblad form of master equations [3]. The entanglement dynamics has frequently been studied within the Markovian approximation: Either formal [4] or rigorously derived via the Davies weak-coupling theory [5]. The possibility of nonmonotonic entanglement of bipartite systems occurs if either the parties interact directly [1] or if they are embedded in a common environment [4,5] acting constructively on the entanglement. However, the environment usually acts destructively upon the entanglement resulting in the noise induced entanglement decay and death [6] or, according to [7], leading to the decoherence of entanglement. It is known that Markovian evolution of two noninteracting systems coupled to their own independent environments cannot result in a nonmonotonic time dependence of their entanglement. The aim of our work is to show that it is not the case for non-Markovian systems, when the time homogeneity is broken.

The Markovian approximation, beyond phenomenological modeling, can rigorously be justified only under very special conditions [3]. Let us mention the so-called weak-coupling, or Davies, approximation. The results obtained within this method cannot be extrapolated to the low-temperature regime. Therefore, the applicability of the weak-coupling approximation for solid-state devices, often operating at deep cold, is truly limited. On the other hand, modeling of reduced dynamics beyond the Markovian approximation requires particular care [8] and expressing it in terms of a standard master equation is in general impossible, except in some very special cases. The key problem concerns the entanglement dynamics in the presence of a real environment derived consistently from the microscopic first principles. The dissipation or dephasing caused by such an environment is, in general, neither Markovian nor weak. In this paper we limit our consideration to a simple exactly solvable model of pure dephasing [9]. As this model allows for an exact analysis of the reduced dynamics, credible results can be obtained for arbitrary model parameters [9,10].

For the purpose of some applications, it is important to overcome the problem of decoherence, e.g., by confining quantum evolution in the decoherence free subspaces [11]. The details of the system-environment interaction are crucial for entanglement of its components, e.g., the quantum systems coupled by a common heat bath can remain asymptotically entangled [12]. One of the aims of our investigations is to show that a pair of qubits, which do not interact with each other but are initially quantum correlated, can remain entangled for arbitrary long time, provided certain conditions are imposed on their environments. We investigate two types of environments: (i) an infinite system of bosonic oscillators that models a dephasing thermal bath and (ii) a finite, controlling, quantum system that may cause the nonmonotonic dynamics of the qubits' entanglement. The exact reduced, non-Markovian, dynamics of the qubits can be derived for such a simplified model [9,10]. Recently, the problem of the non-Markovian entanglement dynamics has been studied in many papers [13–16]. In particular, it has been shown that the entanglement dynamics of two qubits coupled to an environment quantitatively and qualitatively depends on the low-frequency properties of the bath [7,13,14]. We start with a pair of entangled qubits and determine the asymptotic behavior of the entanglement. In particular, we formulate the necessary and sufficient condition for the qubits to remain entangled forever. We also propose a method of design for the entanglement dynamics. It is achieved by means of an external finite quantum system. Although the qubits are uncoupled, the resulting entanglement is a nonmonotonic function of time and strongly depends on the initial preparation of the controlling system. This behavior does not contradict the common wisdom that LOCC cannot increase the entanglement. The nonmonotonicity originates from the non-Markovian character of the qubits' evolution caused by their environments. As the time homogeneity is essentially broken, every local transformation of the two-qubit system must contain the information about the whole history of the quantum evolution. Therefore, adjusting the initial state of the controlling systems allows one to design, in advance, instants of time when the qubits are relatively strongly entangled and next almost disentangled, i.e., quantum correlations between them are essentially negligible.

The layout of the present work is as follows: In Sec. II, we present the two-qubit model. Next, in Sec. III, we elucidate entanglement dynamics for initially depolarized Bell states. In Sec. IV, we work out the case when only one of the qubits is coupled to the environment. Rigorous results are derived for both zero and nonzero temperature. In Sec. V, we elaborate on a single-mode controlling quantum environment, whereas a two-mode case is discussed in Sec. VI. Section VII provides a summary and conclusions. In the Appendix we discuss another aspect of the entanglement dynamics, namely, the effect of classicization [17] of one of the baths.

II. MODEL OF OPEN TWO-QUBIT SYSTEM

We study an open system consisting of two qubits S_1 and S_2. The Hamiltonian is assumed to be of the form

$$H = H_1 + H_2,$$

$$H_1 = S_1^z + S_1^z \otimes \sum_{k=1}^{\infty} g_k (a_k^\dagger + a_k) + \sum_{k=1}^{\infty} \omega_k a_k^\dagger a_k,$$

$$H_2 = \frac{1}{2}|0\rangle\langle 0| \otimes H_+ + \frac{1}{2}|1\rangle\langle 1| \otimes H_-. \qquad (1)$$

In this dimensionless form, energies are rescaled to the energy splitting $\varepsilon_0 = \hbar\omega_0$ of the qubit S_1, frequencies are rescaled to the frequency ω_0, and time is rescaled to $t_0 = 1/\omega_0$. The first qubit S_1 is represented by the spin-1/2 operator S_1^z. It interacts with a heat bath R_1 modeled by an infinite quasifree reservoir composed of bosonic harmonic oscillators of angular frequencies ω_k. a_k and $a_k^\dagger$ are Bose annihilation and creation operators. The strength of the interaction between the qubit and the kth mode of the heat bath is described by g_k.

For the second qubit S_2, we denote by $\{|0\rangle, |1\rangle\}$ its standard basis. In this basis, $S_2^z = (|0\rangle\langle 0| - |1\rangle\langle 1|)/2$. The qubit S_2 is coupled to its own environment R_2 represented by a finite (or infinite) quantum system and the interaction is described in terms of the operators $H_\pm$ which are elements of a Lie algebra $\mathcal{G}$ generating the symmetry group G [18],

$$H_\pm = \sum_{k=1}^{N} h_\pm^k(t) X_k \pm 1,$$

$$[X_i, X_j] = \sum_l C_{ij}^l X_l, \qquad (2)$$

where $h_\pm^k(t)$ are scalar control functions and X_k are basis elements of the Lie algebra with the structural constants C_{ij}^l.

Let us note two important features of the model (1) which describes two subsystems defined by the Hamiltonians H_1 and H_2 with $[H_1, H_2] = 0$. First, the qubits S_1 and S_2 do not directly interact with each other. However, their correlations are completely specified by initial conditions and presence of the environments. Second, the qubits do not exchange energy with their own environments. Therefore, the interactions with the environments are purely dephasing and result in an irreversible process of information loss [19]. As the qubits do not interact with each other, the only connection between them is information which is encoded in the initial bipartite state. Different aspects of similar systems are discussed in Ref. [6].

The second qubit can be steered by the driving $h_\pm^k(t)$ of the controlling quantum system $H_\pm$. The dynamical symmetry allows for an exact formulation of an evolution of the controlled qubit S_2 [18]. Initially this qubit is assumed to be separated from its controlling environment R_2 and the initial wave function of the subsystem H_2 is

$$|\psi(0)\rangle = (\alpha_+|0\rangle + \alpha_-|1\rangle) \otimes |\Omega\rangle, \qquad (3)$$

where α_+ and α_- are complex numbers defining the initial state for the qubit S_2, and $|\Omega\rangle$ is the initial state of the controlling system R_2 that is described by $H\pm$. The time evolution of the H_2 subsystem is given by the relation

$$|\psi(t)\rangle = \alpha_+ e^{it/2}|0\rangle \otimes T(g_+(t))|\Omega\rangle + \alpha_- e^{-it/2}|1\rangle \otimes T(g_-(t))|\Omega\rangle, \qquad (4)$$

where $T(\cdots)$ is a representation of the group G acting in the space of the controlling system R_2 and functions $g_\pm(t) \in G$ depend on the specific form of (2). A particularly simple situation corresponds to the case when the initial state $|\Omega\rangle$ is a generalized coherent state related to the symmetry group G since then it evolves into some other coherent state [18].

The reduced dynamics of qubits can be determined exactly for arbitrary model parameters [9,10] provided the initial state $\varrho(0)$ of the total system H can be factorized into the two-qubit state $\rho(0)$ and the states ρ^{R_1} and ρ^{R_2} of the corresponding environments R_1 and R_2, namely, $\varrho(0) = \rho^{R_1} \otimes \rho(0) \otimes \rho^{R_2}$. Simplicity of the model allows for an exact, rigorous treatment of entanglement dynamics beyond weak coupling and at arbitrary temperatures of the heat bath.

We assume that the state ρ^{R_1} of the heat bath R_1 is an equilibrium Gibbs state and the initial state of the environment R_2 is $\rho^{R_2} = |\Omega\rangle\langle\Omega|$. The initial state $\rho(0)$ of two qubits is an arbitrary state for the bipartite system. For time $t > 0$, the state $\rho(t)$ of two qubits has the form [cf. Eq. (5.19) in [9]]

$$\rho(t) = \Lambda(t)\rho(0), \qquad (5)$$

where the nonunitary evolution operator

$$\Lambda(t) = \Lambda_1(t) \otimes \Lambda_2(t),$$

$$\Lambda_n(t)\rho = C_1^n(t)\rho + 2C_2^n(t)[S_n^z,\rho] + 4C_3^n(t)S_n^z\rho S_n^z, \quad n=1,2, \tag{6}$$

for an arbitrary operator ρ. The functions

$$C_1^n(t) = \frac{1}{2}[1 + A_n(t)\cos\,\phi_n(t)],$$

$$C_2^n(t) = \frac{1}{2}iA_n(t)\sin\,\phi_n(t),$$

$$C_3^n(t) = \frac{1}{2}[1 - A_n(t)\cos\,\phi_n(t)], \tag{7}$$

for $n=1,2$. For the first qubit [9],

$$\phi_1(t) = t, \tag{8}$$

$$A_1(t) = \exp[-f(t)], \tag{9}$$

$$f(t) = \int_0^\infty d\omega \frac{J(\omega)}{\omega^2}\coth(\hbar\omega\beta_0/2)(1-\cos\,\omega t), \tag{10}$$

where $\beta_0=1/k_BT$, k_B is the Boltzmann constant, and T is temperature of the heat bath R_1. The relaxation function $A_1(t)$ is a decreasing function of time and describes the decay of mean values of the x and y components of the spin S_1.

The frequency spectrum of heat bath fluctuations is determined by the spectral function $J(\omega)=\Sigma_k g_k^2\delta(\omega-\omega_k)$. In the thermodynamic limit for R_1, it is assumed to take the form [8,20]

$$J(\omega) = \lambda\omega^{1+\mu}\exp(-\omega/\omega_c), \quad \mu > -1, \tag{11}$$

where the cutoff frequency ω_c determines the largest energy scale of R_1 (it removes possible problems at high frequencies) and λ corresponds to the coupling constant of the qubit S_1 and the environment R_1. The spectral exponent μ characterizes low-frequency properties of the heat bath and defines its various types. According to the classification proposed in Ref. [20], the heat bath is called sub-Ohmic for $\mu\in(-1,0)$, Ohmic for $\mu=0$, and super-Ohmic for $\mu\in(0,\infty)$. This classification shall be reflected in the dynamical properties of entanglement.

As one can infer from Eq. (4), after taking trace with respect to the R_2 variables one obtains for the qubit S_2,

$$\phi_2(t) = t + \arg[\langle\Omega|T^\dagger(g_-(t))T(g_+(t))|\Omega\rangle], \tag{12}$$

$$A_2(t) = |\langle\Omega|T^\dagger(g_-(t))T(g_+(t))|\Omega\rangle|. \tag{13}$$

These functions depend essentially on the choice of the initial preparation $|\Omega\rangle$. It is worth to stress that the corresponding generator of the reduced dynamics is of the Kossakowski-Lindblad form [9] and hence the complete positivity is preserved [3].

III. ENTANGLEMENT DYNAMICS

In this section we discuss evolution and long-time asymptotics of entangled states of two qubits S_1 and S_2. There are several computable measures of entanglement degree of bipartite systems [21]. Two of them, namely, concurrence [22] and negativity [23], have been extensively exploited. The relations between these two measures are discussed in Ref. [24]. In the paper, we choose the negativity $N(\rho)=\max(0,-\Sigma_i\lambda_i)$ [23], where λ_i are negative eigenvalues of the partially transposed density matrix $\rho(t)$ of a pair of qubits [25]. For entangled mixed states, the negativity is positive whereas it vanishes for unentangled states. Moreover, as it is an entanglement monotone [23], it can be used to quantify the degree of entanglement in composite systems. As we limit our attention to the 2×2 dimensional systems, the use of one of several other known entanglement monotones [21] such as, e.g., the concurrence leads to similar results.

We assume that initially the qubits are prepared in states belonging to the family of the depolarized Bell states [26]

$$\rho(0) = (1-p)|B_k\rangle\langle B_k| + \frac{p}{4}I, \tag{14}$$

where $k=1,2,3,4$, $p\in[0,1]$, and I is the unit operator. As $|B_k\rangle$ we take the maximally entangled states

$$|B_{1/2}\rangle = (|11\rangle \pm |00\rangle)/\sqrt{2},$$

$$|B_{3/4}\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}. \tag{15}$$

The depolarization accounts for an imperfect preparation of the initial state. The experimentally accessible states are always mixed due to quantum or classical noise. In this simplest approach, it is modeled by a single parameter p.

For the system of two qubits evolving under exact reduced dynamics given by Eqs. (5) and (14) one can evaluate the negativity at arbitrary time. For the above four Bell states, it has an appealing form, namely,

$$N(\rho(t)) = \max\left(0, \frac{1-p}{2}A_1(t)A_2(t) - \frac{p}{4}\right). \tag{16}$$

It is our main result, which through its simple product form $A_1(t)A_2(t)$, allows us to consider a variety of controlling environments R_2 defined by a large set of Lie algebras $\mathcal{G}$ as in Eq. (2). Inspection of this formula allows us to determine a critical value $p_c(t)$ of the probability p below which the system remains entangled at the given time t, i.e., for $p<p_c(t)$ one gets $N(\rho(t))\neq 0$. This critical value reads as

$$p_c(t) = \frac{2A_1(t)A_2(t)}{1+2A_1(t)A_2(t)}. \tag{17}$$

Notice that for $t=0$, when $A_1(0)=1$ and $A_2(0)=1$, one gets the condition for entanglement of a depolarized Bell state, i.e., $p<2/3$.

In the following we discuss entanglement in the limiting case $t\to\infty$. The asymptotic entanglement is possible in the case when

$$N_\infty = \lim_{t\to\infty} N(\rho(t)) \neq 0. \tag{18}$$

It is clear from Eq. (16) that the condition

$$A_1(\infty)A_2(\infty) = \lim_{t\to\infty} A_1(t)A_2(t) \neq 0 \tag{19}$$

is necessary for nonvanishing asymptotic entanglement. This condition becomes also sufficient for pure initial states, i.e., when $p=0$. For depolarized initial states (14) the relation $p < p_c(\infty)$ imposes the upper bound on the amount of initial noise.

IV. ENVIRONMENT-ASSISTED ENTANGLEMENT

As a first example, let us consider a simplified model with $A_2(t)=1$ [13], i.e., the second qubit S_2 does not interact with the environment R_2. However, the first qubit S_1 interacts with the heat bath R_1. Interestingly enough, this case has been analyzed in a recent work [13]. Notably, the previous study identifies the long-time asymptotics of entanglement by means of numerical calculations only. Here, we provide analytical results and prove that for the assumed class of initial states (14), the conjecture on nonvanishing long-time entanglement holds true. To this end, we must know explicit forms of the function $A_1(t)$ which are presented in Ref. [9]. We shall intensively use the results therein.

A. Zero-temperature limit

Let us first consider the zero-temperature case, $T=0$. In this case, the quantum-mechanical features should be more distinct because the classical sources of dissipation, decoherence, and dephasing are frozen. However, there are still zero-temperature fluctuations due to vacuum fluctuations of the environment R_1.

In order to make the paper self-contained we quote here the formulas derived in [9]. For the Ohmic bath ($\mu=0$), one obtains

$$A_1(t) = (1+\omega_c^2 t^2)^{-\lambda/2}. \tag{20}$$

For the sub-Ohmic and super-Ohmic baths one obtains

$$A_1(t) = \exp(-\lambda\Gamma(\mu)\omega_c^{\mu}\{1-(1+\omega_c^2t^2)^{-\mu/2} \times\cos[\mu \arctan(\omega_c t)]\}), \tag{21}$$

where $\Gamma(z)$ is the Euler Γ function. One can see that for both Ohmic ($\mu=0$) and sub-Ohmic [$\mu\in(-1,0)$] reservoirs, $\lim_{t\to\infty} A_1(t)=0$. In consequence, the negativity is zero, $N_\infty=0$, and the two-qubit states become unentangled in the long-time limit. For the super-Ohmic bath ($\mu>0$),

$$A_1(\infty) = \lim_{t\to\infty} A_1(t) = \exp[-\lambda\Gamma(\mu)\omega_c^{\mu}] \neq 0. \tag{22}$$

It follows that in this case the negativity is positive, $N_\infty>0$ and the two-qubit state is entangled for any time $t>0$, also when $t\to\infty$. It proves the conjecture put in Ref. [13] that only for the super-Ohmic environment, entanglement remains nonzero for arbitrary long times.

In Fig. 1 we visualize time evolution of entanglement of the qubit-qubit system quantified by the negativity. We assume that the first qubit is coupled to the infinite heat bath of oscillators whereas the second qubit evolves freely, i.e., $A_2(t)=1$. For all four initial Bell states (15) with $p=0$ in Eq.

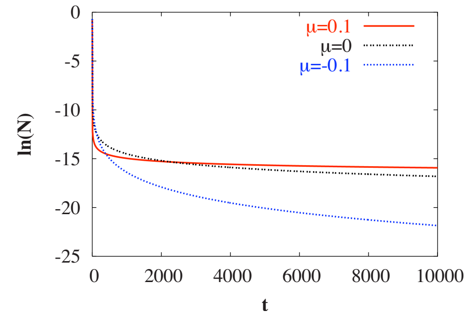

FIG. 1. (Color online) Time evolution for negativity from all four Bell states (15). The first qubit is coupled to the sub-Ohmic ($\mu=-0.1$), Ohmic ($\mu=0$), or super-Ohmic ($\mu=0.1$) bath and the second qubit evolves freely, i.e., $H_\pm=\pm1$. The cutoff frequency $\omega_c=10^3$, the coupling constant $\lambda=0.1$, temperature $T=0$, and the imperfect preparation parameter $p=0$, cf. Eq. (14). Time is given in units of t_0.

(14) (perfect preparation), one can observe two qualitatively distinct regimes [13]. In the first regime, Ohmic and sub-Ohmic, the negativity decays and vanishes in the limit $t\to\infty$. In the second regime, when the heat bath is super-Ohmic, the negativity tends to a nonzero value. Now, let the system be embedded in the super-Ohmic bath and consider the depolarized initial states (14). If the imperfect preparation parameter $p>0$, then one can expect three qualitatively different scenarios. The first, occurring for $p<p_c(\infty)$, is characterized by nonvanishing asymptotic entanglement. In the second scenario, for $p=p_c(\infty)$, the entanglement diminishes with time and asymptotically vanishes. The third scenario is the so-called entanglement sudden death [6] characterized by complete entanglement decoherence in a finite time. The moment of death $t_d<\infty$ can be implicitly defined by the relation $p=p(t_d)>p(\infty)$.

Let us notice that our discussion can follow along the same line as for the generalized case when both qubits interact with two infinite environments. Such two environments can differ from each other with respect to microscopic parameters which determine their spectral properties. It is clear that asymptotic entanglement does not vanish at $T=0$ if and only if both environments are super-Ohmic. This conclusion generalizes the results of Ref. [13].

B. Nonzero temperature

When temperature of the heat bath R_1 is nonzero, $T>0$, then entanglement survives under stronger conditions: The environment must be super-Ohmic and additionally its low-frequency spectrum $J(\omega)\propto\omega^{\mu}$ with $\mu>1$. If $\mu>1$, then the relaxation function $A_1(t)$ tends to a positive value given by the formula

$$A_1(\infty) = \exp\{-\lambda\Gamma(\mu)\omega_c^{\mu}[1+2B(T)]\}, \tag{23}$$

where the temperature contribution $B(T)$ reads as

$$B(T) = \sum_{n=1}^{\infty} (1+n\omega_c/k_BT)^{-\mu}. \tag{24}$$

For low temperature, the dominating term has the power-law form,

$$B(T) = \zeta(\mu)(k_B T/\omega_c)^{\mu}, \tag{25}$$

where $\zeta(\mu)$ is the Riemann ζ function [27]. So, the relaxation function $A_1(\infty)$ depends on temperature in the stretched exponential way: $A_1(\infty) \propto \exp(-\text{const } T^{\mu})$.

The persistence of asymptotic entanglement at nonzero temperature, i.e., $A_1(\infty) \neq 0$, holds true always for super-Ohmic baths satisfying $\mu > 1$. For $\mu \leq 1$ the relaxation function decays to zero either exponentially or algebraically and the qubits become disentangled.

V. SINGLE-MODE CONTROL

For the system considered in the preceding section, the negativity (and entanglement) is a monotonically decreasing function of time. If one wants to manipulate the entanglement in a desired way (e.g., modulate or maintain the entanglement in a desired interval), a method for how to do it must be worked out. Below, we propose possible scenarios for how to decrease and increase in time the entanglement of a pair of qubits. We will concentrate on the dynamics with respect to the role played by the coupling of the second qubit to a finite environment. We assume that the first qubit S_1 is dephasingly coupled to the super-Ohmic bath and the second qubit S_2 is controlled by a quantum system having the Heisenberg-Weyl symmetry. Such a symmetry, typical for quantum oscillators, is related to the system of standard coherent states in the Euclidean phase space. Our model is also valid for controlling systems with different dynamical symmetries. An example of a control with the rotational symmetry is briefly presented in the Appendix.

Experimentally, an oscillatorlike controlling system R_2 can be prepared by placing one of the qubits in the n-mode cavity. We show that by choosing an initial state of the cavity and the form of its interaction with the qubit one can design dynamical properties of the two-qubit entanglement.

We start with a single-mode coupling. Namely, let in Eq. (2) the operators $H_{\pm}$ be of the form

$$H_{\pm} = a^{\dagger}a \pm \gamma_{\pm}(a + a^{\dagger}) \pm 1. \tag{26}$$

For $\gamma_{\pm} = \gamma$=const we arrive at a single-mode limit of the Hamiltonian of the environment R_2. The exact time-dependent solution generated by the Hamiltonian (26) is known for a general, time-dependent coupling $\gamma = \gamma(t)$ [18,28]. In the following we limit ourselves to the isotropic coupling $\gamma_{\pm} = \gamma$. The corresponding function $A_2(t)$ reads as

$$A_2(t) = |\langle \Omega | D(\alpha(t)) | \Omega \rangle|, \tag{27}$$

where $\alpha(t) = \gamma[1 - \exp(it)]$ and $D(x) = \exp(xa^{\dagger} - x^* a)$ is the displacement operator [18,29]. This operator generates the set of standard coherent states. The function $A_2(t)$ is the Weyl function studied intensively in the context of interference phenomena [30] and mesoscopic devices controlled by nonclassical external fields [31].

A. Pure initial states

In the following we study the effect of a particular choice of initial preparation. Various states of quantum systems can

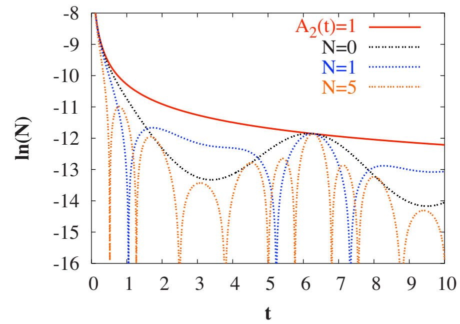

FIG. 2. (Color online) Negativity of the qubit-qubit system. The single-mode control is determined by (26) and (27) with $\gamma_{\pm} = 1$. The initial state of the controlling quantum system is the number state $|\Omega\rangle = |N\rangle$ with $N = 0, 1, 5$. The reservoir R_1 is super-Ohmic at $T = 0$. Remaining parameters are the same as in Fig. 1.

be, under carefully performed quantum engineering, experimentally prepared in the context of producing nonclassical electromagnetic fields [32]. In this section we limit our attention to number states, i.e., $|\Omega\rangle = |N\rangle$ and $a^{\dagger}a|N\rangle = N|N\rangle$. Choice of the coherent state as an initial state of the controlling system R_2 ($|\Omega\rangle = |z\rangle$) results in modification of the phase $\phi_2(t)$ in Eq. (12) without changing $A_2(t)$. The qubit-qubit entanglement is then the same as for the initial ground number state $|\Omega\rangle = |0\rangle$.

Numerical results for the entanglement dynamics controlled by the single-mode bosonic field are depicted in Fig. 2. Initially the two-qubit system is maximally entangled. The controlling system is assumed to be prepared in the number eigenstate $|N\rangle$. There are two features induced by a controlling system. First, the time evolution results in damped periodic oscillations of the entanglement. Second, the entanglement is weaker (not stronger) compared to the case $H_{\pm} = \pm 1$ (depicted by the uppermost line in Fig. 2) and the maximal values are limited by $A_1(t)$ as the envelope function. This amplitude can be controlled by a suitable depolarization of the initial state. For $p > p_c(\infty)$ one can expect the entanglement sudden death. Moreover, time distances between successive extrema of entanglement are smaller and smaller when N increases. It allows us to manipulate the degree of entanglement at certain time intervals. As the entanglement establishes strong quantum correlations between subsystems, the action carried out on one of the components is felt by the other. This is an essence of most of quantum protocols. By controlling entanglement oscillations one can predict and design when the system becomes almost disentangled allowing for local operations which do not affect the remaining subsystem.

Let us notice that nonmonotonicity of the negativity caused by interaction with a system locally coupled to one of the qubits is not in contradiction with the common wisdom concerning nonincreasing entanglement under local operations or classical communication (LOCC). As the control is non-Markovian, the only local operation transforms the system from $t_i = 0$ to $t_f = t > 0$. This transformation carries the information about the history of time evolution and certainly does not increase entanglement for every $t_f > 0$, i.e., $N(\rho(0)) \geq N(\rho(t_f))$.

B. Mixed initial states

In this section we consider the influence of the controlling quantum system $H_{\pm}$ prepared initially in a mixed state. Without loss of generality we work in the basis where the initial state of the controlling system is diagonal, i.e., $\rho^{R_2} = \Sigma_k p_k |\phi_k\rangle\langle\phi_k|$. The methods discussed in the preceding sections can be extended if we consider the following purification of the initial state:

$$|\Omega^{\mathrm{pur}}\rangle = |\Psi(0)\rangle = \sum_k \sqrt{p_k}|\phi_k\rangle|a_k\rangle, \tag{28}$$

where the set $\{a_k\}$ spans the Hilbert space of an ancilla system [26]. The probabilities $p_k \geq 0$ and $\Sigma_k p_k = 1$. We assume that the ancilla system does not evolve in time and the Hamiltonian of the purified system is

$$H_{\pm}^{\mathrm{pur}} = H_{\pm}(t) \otimes I, \tag{29}$$

where I is the identity operator in the Hilbert space of an ancilla system. In such a case, the resulting state is

$$|\Psi_{\pm}(t)\rangle = [T(g_{\pm}(t)) \otimes I]|\Psi(0)\rangle. \tag{30}$$

Its impact on the entanglement is quantified by the function

$$A_2(t) = |\langle\Psi_-(t)|\Psi_+(t)\rangle| = \left|\sum_k p_k \langle\phi_k|T^{\dagger}(g_-(t))T(g_+(t))|\phi_k\rangle\right|. \tag{31}$$

Because the probabilities $p_k \leq 1$, $A_2(t)$ is smaller in comparison with the cases of pure initial states. In consequence, it causes lowering of the entanglement degree [30].

VI. TWO-MODE CONTROL

In this section we assume that the controllable qubit is exposed to the two-mode controlling system characterized by the Hamiltonian

$$H_{\pm} = \sum_{i=1}^{2} [a_i^{\dagger}a_i \pm \gamma_i(a_i + a_i^{\dagger})] \pm 1. \tag{32}$$

Such a control results in dephasing governed by the function

$$A_2(t) = \left|\langle\Omega|\prod_{i=1}^{2} D(\alpha_i(t))|\Omega\rangle\right|. \tag{33}$$

For the state which initially is factorizable, i.e., for $|\Omega\rangle = |\Omega_1,\Omega_2\rangle = |\Omega_1\rangle|\Omega_2\rangle$, it factorizes into the product form

$$A_2(t) = \prod_{i=1}^{2} |\langle\Omega_i|D(\alpha_i(t))|\Omega_i\rangle|. \tag{34}$$

This property holds true for an arbitrary finite number of factorizable modes.

In the following we concentrate on two groups of initially entangled states. The first group consists of quantum correlated number states,

$$|\Omega\rangle = \frac{1}{\sqrt{2}}[|0,N\rangle \pm |N,0\rangle], \tag{35}$$

$$|\Omega\rangle = \frac{1}{\sqrt{2}}[|0,0\rangle \pm |N,N\rangle]. \tag{36}$$

The second group consists of entangled coherent states,

$$|\Omega\rangle = \frac{1}{\sqrt{\mathcal{N}}}[|0,z\rangle \pm |z,0\rangle], \tag{37}$$

where $\mathcal{N} = 2[1+\exp(-|z|^2)]$ and z is a complex number. The results for the entanglement of four Bell states (35) and (36) are presented in Fig. 3 for the case $N=1$. It is seen that two of the initial states result in the same entanglement evolution. Increase of N modifies oscillatory behavior of the negativity evolving in time (Fig. 4) in the similar way as in the single-mode case.

The controlling system prepared in the entangled state (37) strongly affects the qubit-qubit entanglement and it is related directly to $|z|$ as shown in Fig. 5. When $|z|$ increases, the entanglement becomes smaller and smaller for all t; however, the amplitude of oscillations becomes larger. The effect of phase of the complex number z is less spectacular and will not be discussed here.

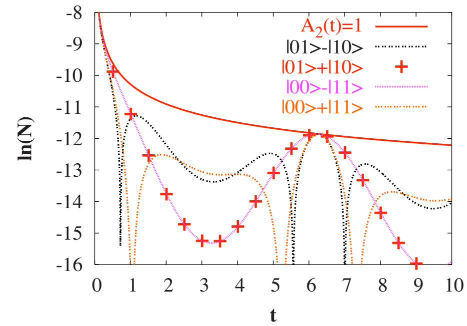

FIG. 3. (Color online) Negativity of the qubit-qubit system. The two-mode control is determined by (32) and (33) with $\gamma_1 = \gamma_2 = 1$. The initial state of the controlling quantum system is the number state given by Eqs. (35) or (36) with $N=1$. The reservoir R_1 is super-Ohmic at $T=0$. Remaining parameters are the same as in Fig. 1.

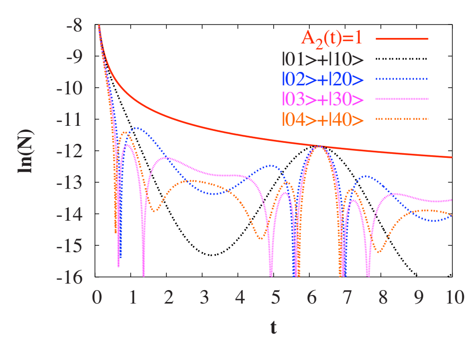

FIG. 4. (Color online) The same as in Fig. 3 for selected values of N of the initial state (35) or (36) of the controlling quantum system.

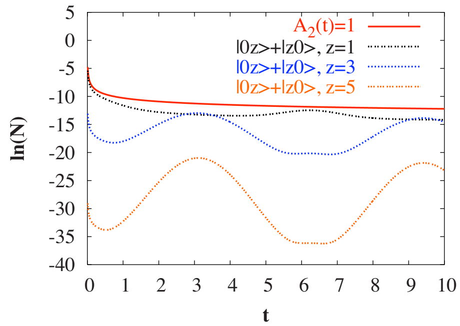

FIG. 5. (Color online) The same as in Fig. 3 for selected values of z of the initially entangled coherent state (37) of the controlling quantum system.

VII. SUMMARY

We have analyzed the non-Markovian entanglement dynamics based on the exact reduced description. In our setup, the system is composed of two qubits which are coupled to their own independent environments: The first qubit is coupled to the infinite heat bath R_1 while the second qubit can undergo the free evolution, can be coupled to infinite or finite environments R_2. The two subsystems (each consisting of the qubit and its own environment) are uncoupled. Our considerations are limited by the choice of a class of the initial qubit-qubit preparation. We assume that the qubits are prepared initially in one of the Bell states. The error in the initial preparation has been modeled in terms of the so-called depolarizing channel.

The entanglement of the bipartite system is shown to be a monotonically decreasing function of time when R_1 is an infinite heat bath and R_2 is absent or infinite. In such a case, the effect of the heat bath R_1 is to decrease the amount of entanglement of the initial state: The stronger the qubit-reservoir coupling the stronger the reduction of the degree of entanglement. We have discussed the influence of the low-frequency spectral properties of the heat bath. We have pointed out that for the super-Ohmic bath, it is possible to maintain entanglement forever not only at zero temperature but also at nonzero temperature. It extends and completes previous studies concerning other systems [14,33] and/or other modeling of environments [15,16]. We have also formulated the criterion which quantifies the maximal amount of initial noise for which the qubits remain asymptotically entangled. This allows us to prove the conjectures of the paper [13] within the assumed class of initial states.

We have found that in the case when R_2 is a finite quantum environment, the dynamics of entanglement can oscillate. This nonmonotonic behavior appears to be linked to the non-Markovian character of the dynamics and does not violate "the principle" of nonincreasing entanglement under LOCC. In the short-time regime this oscillating behavior is modified by the infinite thermostat R_1 that is responsible for a reduction of negativity. However, asymptotically $A_1(t) \to \mathrm{const}$ and the negativity becomes a periodic function of time. Both the amplitude and the period of these oscillations are very sensitive to the initial state of the quantum environment R_2. The difference between the minimal and the maximal values of negativity can be of a few orders of magnitude. It means that qubits evolve quasiperiodically between entangled and almost disentangled states. One cannot use this method to control the entanglement during its evolution. However, both studied cases (one-mode and two-mode control) are found promising to design *a priori* a desired sequence of times, when qubits are almost disentangled. It can be achieved by means of an appropriate choice of the initial state of the quantum environment. This, in turn, could help to work out a method to manipulate the degree of entanglement at certain time intervals. It additionally provides useful information for experimental design of quantum-information protocols.

The model considered in this work is (one of) the simplest. However, its features are generic for a large class of quantum open systems. We have limited our studies to the Heisenberg-Weyl dynamical symmetry due to its potential implementations by means of optical devices such as, e.g., quantum cavities. There is no fundamental obstruction to study other symmetries. An example, the rotational SU(2) symmetry, is briefly discussed in the Appendix. Generalization of the discussion to the noncompact groups [e.g., to SU(1,1) related to the squeezed states] with nonequivalent representation series [18] is a natural open problem.

The approach presented in this paper can be extended to describe systems composed of more than two qubits. The dynamics given in Eq. (6) becomes then generated by a finite product of dephasing generators. However, in such a case one faces a highly nontrivial problem of the many-body entanglement where the results depend strongly on the chosen entanglement measure [34]. Such an extension would be useful for the solid-state quantum-information processing. Moreover, it would also be of importance to extend the model toward more realistic decoherence scenarios.

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APPENDIX: ENVIRONMENT R_2 WITH ROTATIONAL SYMMETRY

Here, we present one more example of the controlling system R_2 which has a rotational dynamical symmetry, i.e., the case when the $H_\pm$ can be expressed in terms of generators of the SU(2) group. Such systems, usually exemplified by quantum tops, have a well-defined classical limit [17]. Instead of the monotonicity, we briefly discuss the effect of classicization of R_2 on the entanglement.

Let us assume that the qubit S_2 is coupled to a particle of angular momentum j in the presence of an external magnetic field $B(t)$. The control is performed by the quantum system of the Hamiltonian

$$H_\pm(t) = -\frac{1}{j}B_\pm(t)J \pm 1 \tag{A1}$$

with the magnetic field

$$B_{\pm}(t) = \{\pm b_0 \hat{z} + b_1[\hat{x}\cos(t) + \hat{y}\sin(t)]\} \tag{A2}$$

and the angular momentum operator $J=(J_x,J_y,J_z)$. When the controlling system is initially in its ground state, the negativity can be expressed in terms of the SU(2) coherent states,

$$A_2(t) = |\langle \xi_-(t)|\xi_+(t)\rangle|, \tag{A3}$$

where [18]

$$\xi_{\pm}(t) = \frac{i\omega_1 \sin(\Omega t)\exp[i(\omega_0 - 1)t]}{2\Omega\cos(\Omega t) - i(1-\omega_0)\sin(\Omega t)} \tag{A4}$$

with $\omega_1 = b_1/j$, $\omega_0 = \pm b_0/j$, and $\Omega = \sqrt{(1-\omega_0)^2 + \omega_1^2}/2$. Increasing j results in dequantization of the top which becomes classical in the limit $j \to \infty$. We have performed numerical calculations, not reproduced here, for time evolution of qubit-qubit entanglement for the system with fixed $b_0 = b_1 = 1$ and increasing integral j. As a result, we formulate the conjecture that the classicization of the controlling system decreases its impact on the entanglement. We expect that the classical limit of infinite j is equivalent to $A_2(t) \to 1$ for arbitrary $t \geq 0$.

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