Effect of spin-orbit interaction on entanglement of two-qubit Heisenberg *XYZ* systems in an inhomogeneous magnetic field

Fardin Kheirandish,* S. Javad Akhtarshenas,† and Hamidreza Mohammadi‡
Department of Physics, University of Isfahan, Hezar Jarib Avenue, Isfahan, Iran

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The role of spin-orbit interaction in the ground state and thermal entanglement of a Heisenberg *XYZ* two-qubit system in the presence of an inhomogeneous magnetic field is investigated. We show that the ground state entanglement tends to vanish suddenly for a certain value of the spin-orbit parameter D and, when D crosses its critical value D_c, the entanglement undergoes a revival. Indeed, when D crosses its critical value (D_c), the ground state entanglement tends to its maximum value ($C=1$). Also, at finite temperatures there are revival regions in the D-T plane. In these regions, entanglement first increases with increasing temperature and then decreases and ultimately vanishes for temperatures above a critical value. We find that this critical temperature is an increasing function of D and that the amount of entanglement in the revival region depends on the spin-orbit parameter. Therefore when spin-orbit interaction is included larger thermal entanglement can exist at higher temperatures. We also show that the rate of enhancement of thermal entanglement by D is not the same for ferromagnetic ($J_z<0$) and antiferromagnetic ($J_z>0$) chains. The entanglement teleportation via the quantum channel constructed by the above system is also investigated, and the influence of the spin-orbit interaction on the fidelity of teleportation and entanglement of replica states is studied. We show that, by introducing spin-orbit interaction, the entanglement of the replica state and fidelity of teleportation can be increased for the case of $J_z<0$. We also argue that a minimal entanglement of the channel is required to realize efficient entanglement teleportation and, in the case of $J_z<0$, this minimal entanglement can be achieved by introducing spin-orbit interaction.

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I. INTRODUCTION

Einstein, Podolsky, and Rosen (EPR), in their famous paradox, argued that in general two quantum system cannot be separated even if they are located far from each other [1]. Schrödinger named this quantum mechanical property entanglement [2]. Today, entanglement is a uniquely quantum mechanical resource that plays a key role in many of the most interesting applications of quantum computation and quantum information [3,4]. Thus a great deal of effort has been devoted to studying and characterizing entanglement in recent years. The central task of quantum-information theory is to characterize and quantify the entanglement of a given system. A mixed state ρ of a bipartite system is said to be separable or classically correlated if it can be expressed as a convex combination of uncorrelated states ρ_A and ρ_B of each subsystem, i.e., $\rho=\Sigma_i\omega_i\rho_A^i\otimes\rho_B^i$ such that $\omega_i\geq 0$ and $\Sigma_i\omega_i=1$; otherwise ρ is entangled [4,5]. Many measures of entanglement have been introduced and analyzed [3,4,6,7], but the one most relevant to this work is entanglement of formation, which is intended to quantify the resources need to create a given entangled state [6]. For the case of a two-qubit system Wootters has shown that the entanglement of formation can be obtained explicitly as

$$E(\rho)=\Xi[C(\rho)]=h\left(\frac{1+\sqrt{1+C^2}}{2}\right), \quad (1)$$

where $h(x)=-x\log_2 x-(1-x)\log_2(1-x)$ is the binary entropy function and $C(\rho)=\max\{0,2\lambda_{\max}-\Sigma_{i=1}^4\lambda_i\}$ is the concurrence of the state, where the λ_i's are positive square roots of the eigenvalues of the non-Hermitian matrix $R=\rho\tilde{\rho}$, and $\tilde{\rho}$ is defined by $\tilde{\rho}:=(\sigma^y\otimes\sigma^y)\rho^*(\sigma^y\otimes\sigma^y)$. The function Ξ is a monotonically increasing function and ranges from 0 to 1 as $C(\rho)$ goes from 0 to 1, so that one can take the concurrence as a measure of entanglement in its own right. In the case that the state of the system is pure, i.e., $\rho=|\psi\rangle\langle\psi|$, $|\psi\rangle=a|00\rangle+b|01\rangle+c|10\rangle+d|11\rangle$, the above formula is simplified to $C(|\psi\rangle)=2|ad-bc|$.

Among the numerous concepts considered to implement quantum bits (qubits), approaches based on semiconductor quantum dots (QDs) offer the great advantage that ultimately a miniaturized version of a quantum computer is feasible. Indeed, Loss and DiViencenzo initially proposed a quantum computer protocol based on electron spins trapped in semiconductor QDs in 1998 [8–10]. Here, the qubit is represented by a single electron in a QD, which can be initialized, manipulated, and read out by extremely sensitive devices. Compared with other systems such as quantum optical systems and nuclear magnetic resonance (NMR), QDs are argued to be more scalable and have a longer decoherence time. Consider a two-qubit system represented by two electrons confined in two coupled quantum dots (CQDs). Because of the weak lateral confinement, electrons can tunnel from one dot to the other, and spin-spin and spin-orbit (SO) interactions between two qubits exist. Thus, we can consider the spin of

*fardin_kh@phys.ui.ac.ir
†akhtarshenas@phys.ui.ac.ir
‡h.mohammadi@phys.ui.ac.ir

the two electrons confined in two CQDs as a two-qubit spin chain. The Heisenberg model (the simplest model for studying and investigating the behavior of spin chains) is suitable for modeling the above two-qubit system if it includes SO interaction. The SO interaction in such nanostructures can be investigated with the help of quantum optical methods.

Nielsen [12] first studied the entanglement of a two-qubit Heisenberg *XXX* chain modeled with the Hamiltonian $H = J\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \mathbf{B} \cdot (\sigma_1^z + \sigma_2^z)$. He showed that entanglement in such systems exists only for the antiferromagnetic $(J>0)$ case below a threshold temperature T_c. After Nielsen, entanglement in two-qubit *XXX*, *XXZ*, and *XY* systems in the presence of homogeneous and inhomogeneous magnetic fields has been investigated [13–18]. For example, the authors of Ref. [18] showed that, for two-qubit Heisenberg systems at zero temperature, the ground state entanglement vanishes for a critical value of external magnetic field (B_c), but for $B > B_c$ and special values of the anisotropy parameter, the ground state entanglement exhibits revival before decreasing to zero asymptotically.

The effect of anisotropy due to spin coupling in the x, y, z directions has also been studied in a number of works [19–21]. In Ref. [21], Yang *et al.* showed that, in *XYZ* Heisenberg systems, a larger revival phenomenon at $T=0$ can be achieved by introducing an inhomogeneous magnetic field. They also showed that the revival phenomenon in the thermal entanglement still exists below a critical temperature (T_c).[1] If the parameters are proper and the inhomogeneity is large enough, in the revival region, the thermal entanglement and the critical temperature can be improved by increasing the inhomogeneity of the magnetic field.

The spin-orbit interaction cause another type of anisotropy [22–24]. The effect of SO interaction on the thermal entanglement of a two-qubit *XX* system in the absence of magnetic field was studied in [25]. In this reference, it was shown that the critical temperature (T_c) increases with the increase of the absolute value of the SO parameter (D). In the same reference it was also shown that the thermal entanglement is the same for antiferromagnetic and ferromagnetic Heisenberg model with SO interaction. However, the entanglement for an *XYZ* Heisenberg system under an inhomogeneous magnetic field in the presence of SO interaction has not been discussed.

On the other hand, a two-qubit entangled system can be used to perform quantum teleportation protocols. Historically, Bennet *et al.* first showed that two entangled spatially separated particles can be used for teleportation [26]. They also argued that states which are less entangled still could be used for teleportation but they reduce "the fidelity of teleportation and/or the range of state $|\phi\rangle$ which can accurately be teleported." Then Popescu, using the hidden variable model, showed that teleportation of a quantum state via pure classical communication cannot be performed with fidelity larger than $\frac{2}{3}$ [27]. Thus, mixed quantum channels which allow transfer of quantum information with fidelity larger than $\frac{2}{3}$ are worthwhile. Horodecki *et al.* calculated the optimal fidelity of teleportation for a bipartite state acting on $C^d \otimes C^d$ by using the isomorphism between quantum channels and a class of bipartite states and twirling operations [28]. Bowen and Bose have shown that standard teleportation with an arbitrary mixed state resource is equivalent to a general depolarizing channel with the probabilities given by the maximally entangled component of the resource. This enables the usage of any quantum channel as a generalized depolarizing channel without additional twirling operations [29]. Using the property of the linearity of teleportation [26], Lee and Kim showed that quantum teleportation preserves the nature of quantum correlation in the unknown entangled state if the channel is quantum mechanically correlated and then they investigated entanglement teleportation via two copies of Werner states [30]. The entanglement teleportation via thermally entangled states of two-qubit Heisenberg *XX* and *XY* chains has been studied by Yeo *et al.* [31,32]. The effect of spin-orbit interaction on entanglement teleportation in a two-qubit *XXX* Heisenberg chain in the absence of a magnetic field is reported by Zhang [33].

In this paper, taking advantage of the tunability of SO strength (see [11] and references therein), we investigate the influence of SO interaction on the entanglement and entanglement teleportation of a two-qubit system at thermal equilibrium. We show that the SO interaction parameter (D) plays the role of the magnetic field inhomogeneity parameter and hence a larger revival phenomenon of ground state entanglement can be achieved by introducing the SO interaction. Indeed, when D crosses its critical value (D_c), the ground state entanglement tends to its maximum values $(C=1)$. Furthermore, we also show that the critical values of other parameters, such as temperature (T), magnetic field (B), and inhomogeneity of the magnetic field (b), are controllable by adjusting the SO parameter (D). In particular, if the parameters are proper and D is large enough, in the revival region, the thermal entanglement and the critical temperature can be improved by increasing D. Therefore larger entanglement can exist at larger temperatures. We also show that, in contrast to the case of the *XX* chain in the absence of magnetic field (as mentioned in Ref. [25]), the rate of enhancement of thermal entanglement by D is not the same for ferromagnetic $(J_z<0)$ and antiferromagnetic $(J_z>0)$ chains. Also in this paper we investigate the possibility of entanglement teleportation by the above mentioned two-qubit *XYZ* system. We show that the SO interaction and the inhomogeneity of the magnetic field have an effect on the entanglement of replica states and the fidelity of the teleportation. We show that a minimal entanglement of the thermal state in the model is required to realize efficient entanglement teleportation and we can attain this minimal entanglement, in the case of $J_z<0$, by introducing the SO interaction.

The paper is organized as follows. In Sec. II we introduce the Hamiltonian of the system and write the thermal density

[1] In finite temperatures, there is a threshold temperature (T_c) above which thermal fluctuations suppress quantum fluctuations, and for that reason entanglement vanishes for all $T>T_c$. Nonetheless, when the ground state is less entangled than the first excited state, entanglement, as measured by entanglement of formation or concurrence, may not be a decreasing function of T for $T<T_c$, and even the vanishing of entanglement plus reentry may occur (revival region at finite temperatures). In the revival region, increasing temperature first increases the entanglement and then begins to diminish it.

matrix of the system related to this Hamiltonian. Ultimately, the ground state entanglement and finite temperature entanglement of the system are computed. The entanglement teleportation of a two-qubit pure state and its fidelity are derived in Sec. III. In Sec. IV a discussion concludes the paper.

II. THE MODEL AND THE HAMILTONIAN

The Hamiltonian of a two-qubit anisotropic Heisenberg XYZ model in the presence of an inhomogeneous magnetic field and spin-orbit interaction is defined as

$$H=\frac{1}{2}[J_x\sigma_1^x\sigma_2^x+J_y\sigma_1^y\sigma_2^y+J_z\sigma_1^z\sigma_2^z+\mathbf{B}_1\cdot\boldsymbol{\sigma}_1+\mathbf{B}_2\cdot\boldsymbol{\sigma}_2+\mathbf{D}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)+\delta\boldsymbol{\sigma}_1\cdot\bar{\boldsymbol{\Gamma}}\cdot\boldsymbol{\sigma}_2], \tag{2}$$

where $\boldsymbol{\sigma}_j=(\sigma_j^x,\sigma_j^y,\sigma_j^z)$ is the vector of Pauli matrices, $\mathbf{B}_j$ ($j=1,2$) is the magnetic field on site j, J_μ ($\mu=x,y,z$) are the real coupling coefficients (the chain is antiferromagnetic for $J_\mu>0$ and ferromagnetic for $J_\mu<0$), $\mathbf{D}$ is the Dzyaloshinski-Moriya vector, which is first order in SO coupling and is proportional to the coupling coefficients (J_μ), and $\bar{\boldsymbol{\Gamma}}$ is a symmetric tensor which is second order in SO coupling [22,23]. For simplicity, we assume $\mathbf{B}_j=B_j\hat{\mathbf{z}}$ such that $B_1=B+b$ and $B_2=B-b$, where b indicates the amount of inhomogeneity of the magnetic field. The vector $\mathbf{D}$ and the parameter δ are dimensionless. In a system like coupled GaAs quantum dots, $|\mathbf{D}|$ is of the order of a few percent, while the order of the last term is 10^{-4} so it is negligible. (In GaAs double QDs, hyperfine interactions have been shown to dominate spin mixing at small magnetic fields, while SO interaction is not relevant in this regime. However, SO interaction and the coupling magnetic fields are expected to be orders of magnitude stronger in InAs compared to GaAs [34].) If we take $\mathbf{D}=J_zD\hat{\mathbf{z}}$ and ignore the second-order SO coupling, then the above Hamiltonian can be expressed as

$$H=J\gamma(\sigma_1^+\sigma_2^++\sigma_1^-\sigma_2^-)+(J+iJ_zD)\sigma_1^+\sigma_2^-+(J-iJ_zD)\sigma_1^-\sigma_2^++\frac{J_z}{2}\sigma_1^z\sigma_2^z+\left(\frac{B+b}{2}\right)\sigma_1^z+\left(\frac{B-b}{2}\right)\sigma_2^z, \tag{3}$$

where $J:=\frac{J_x+J_y}{2}$, is the mean coupling coefficient in the XY plane, $\gamma:=\frac{J_x-J_y}{J_x+J_y}$ specifies the amount of anisotropy in the XY plane (partial anisotropy, $-1\leq\gamma\leq1$), and $\sigma^\pm=\frac{1}{2}(\sigma^x\pm i\sigma^y)$ are lowering and raising operators. The Hamiltonian (3), in the standard basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$, has the following matrix form:

$$H=\begin{pmatrix}\frac{1}{2}J_z+B & 0 & 0 & J\gamma\\ 0 & -\frac{1}{2}J_z+b & J+iJ_zD & 0\\ 0 & J-iJ_zD & -\frac{1}{2}J_z-b & 0\\ J\gamma & 0 & 0 & \frac{1}{2}J_z-B\end{pmatrix}. \tag{4}$$

The spectrum of H is easily obtained as

$$H|\psi^\pm\rangle=\varepsilon_{1,2}|\psi^\pm\rangle,$$

$$H|\Sigma^\pm\rangle=\varepsilon_{3,4}|\Sigma^\pm\rangle, \tag{5}$$

where the eigenstates and the corresponding eigenvalues are, respectively,

$$|\psi^\pm\rangle=N^\pm\left[-\left(\frac{b\pm\xi}{J-iJ_zD}\right)|01\rangle+|10\rangle\right],$$

$$|\Sigma^\pm\rangle=M^\pm\left[-\left(\frac{B\pm\eta}{J\gamma}\right)|00\rangle+|11\rangle\right], \tag{6}$$

and

$$\varepsilon_{1,2}=-\frac{1}{2}J_z\pm\xi,$$

$$\varepsilon_{3,4}=\frac{1}{2}J_z\pm\eta. \tag{7}$$

In the above equations $N^\pm=1/\sqrt{1+\frac{(b\pm\xi)^2}{J^2+(J_zD)^2}}$ and $M^\pm=1/\sqrt{1+(\frac{B\pm\eta}{J\gamma})^2}$ are the normalization constants. Here we define $\xi:=\sqrt{b^2+J^2+(J_zD)^2}$ and $\eta:=\sqrt{B^2+(J\gamma)^2}$, for later convenience.

The state (density matrix) of a system in equilibrium at temperature T is $\rho=Z^{-1}\exp(-\frac{H}{k_BT})$, where Z is the partition function of the system and k_B is the Boltzmann constant. For simplicity we take $k_B=1$. In the standard basis, the density matrix of the system in thermal equilibrium can be written as

$$\rho_T=\begin{pmatrix}\mu_+ & 0 & 0 & \nu\\ 0 & w_1 & z & 0\\ 0 & z^* & w_2 & 0\\ \nu & 0 & 0 & \mu_-\end{pmatrix}, \tag{8}$$

where

$$\mu_\pm=\frac{e^{-\beta J_z/2}}{Z}\left(\cosh\beta\eta\mp\frac{B}{\eta}\sinh\beta\eta\right),$$

$$w_{1,2}=\frac{e^{\beta J_z/2}}{Z}\left(\cosh\beta\xi\mp\frac{b}{\xi}\sinh\beta\xi\right),$$

$$\nu=-\frac{J\gamma e^{-\beta J_z/2}}{Z\eta}\sinh\beta\eta,$$

$$z=-\frac{(J+iJ_zD)e^{\beta J_z/2}}{Z\xi}\sinh\beta\xi, \tag{9}$$

and $Z=2e^{\beta J_z/2}(\cosh\beta\xi+e^{-\beta J_z}\cosh\beta\eta)$.

In what follows, our purpose is to quantify the amount of entanglement of the above two-qubit system versus the parameters of the system, with the main focus on D. For a density matrix in the form (8), one can show, by straightforward calculations, that the square roots of the eigenvalues of matrix $R=\rho\tilde{\rho}$ are

$$\lambda_{1,2} = |\sqrt{w_1 w_2} \pm |z|| = \frac{e^{\beta J_z/2}}{\xi Z} |\sqrt{\xi^2 + J^2 + (J_z D)^2 \sinh^2 \beta\xi} \pm \sqrt{J^2 + (J_z D)^2} \sinh \beta\xi|,$$

$$\lambda_{3,4} = |\sqrt{\mu_+\mu_-} \mp \nu| = \frac{e^{-\beta J_z/2}}{\eta Z} | \sqrt{\eta^2 + (J\gamma)^2 \sinh^2 \beta\eta} \mp J\gamma \sinh \beta\eta| . \tag{10}$$

Now, it is easy to calculate the concurrence. Without loss of generality we can assume $J>0$ and $\gamma>0$, since the above formulas are invariant under the substitution $J\to -J$ and $\gamma \to -\gamma$. For the special case $D=0$, these equations give the same results as in Ref. [21]. The obtained results are given in the following sections.

A. Ground state entanglement and critical parameters

The behavior of the system at the quantum phase transition (QPT) point,[2] such as $B=B_c$, $b=b_c$, and $D=D_c$, may be determined from the density matrix at $T=0$, where the system is in its ground state. If we consider $T\to 0$ (or $\beta\to\infty$) then the concurrence C can be written analytically as

$$C(T=0) = \begin{cases} \left|\frac{J\gamma}{\eta}\right| & \text{if } \xi < \eta - J_z, \\ \frac{1}{2}\left|\left|\frac{J\gamma}{\eta}\right| - \frac{\sqrt{J^2+(J_zD)^2}}{\xi}\right| & \text{if } \xi = \eta - J_z, \\ \left|\frac{\sqrt{J^2+(J_zD)^2}}{\xi}\right| & \text{if } \xi > \eta - J_z. \end{cases} \tag{11}$$

Also, this formula can be derived directly, by calculating the entanglement of the ground state of H: When $\epsilon_4<\epsilon_2$ (or $\xi<\eta-J_z$) the ground state of H is $|\Sigma^-\rangle$ and then $C=|\frac{J\gamma}{\eta}|$ and when $\epsilon_2<\epsilon_4$ (or $\xi>\eta-J_z$) the ground state of is $|\psi^-\rangle$ and then $C=|\frac{\sqrt{J^2+(J_zD)^2}}{\xi}|$. On the other hand, at the critical point (where $\epsilon_4=\epsilon_2$ or $\xi<\eta-J_z$), the ground state of the system is an equal mixture of $|\Sigma^-\rangle$ and $|\psi^-\rangle$, i.e., $|\text{g.s.}\rangle=(|\Sigma^-\rangle+|\psi^-\rangle)/2$, and therefore $C=\frac{1}{2}||\frac{J\gamma}{\eta}|-\frac{\sqrt{J^2+(J_zD)^2}}{\xi}|$.

The concurrence C as a function of b (at $T=0$) for three values of D ($D=0,0.8,1$) are plotted in Fig. 1. With increasing b, the concurrence C is initially constant and equal to $C=|\frac{J\gamma}{\eta}|=0.53$, and then drops suddenly at a critical value of b $\{b_c=\sqrt{(\eta-J_z)^2-[(J_zD)^2+J^2]}\}$. At this point ($T=0$, $b=b_c$), the concurrence becomes a nonanalytical function of b and the QPT occurs [21]. For $b>b_c$, the concurrence C undergoes a revival before decreasing to zero. The amount of concurrence in the revival region depends on D in the sense that when D increases the revival is greater. Furthermore, when D increases, b_c decreases, i.e., for larger D, the critical point and hence the revival phenomenon occur at smaller b. In order to better illustrate the effects of D on the entanglement of the ground state, the concurrence is plotted in terms of D in Figs. 2(a)–2(c) for different values of b, γ, and J_z, respectively. The results show that until D reaches its critical value, defined by $D_c=\sqrt{(\eta-J_z)^2-(b^2+J^2)}/|J_z|$, the concurrence drops, and it exhibits a revival when D crosses its critical value D_c. It is also seen from these figures that the critical point D_c is increased by increasing γ and $|J_z|$, but decreased by increasing b, and also the size of γ determines the value of the entanglement before the critical point is reached. Hence, we can control the value of D_c by adjusting the parameters of the system such as b, γ, and J_z. We want to decrease D_c because for $D>D_c$ the concurrence reaches its maximum value ($C=1$) as D increases. Therefore we should choose $|J_z|=1$, γ small, and b as large as possible.

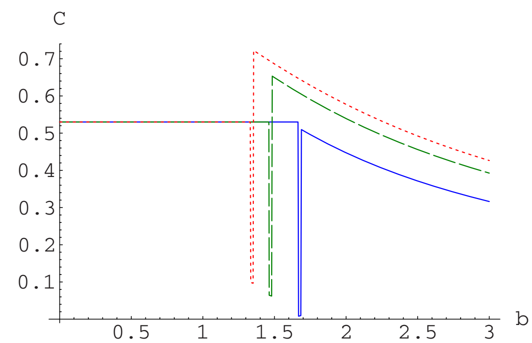

FIG. 1. (Color online) Ground state concurrence vs b for $D=0$ (blue solid line), 0.8 (green dashed line), and 1 (red dotted line), where $B=0.8$, $J=1$, $J_z=-1$, and $\gamma=0.5$. All the parameters are dimensionless [see Eq. (11)].

B. Thermal entanglement

Since the relative magnitudes of λ_i ($i=1,2,3,4$) [see Eq. (10)] depend on the parameters involved, they cannot be ordered by magnitude without knowing the values of the parameters. This prevents one from writing an analytical expression for the concurrence $C(\rho)=\max\{0,2\lambda_{\max}-\Sigma_{i=1}^4\lambda_i\}$. For particular parameters, C can be evaluated numerically from Eq. (10) and the definition of concurrence. The role of each parameter can be seen by fixing the other parameters and drawing the variation of entanglement for specific values of that parameter. The cross influence of two parameters can be shown in three-dimensional (3D) plots of entanglement. The thermal entanglement versus the system's parameters is depicted in Figs. 3–5. The concurrence as a function of T and D is shown in Fig. 3. An analysis of the results in this figure is given in the following. Let $C'_j=2\lambda_j-\Sigma_{i=1}^4\lambda_i$ ($j=1,2,3,4$); it is evident that the function $C_j=\max\{0,C'_j\}$ is the concurrence of the system if and only if $\lambda_j=\lambda_{\max}=\max\{\lambda_1,\lambda_4\}$. (Since $J,\gamma>0$ we have already $\lambda_4>\lambda_3$ and $\lambda_1>\lambda_2$.) We can divide Fig. 3 into four parts.

[2] Note that a phase transition can, strictly speaking, occur only in the thermodynamic limit of $\sim 10^{23}$ particles, but one may see traces of what may become a phase transition in smaller systems. However, it is important to emphasize that no actual phase transition can occur in this system. In fact it has become accepted practice in this field to borrow terms such as "critical" and "quantum phase transition" from the terminology of phase transition theory [35,36].

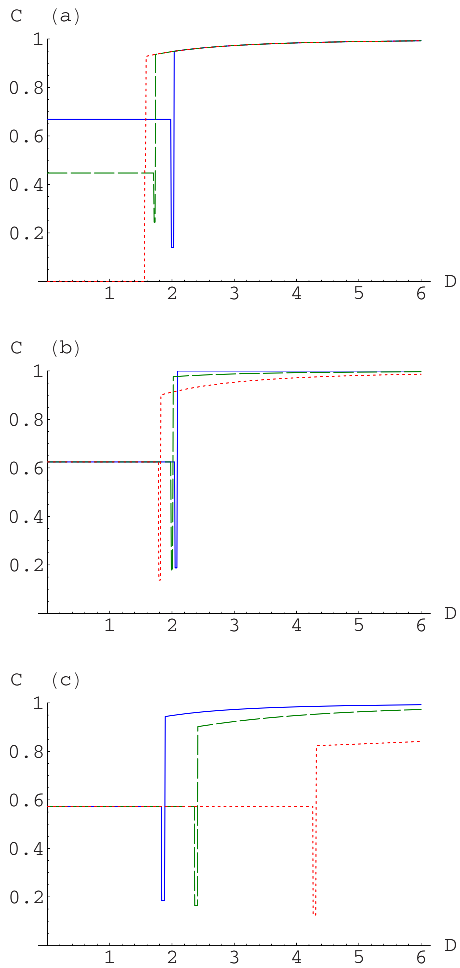

FIG. 2. (Color online) Concurrence of the ground state vs D. (a) $b=0$ (blue solid line), 0.5 (green dashed line), and 1 (red dotted line), $J_z=-1$, and $\gamma=0.8$. (b) $\gamma=0.9$ (blue solid line), 0.5 (green dashed line), and $\simeq 0$ (red dotted line), $J_z=-1$, and $b=0.75$. (c) $J_z=-1$ (blue solid line), -0.5 (green dashed line), and -0.1 (red dotted line), $b=0.75$, and $\gamma=0.7$. All the parameters are dimensionless [see Eq. (11)].

(i) The region for which $T<T_{c1}$ (see below) and $D<D_c$; in this region $\lambda_{max}=\lambda_4$ and thus $C_4'=\lambda_4-\lambda_3-\lambda_2-\lambda_1$ determines the amount of entanglement. According to Eq. (10), the value of λ_4 depends on γ and η (or equivalently γ and B). Thus in this region we can manipulate the amount of entanglement by adjusting γ and B. The first critical temperature (T_{c1}) is the point at which $\lambda_{max}=\lambda_4=\lambda_1$ and hence $C=0$.

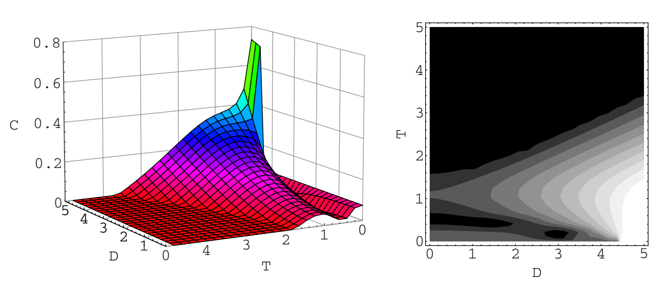

FIG. 3. (Color online) Thermal concurrence vs T and D, where $B=4$, $b=2.5$, $J=1$, and $\gamma=0.3$. All the parameters are dimensionless [see Eq. (10) and the definition of concurrence].

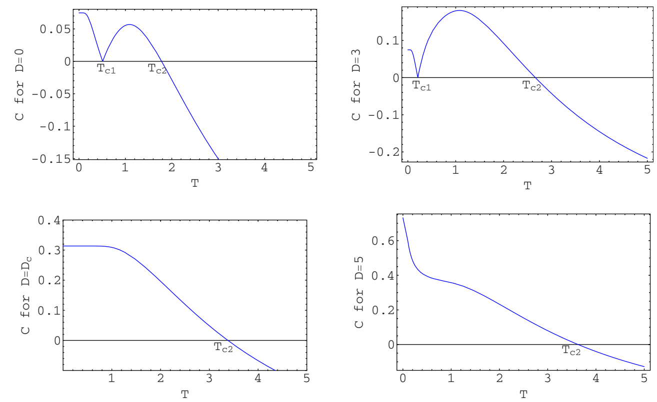

FIG. 4. (Color online) Thermal concurrence vs for different values of . The parameters are the same as in Fig. 3. In this case $D_c \simeq 4.51$. All the parameters are dimensionless [see Eq. (10) and the definition of concurrence].

(ii) The revival region, for which $T_{c1}<T<T_{c2}$ (see below) and $D<D_c$; in this region $\lambda_{max}=\lambda_1$ and hence C_1' determines the amount of entanglement. According to Eq. (10), the value of λ_1 is adjustable by changing the values of D and ξ (or equivalently D and b) and hence the parameters D and b play an important role in quantifying the amount of entanglement in this region. For $T>T_{c1}$, the value of λ_1 and also the rate of enhancement of the function λ_1 with T increases as D increases. Enhancement of the function λ_1 with T causes C_1' to rise to a positive number and thus the entanglement undergoes a revival. Since the rate of enhancement of the function λ_1 with T is an increasing function of D, the amount of revival increases as D increases. When T reaches the value $T=T_{c2}$ (second critical temperature), λ_1 tends to zero again and thus the entanglement vanishes.

(iii) The region for which $D>D_c$ for all values of T; in this region $\lambda_{max}=\lambda_1$ and C_1' is the entanglement indicator. The maximum value of the entanglement occurs in this region. At zero temperature the entanglement has its maximum value and when the temperature increases the system loses its entanglement which ultimately vanishes at $T=T_{c2}$. In this region no revival phenomenon occurs.

(iv) The region for which $T\geq T_{c2}$ for all values of D; in this region all values of C_j' $(j=1,2,3,4)$ are negative and the entanglement is zero for all values of D and the other parameters.

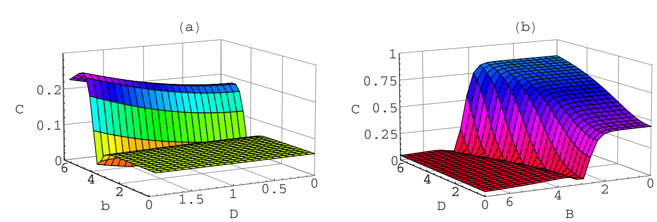

FIG. 5. (Color online) (a) Thermal concurrence vs b and , where $B=5$. (b) Thermal concurrence vs B and , where $b=2$. Here $T=0.3$, $J=1$, $J_z=0.5$, and $\gamma=0.3$. All the parameters are dimensionless [see Eq. (10) and the definition of concurrence].

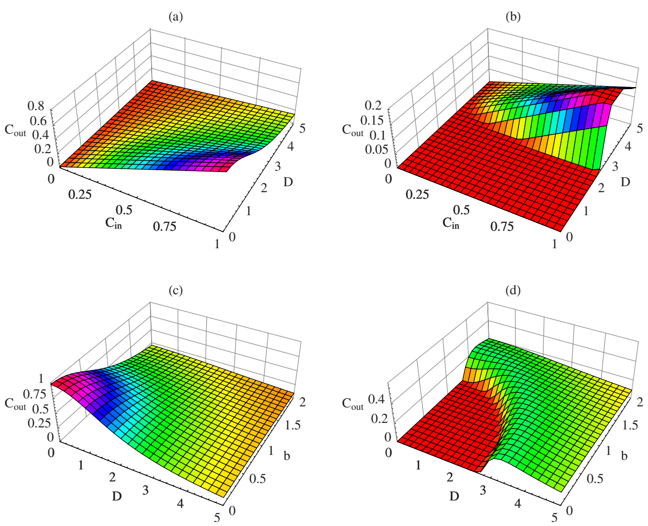

FIG. 6. (Color online) Entanglement of output state (C_{out}) vs the channel parameters and C_{in}. C_{out} vs D and C_{in} for (a) $b=0.5$ and $J_z>0$ and (b) $b=0.5$ and $J_z<0$. Also, C_{out} vs b and D for (c) $C_{\text{in}}=1$ and $J_z>0$ and (d) $C_{\text{in}}=1$ and $J_z<0$. $J=1$, $T=0.1$, $B=1$, and $\gamma=0.3$. All the parameters are dimensionless [see Eq. (16) and the definition of concurrence].

Notice that T_{c1} and T_{c2} are sensitive functions of D. In Fig. 4, we try to demonstrate these facts, by illustrating the function of thermal concurrence vs T for a few values of D. This figure shows that, for $D<D_c$, T_{c1} is a decreasing function of D but, for $D>D_c$, T_{c1} is undefined. In contrast, T_{c2} is an increasing function of D for all values of T, i.e., we can create and maintain nonzero entanglement at larger temperatures. In summary, Fig. 3 shows that, for $T_{c1}<T<T_{c2}$ and $D<D_c$, there are regions in the D-T plane where increase of temperature first increases the entanglement (revival region) and then tends to decrease the entanglement; ultimately for $T>T_{c2}$ the entanglement vanishes. The maximum entanglement exists at zero temperature and for large D. In the revival region and the region for which $D>D_c$, increase of D causes the entanglement to increase. In this region, T_{c1} decreases and T_{c2} increases as D increases and hence the width of the revival region increases. In all regions T_{c2} is an increasing function of D; thus, when D is large enough, the entanglement can exist for larger temperatures. Furthermore, the parameter D plays the role of the parameter b. Figure 5(a) shows the variation of entanglement as a function of b (inhomogeneity of the magnetic field) and D. For fixed D, there are three regions in this figure: (i) The main region where $b<b_c$; in this region the entanglement is constant; (ii) the collapse region where $b=b_c$; in this region the entanglement decreases suddenly; (iii) the revival region where $b>b_c$; in this region the entanglement undergoes a revival.

The size of D determines b_c and hence the edge of the revival region. Furthermore, in the revival region, increasing D increases the entanglement. In Fig. 5(b), the thermal entanglement is plotted vs magnetic field (B) and D. The role of D is similar to its role in Fig. 5(a), i.e., enhancement of D improves B_c and increases the amount of entanglement in the revival region.

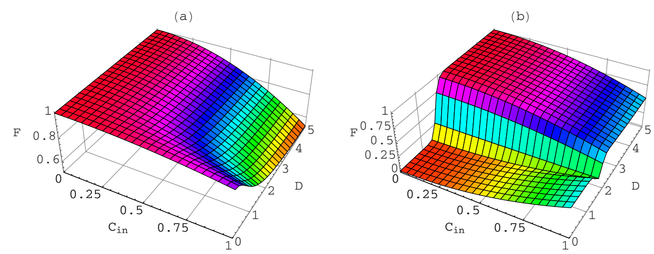

FIG. 7. (Color online) $F(\rho_{\text{in}},\rho_{\text{out}})$ vs C_{in} and D for (a) $J_z>0$ and (b) $J_z<0$, where $J=1$, $T=0.1$, $B=1$, $b=0.5$, and $\gamma=0.3$. All the parameters are dimensionless [see Eq. (19)].

One can use the above entangled two-qubit system to perform teleportation protocols. The next section discusses this subject.

III. THERMAL ENTANGLEMENT TELEPORTATION

For the entanglement teleportation of a whole two-qubit system, a thermal mixed state of the Heisenberg spin chain can be considered as a general depolarizing channel. In the following, we consider Lee and Kim's two-qubit teleportation protocol, and use two copies of the above two-qubit thermal state, $\rho_T\otimes\rho_T$, as resource [30]. Similarly to standard teleportation, entanglement teleportation for the mixed channel of an input entangled state is destroyed and its replica state appears at the remote place after applying a local measurement in the form of linear operators. We consider as input a two-qubit state in the special pure state $|\psi_{\text{in}}\rangle=\cos\,\theta/2|10\rangle+e^{i\phi}\sin\,\theta/2|01\rangle\quad(0\leq\theta\leq\pi,0\leq\phi\leq2\pi)$. The density matrix related to $|\psi_{\text{in}}\rangle$ is in the form

$$\rho_{in}=\begin{pmatrix}0&0&0&0\\0&a&c&0\\0&c^*&b&0\\0&0&0&0\end{pmatrix},\tag{12}$$

where $a=\sin^2\,\theta/2$, $b=\cos^2\,\theta/2$, and $c=\frac{1}{2}e^{-i\phi}\sin\,\theta$. Therefore the concurrence of the initial state is $C_{\text{in}}=2|e^{i\phi}\sin\,\theta/2\,\cos\,\theta/2|=\sin\,\theta$. The output (replica) state ρ_{out} can be obtained by applying a joint measurement and local unitary transformation on the input state ρ_{in}. Thus the output state is given by [29]

$$\rho_{\text{out}}=\sum_{\mu,\nu}p_{\mu\nu}(\sigma_\mu\otimes\sigma_\nu)\rho_{\text{in}}(\sigma_\mu\otimes\sigma_\nu),\tag{13}$$

where $\mu,\nu=0,x,y,z$ $(\sigma^0=I)$, $p_{\mu\nu}=\text{tr}(E^\mu\rho_{\text{ch}})\text{tr}(E^\nu\rho_{\text{ch}})$ such that $\sum_{\nu,\mu}p_{\mu\nu}=1$, and ρ_{ch} represents the state of the channel that is used for the teleportation. Here $E^0=|\Psi^-\rangle\langle\Psi^-|$, $E^1=|\Phi^-\rangle\langle\Phi^-|$, $E^0=|\Phi^+\rangle\langle\Phi^+|$, and $E^0=|\Psi^+\rangle\langle\Psi^+|$, where $|\Psi^\pm\rangle=\frac{(|01\rangle\pm|10\rangle)}{\sqrt{2}}$ and $|\Phi^\pm\rangle=\frac{(|00\rangle\pm|11\rangle)}{\sqrt{2}}$ are the Bell states.

If the two-qubit spin system is considered as a quantum channel, the state of the channel is $\rho_{\text{ch}}=\rho_T$, given in Eq. (8), and hence one can obtain ρ_{out} as

$$\rho_{\text{out}} = \begin{pmatrix} \alpha & 0 & 0 & \kappa \\ 0 & a' & c' & 0 \\ 0 & c'^* & b' & 0 \\ \kappa & 0 & 0 & \alpha \end{pmatrix}, \tag{14}$$

where

$$\alpha = (w_1 + w_2)(\mu^+ + \mu^-),$$

$$\kappa = 4\,\mathrm{Re}(z)\nu \cos\phi \sin\theta,$$

$$a' = (\mu^+ + \mu^-)^2 \cos^2\frac{\theta}{2} + (w_1 + w_2)^2 \sin^2\frac{\theta}{2},$$

$$b' = (w_1 + w_2)^2 \cos^2\tfrac{\theta}{2} + (\mu^+ + \mu^-)^2 \sin^2\tfrac{\theta}{2},$$

$$c' = 2e^{-i\phi}\{[\mathrm{Re}(z)]^2 + e^{2i\phi}\nu^2\}\sin\theta. \tag{15}$$

Now we can determine the concurrence of the output state by calculating the positive square roots of $R_{\text{out}}=\rho_{\text{out}}\tilde{\rho}_{\text{out}}$, i.e., the λ_i''s. It is easy to show that

$$\lambda'_{1,2} = \left| \frac{\sqrt{C_{\text{in}}^2(\cosh^2\beta\eta - e^{2\beta J_z}\cosh^2\beta\xi)^2 + 4e^{2\beta J_z}\cosh^2\beta\eta\cosh^2\beta\xi}}{2(\cosh\beta\eta + e^{\beta J_z}\cosh\beta\xi)^2} \pm \frac{\left|C_{\text{in}}\left[\left(\frac{J\gamma}{\eta}\right)^2 e^{2i\phi}\sinh^2\beta\eta + \left(\frac{J}{\xi}\right)^2 e^{2\beta J_z}\sinh^2\beta\xi\right]\right|}{2(\cosh\beta\eta + e^{\beta J_z}\cosh\beta\xi)^2} \right|,$$

$$\lambda'_{3,4} = \left| \frac{\cosh\beta\eta\cosh\beta\xi \pm C_{in}\left(\frac{J\gamma}{\eta}\right)\left(\frac{J}{\xi}\right)\sinh\beta\eta\sinh\beta\xi\cos\phi}{(\cosh\beta\eta + e^{\beta J_z}\cosh\beta\xi)^2} \right| e^{\beta J_z}. \tag{16}$$

Thus $C_{\text{out}}=C(\rho_{\text{out}})=\max\{0,2\lambda_{\max}-\Sigma_{i=1}^4\lambda_i'\}$ is computable when the parameters of the channel are known. The function C_{out} is dependent on the entanglement of the initial state and the parameters of the channel [which determine the entanglement of the channel (C_{ch})]. The C_{out} is nonzero only for a particular choice of the channel parameters for which C_{ch} is greater than a critical value. Figures 6 and 8 depict the behavior of C_{out} versus the parameters of the channel and C_{in}. For the case that $J_z>0$, the entanglement of the replica state (C_{out}) increases linearly as C_{in} increases. The rate of this enhancement is determined by D [indeed, by $C_{\text{ch}}(D)$]. But in the case of $J_z<0$ C_{out} is zero for small values of D. As D crosses a threshold value, C_{out} increases when C_{in} increases with a rate determined by the size of D [equivalently $C_{\text{ch}}(D)$]. Figures 6(b) and 6(c) show that the inhomogeneity parameter b can play the role of D.

The fidelity of entanglement teleportation

The fidelity between ρ_{in} and ρ_{out} characterizes the quality of the teleported state ρ_{out}. When the input state is a pure state, we can apply the concept of fidelity as a useful indicator of teleportation performance of a quantum channel [31,37]. The maximum fidelity of ρ_{in} and ρ_{out} is defined to be

$$F(\rho_{\text{in}},\rho_{\text{out}}) = \{\mathrm{Tr}[\sqrt{(\rho_{\text{in}})^{1/2}\rho_{\text{out}}(\rho_{\text{in}})^{1/2}}]\}^2 = \langle\psi_{\text{in}}|\rho_{\text{out}}|\psi_{\text{in}}\rangle. \tag{17}$$

By substituting ρ_{in} and ρ_{out} from above, we have

$$F(\rho_{\text{in}},\rho_{\text{out}}) = a'\sin^2\frac{\theta}{2} + b'\cos^2\frac{\theta}{2} + \mathrm{Re}(c'e^{-i\phi})\sin\theta. \tag{18}$$

Simplifying the above formula, we find that the maximum fidelity $F(\rho_{\text{in}},\rho_{\text{out}})$ depends on the initial entanglement (C_{in}):

$$F(\rho_{\text{in}},\rho_{\text{out}}) = f^c + f^q C_{\text{in}}^2, \tag{19}$$

where $f^c=(w_1+w_2)^2$ and $f^q=\frac{1}{2}-(w_1+w_2)+2\{\nu^2\cos 2\phi+[\mathrm{Re}(z)]^2\}$. The functions f^c and f^q depend on the channel parameters only (we consider $\phi=0$) and hence they are related to the entanglement of the channel. This formula is the same as the results of Ref. [30],[3] but in spite of the Werner states, f^q can be a positive number for Heisenberg chains. This means that there exists a channel such that it can teleport a more entangled initial state with more fidelity, but this is not a useful claim because, when we choose the parameters of the channel such that $f^q>0$, then f^c decreases and ultimately $F(\rho_{\text{in}},\rho_{\text{out}})$ becomes smaller than $\frac{2}{3}$, which means that the entanglement teleportation of the mixed state is inferior to classical communication. Thus, to obtain the same proper fidelity, a larger entangled channel is required for a larger entangled initial state. Figure 7 emphasizes the above points.

The average fidelity F_A is another useful concept for characterizing the quality of teleportation. The average fidelity F_A of teleportation can be obtained by averaging $F(\rho_{\text{in}},\rho_{\text{out}})$ over all possible initial states,

$$F_A = \frac{\int_0^{2\pi} d\phi \int_0^{\pi} F(\rho_{\text{out}},\rho_{\text{in}})\sin\theta\, d\theta}{4\pi}, \tag{20}$$

and for our model F_A can be written as

[3] In Ref. [30], Lee and Kim use the negativity as a measure of entanglement and the Werner states as resource. In 2×2 systems, the negativity coincides with the concurrence for pure states and Werner states [7].

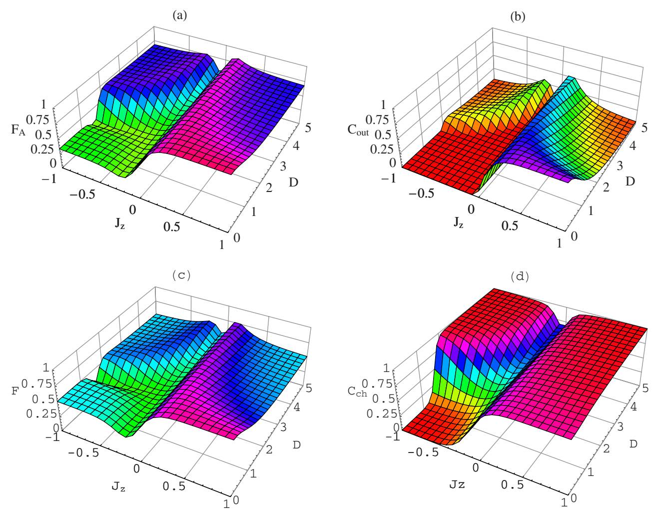

FIG. 8. (Color online) (a) Average fidelity [see Eq. (21)]. b) Output entanglement [see Eq. (16) and the definition of concurrence]. c) Fidelity [see Eq. (19)] between ρ_{in} [see Eq. (12)] and ρ_{out} [see Eqs. (14) and (15). (d) The entanglement of the channel [see Eq. (10) and the definition of concurrence] versus J_z and D. $J=1$, $T=0.1$, $B=1$, $b=0.5$, $\gamma=0.3$, and $C_{\text{in}}=1$. All the parameters are dimensionless.

$$F_A = \frac{\xi^2 \cosh^2 \beta\eta + e^{2\beta J_z}(2\xi^2 \cosh^2 \beta\xi + J^2 \sinh^2 \beta\xi)}{3\xi^2(\cosh \beta\eta + e^{\beta J_z} \cosh \beta\xi)^2}. \tag{21}$$

In the case of an isotropic *XXX* Heisenberg chain in the absence of magnetic field with spin-orbit interaction, this equation gives the same results as those of Ref. [33]. The function F_A is dependent on the channel parameters. Figure 8 gives a plot of F_A, C_{out}, $F(\rho_{\text{in}},\rho_{\text{out}})$, and C_{ch} versus the channel parameters. This figure shows that, in the case of $J_z>0$, F_A, $F(\rho_{\text{in}},\rho_{\text{out}})$, and C_{out} decrease when D increases. In this case F_A approaches $\frac{2}{3}$ from above for large values of D. In contrast, in the case of $J_z<0$ and for small values of D, F_A and $F(\rho_{\text{in}},\rho_{\text{out}})$ have a constant value (smaller than $\frac{2}{3}$) and C_{out} is zero. As D becomes larger than a threshold value, undergoes a revival and then decreases for larger values of D. Since in the revival region C_{ch} has its maximum value, F_A, $F(\rho_{\text{in}},\rho_{\text{out}})$ and C_{out} increase in this region such that, for a particular interval of D, F_A becomes larger than $\frac{2}{3}$ and ultimately tends to $\frac{2}{3}$ for larger D. In summary, the fidelity of teleportation and entanglement of the replica state are dependent on the entanglement of the channel, which is tunable by the channel parameters (such as $D,b,B,\ldots$). The effect of D is more desirable in the case of $J_z<0$. In this case, for certain values of , F_A becomes greater than $\frac{2}{3}$; this make the channel useful for performance in a teleportation protocol. For large D, F_A tends to $\frac{2}{3}$ in both cases $J_z>0$ and $J_z<0$.

IV. DISCUSSION

The entanglement of a two-qubit *XYZ* Heisenberg system in the presence of an inhomogeneous magnetic field and spin-orbit interaction is investigated. We have shown that, by turning on the spin-orbit interaction, we can change the behavior of the system without manipulating the other parameters. Therefore, the critical values of B, b, and T are adjustable using D, and by introducing D we can improve the critical values of B, b, and T. At zero temperature, the ground state entanglement undergoes a large revival when D crosses its critical value D_c and tends to its maximum for larger D. At finite temperatures, an increase in D causes volume enhancement of the revival region and also enhancement of the thermal entanglement in this region. This means that by increasing D we can obtain larger entanglement at larger temperatures. Also, entanglement teleportation via two copies of the above two-qubit system is studied. We show that, by introducing the SO interaction, the entanglement of the replica state and the fidelity of teleportation can be increased for the case of $J_z<0$, but are decreased for the case $J_z>0$. For both $J_z>0$ and $J_z<0$, when D becomes very large, the fidelity approaches $\frac{2}{3}$ from above; this is the maximum fidelity for the classical communication of a quantum state. Thus, for very large D, our quantum channel is still superior to classical channels in performing the entanglement teleportation protocol. We also argue that a minimal entanglement of the channel is required to realize efficient entanglement teleportation. In the absence of SO interaction the channel with $J_z<0$ has zero entanglement and hence is not suitable for performance of the entanglement teleportation protocol. We show that we can attain this minimal channel entanglement for the case $J_z<0$ by introducing the SO interaction.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] E. Schrödinger, Naturwiss. **23**, 807 (1935).
[3] M. A. Neilsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, U.K., 2004).
[4] J. Audretsch, *Entangled Systems*, (Wiley-VCH, Weinheim, 2007).
[5] N. Canosa and R. Rossignoli, Phys. Rev. A **69**, 052306 (2004).
[6] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997); W. K. Wootters, *ibid.* **80**, 2245 (1998).
[7] J. Eisert, Ph.D thesis, The University of Postdam, 2001 (unpublished).
[8] D. Loss and D. P. DiVincenzo, Phys. Rev. A **57**, 120 (1998).
[9] D. P. DiVincenzo, Phys. Rev. A **51**, 1015 (1995).
[10] W. A. Coish and D. Loss, e-print arXiv:cond-mat/0606550; G. Burkard, D. Loss, and D. P. DiVincenzo, Phys. Rev. B **59**, 2070 (1999).
[11] N. Zhao, L. Zhong, J. L. Zhu, and C. P. Sun, Phys. Rev. B **74**, 075307 (2006).
[12] M. A. Nielsen, Ph.D. dissertation, University of New Mexico, 1998, e-print arXiv:quant-ph/0011036.

[13] D. Gunlycke, V. M. Kendon, V. Vedral, and S. Bose, Phys. Rev. A **64**, 042302 (2001).
[14] M. C. Arnesen, S. Bose, V. Vedral, and S. Bose, Phys. Rev. Lett. **87**, 017901 (2001).
[15] Y. Sun, Y. Chen, and H. Chen, Phys. Rev. A **68**, 044301 (2003).
[16] G. F. Zhang and S. S. Li, Phys. Rev. A **72**, 034302 (2005).
[17] M. Asoudeh and V. Karimipour, Phys. Rev. A **71**, 022308 (2005).
[18] G. L. Kamta and A. F. Starace, Phys. Rev. Lett. **88**, 107901 (2002).
[19] L. Zhou, H. S. Song, Y. Q. Guo, and C. Li, Phys. Rev. A **68**, 024301 (2003).
[20] G. Rigolin, Int. J. Quantum Inf. **2**, 393 (2004).
[21] G. H. Yang, W. B. Gao, L. Zhou, and H. S. Song, e-print arXiv:quant-ph/0602051.
[22] I. Dzyaloshinsky, J. Phys. Chem. Solids **4**, 241 (1958).
[23] T. Moriya, Phys. Rev. **117**, 635 (1960); T. Moriya, Phys. Rev. Lett. **4**, 228 (1960); T. Moriya, Phys. Rev. **120**, 91 (1960).
[24] N. E. Bonesteel, D. Stepanenko, and D. P. DiVincenzo, Phys. Rev. Lett. **87**, 207901 (2001); G. Burkard and D. Loss, *ibid.* **88**, 047903 (2002); L. A. Wu and D. A. Lidar, Phys. Rev. A **66**, 062314 (2002).
[25] X. Wang, Phys. Lett. A **281**, 101 (2001).
[26] C. H. Bennett, G. Brassard, C. Crepeau, R. Josza, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[27] S. Popescu, Phys. Rev. Lett. **72**, 797 (1994).
[28] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **60**, 1888 (1999).
[29] G. Bowen and S. Bose, Phys. Rev. Lett. **87**, 267901 (2001).
[30] J. Lee and M. S. Kim, Phys. Rev. Lett. **84**, 4236 (2000).
[31] Y. Yeo, Phys. Rev. A **66**, 062312 (2002); Y. Yeo, e-print arXiv:quant-ph/023014.
[32] Y. Yeo, T. Liu, Y. Lu, and Q. Yang, J. Phys. A **38**, 3235 (2005).
[33] G. F. Zhang, Phys. Rev. A **75**, 034304 (2007).
[34] A. Pfund, I. Shorubalko, K. Ensslin, and R. Leturcq, Phys. Rev. B **76**, 161308(R) (2007).
[35] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, U.K., 1999).
[36] S. Olav Skrøvseth, e-print arXiv:quant-ph/0612133.
[37] R. Jozsa, J. Mod. Opt. **41**, 2315 (1994).