Environment-induced two-mode entanglement in quantum Brownian motion

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(Received 4 October 2007; published 4 April 2008)

The time evolution of quantum correlations of entangled two-mode continuous variable states is examined in single-reservoir as well as two-reservoir models, representing noisy correlated or uncorrelated non-Markovian quantum channels. For this purpose the model of quantum Brownian motion is extended. Various separability criteria for Gaussian continuous variable systems are applied. In both types of reservoir models moderate non-Markovian effects prolong the separability time scales. However, in these models the properties of the stationary state may differ. In the two-reservoir model the initial entanglement is completely lost and both modes are finally uncorrelated. In a common reservoir both modes interact indirectly via the coupling to the same bath variables. Below a critical bath temperature entanglement between the two modes is preserved even in the steady state. A separability criterion is derived, which depends on the bath temperature and the response function of the open quantum system. Thus, the extended quantum Brownian motion model of a two-mode continuous variable system in a common reservoir provides an example of environment-induced entanglement.

DOI: 10.1103/PhysRevA.77.042305 PACS number(s): 03.67.Mn, 03.65.Ud

I. INTRODUCTION

During the last decade quantum information and computation has been extended from discrete systems to quantum systems with continuous variables such as position and momentum or the amplitudes of electromagnetic field modes. This quantum information theory of continuous variable systems has received much attention in the past few years [1–3] and has found various applications in quantum cryptography and quantum teleportation [4,5]. In this context, Gaussian states play a prominent role since they can be easily created and controlled experimentally and are less affected by decoherence. Great advances have been made in characterizing the entanglement properties of two-mode Gaussian states by determining the necessary and sufficient criteria for their separability [6,7] and by developing quantitative entanglement measures [8,9]. The prototype of these states are the two-mode squeezed states, which have been successfully produced via nonlinear parametric down conversation and applied to quantum teleportation [10,11].

Due to the unavoidable interaction with the environment, any pure quantum state used in some quantum information process evolves into a mixed state. Thus, a realistic analysis of continuous variable quantum channels must take decoherence and dissipation into account. Within the theory of open quantum systems [12,13] the dissipative dynamics are mainly described by master equations of the reduced density matrix. Initial quantum superpositions are destroyed and quantum correlations are lost during characteristic decoherence and separability time scales. The Markovian time evolution of quantum correlations of entangled two-mode continuous variable states has been examined in single-reservoir [14,15] and two-reservoir models [7,16–18], representing noisy correlated or uncorrelated Markovian quantum channels. Quantum correlations are found to be better preserved in a common reservoir. Additionally the coupling to the same bath variables might generate new quantum correlations between the parts of the subsystem. This effect of environment-induced entanglement has already been studied for discrete systems [15,19,20] including revivals of entanglement [21,22] and even asymptotically entangled states [23–26]. The underlying Born-Markov approximation assumes weak coupling between the system and the environment to justify a perturbative treatment and neglects short-time correlations between the system and the reservoir. This approach has been widely and successfully employed in the field of quantum optics [27] where the characteristic time scales of the environmental correlations is much shorter compared to the internal system dynamics. Challenged by new experimental evidence a growing interest in non-Markovian descriptions can be observed. Very recently some phenomenological [28,29] and microscopic models [30–33] of non-Markovian quantum channels have been proposed. Using the analogy between the Hilbert space of quantized electromagnetic fields and the Hilbert space of quantum harmonic oscillators, the Caldeira-Leggett model of quantum Brownian motion [34–36] can be extended to describe the entanglement dynamics of two-mode squeezed states.

In this paper, the time evolution of quantum correlations of initially entangled two-mode continuous variable states is numerically examined in a common reservoir model. The focus is on non-Markovian influences and strong coupling effects. The non-Markovian dynamics are described by an extended two-mode version of Hu-Paz-Zhang master equation of quantum Brownian motion. In the single or common reservoir model both oscillators (modes) are coupled to the same reservoir variables, whereas in the two-reservoir model each oscillator (mode) is interacting with its own independent reservoir. In both types of reservoir models moderate non-Markovian effects prolong the separability time scales which also depend on the interaction strength between the system and the environment. However, in these models the properties of the stationary state may differ. In the two-reservoir model the initial entanglement is completely lost and both modes are finally uncorrelated. In a common reservoir both modes interact indirectly via the coupling to the

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same bath variables. Therefore, new quantum correlations may emerge between the two modes. Below a critical bath temperature entanglement is preserved even in the steady state. A separability criterion is derived, which depends on the bath temperature and the response function of the open quantum system. Thus, the extended quantum Brownian motion model of a two-mode continuous variable system in a common reservoir provides an example of environment-induced quantum two-mode entanglement.

The paper is organized as follows. In Sec. II, we briefly review various separability criteria for two-mode Gaussian states and the Markovian separability times for two-mode squeezed states in single- and two-reservoir models. In Sec. III we shortly describe the Hu-Paz-Zhang master equation of quantum Brownian motion [36] which is the basis for studying non-Markovian effects. We resume the extended, two-mode-version of the Caldeira-Leggett model for single and two-reservoir models and introduce a modified common-reservoir model. In Sec. IV we present and discuss the numerical results of the entanglement dynamics of two-mode squeezed states in the modified common-reservoir model. Different scenarios are analyzed, including the case of noise-induced steady state entanglement. A simple separability criteria for the stationary two-mode state is derived. Finally, a brief summary is given in Sec. V.

II. CONTINUOUS VARIABLE SYSTEMS

A. Separability criteria for two-mode Gaussian states

In the following we review the separability criteria for a special class of continuous variable systems—the two-mode Gaussian states. A Gaussian two-mode state with coordinates q_1, q_2 and momenta p_1, p_2 has a Gaussian Wigner function in semiclassical phase space

$$W(\mathbf{X}) = \frac{1}{4\pi^2\sqrt{\det \mathbf{V}}}\exp\left[-\frac{1}{2}\mathbf{X}\cdot\mathbf{V}^{-1}\cdot\mathbf{X}^{\mathrm{T}}\right] \quad (1)$$

with $\mathbf{X}=(q_1,p_1,q_2,p_2)$ and commutation relation $[q_i,p_j]=i\hbar\delta_{ij}$ for $i,j=1,2$. It is completely characterized by its first and second moments.

The 4×4-covariance matrix

$$\mathbf{V} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad V_{ij} = \frac{1}{2}\langle X_iX_j + X_jX_i\rangle \quad (2)$$

(where all first moments $\langle X_j\rangle$ have been set to zero) contains four local symplectic invariants in form of the determinants of the block-matrices **A**, **B**, **C**, and **V**. A Gaussian continuous variable state is separable if and only if the partial transpose $\tilde{\rho}$ of its density matrix ρ is non-negative [positive partial transpose (PPT) criterion]. Based on the above invariants Simon [6] has derived a PPT criterion for bipartite Gaussian continuous variable states

$$S(t) = \det \mathbf{V} - \frac{1}{4}(\det \mathbf{A} + \det \mathbf{B} + 2|\det \mathbf{C}|) + \frac{1}{16} \geq 0 \quad (3)$$

which is also a necessary separability criterion for non-Gaussian states. The PPT criterion has a geometrical interpretation as mirror reflection of the Wigner function in phase space. In case of a Gaussian two-mode state the partial transpose coincides with a change of the signs in those elements of the covariance matrix, which connect the momentum of the first mode to the coordinate of the second mode. Equivalent to that is the criterion $\tilde{\nu}_- \geq \frac{1}{2}$ for separability where $\tilde{\nu}_\pm$ are the symplectic eigenvalues of the partial transposed density matrix $\tilde{\rho}$:

$$\tilde{\nu}_\pm = \frac{1}{\sqrt{2}}[\tilde{\Delta}_V \pm \sqrt{\tilde{\Delta}_V^2 - 4\det V}]^{1/2} \quad (4)$$

with $\tilde{\Delta}_V=\det A+\det B-2\det C$. With the symplectic eigenvalue $\tilde{\nu}_-$ the logarithmic negativity as a quantitative measure of entanglement can be defined by [8,37]

$$E_{\mathcal{N}}(\rho) = \max\{0, -\ln 2\tilde{\nu}_-\}. \quad (5)$$

In order to apply the PPT criterion the complete knowledge of all second moments is required. For practical purposes there are also weaker separability criteria based on linear or quadratic combinations of only a few elements of the covariance matrix. Duan *et al.* [7] derived such a criterion. Starting point is the definition of a pair of EPR-like operators $\hat{u}=|a|\hat{q}_1+\frac{1}{a}\hat{q}_2$ and $\hat{v}=|a|\hat{p}_1-\frac{1}{a}\hat{p}_2$ with $a\in\mathbb{R}\setminus\{0\}$ fulfilling the commutation relations $[\hat{q}_k,\hat{p}_j]=i\delta_{kj}$, $j,k=1,2$. Then, for every separable bipartite quantum state ρ, the sum of the variances fulfills the relation

$$\langle(\Delta\hat{u})^2\rangle_\rho + \langle(\Delta\hat{v})^2\rangle_\rho \geq a^2 + \frac{1}{a^2} \quad (6)$$

For Gaussian states this set of inequalities (for all possible values of a) completely characterizes the set of separable states and is equivalent to Eq. (3). The product version of this inequality for $a=1$ is given by [38–40]

$$\langle(\Delta\hat{u})^2\rangle_\rho\langle(\Delta\hat{v})^2\rangle_\rho \geq 1, \quad (7)$$

which is a special case of the generalized curved quadratic entanglement witnesses [41]. In general, the product witnesses are stronger tests than the respective linear tests. Thus, the set of entangled covariance matrices detected by this quadratic test is strictly larger than that detected by linear combinations of second moments [41].

We also will apply a classification scheme of quantum states based on marginal and global purities of a bipartite Gaussian quantum system whereas $\mu=\mathrm{Tr}[\rho^2]$ is the purity of the total system and $\mu_1=\mathrm{Tr}[\rho_1^2]$, $\mu_2=\mathrm{Tr}[\rho_2^2]$ are the purities of the reduced density matrices $\rho_1=\mathrm{Tr}_2[\rho]$, $\rho_2=\mathrm{Tr}_1[\rho]$ of the two-mode system. According to Refs. [42,43], a two-mode Gaussian mixed state is separable if

$$\mu_1\mu_2 \leq \mu \leq \frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2} \quad (8)$$

and it is entangled for the case that

$$\frac{\mu_1\mu_2}{\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}} < \mu \leq \frac{\mu_1\mu_2}{\mu_1\mu_2-|\mu_1-\mu_2|}. \quad (9)$$

In between there is a coexistence region

$$\frac{\mu_1\mu_2}{\mu_1+\mu_2-\mu_1\mu_2} < \mu \le \frac{\mu_1\mu_2}{\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}}, \tag{10}$$

where both separable and entangled states can be found. The calculation of the global and marginal purities thus provides analytical bounds on the entanglement of Gaussian states.

B. Markovian separability times for two-mode squeezed states

The separability criterion (3) is simplified if the covariance matrix V is transformed to standard form by symplectic transformations

$$V_{\mathrm{st}} = \begin{pmatrix} a & 0 & c_+ & 0 \\ 0 & a & 0 & c_- \\ c_+ & 0 & b & 0 \\ 0 & c_- & 0 & b \end{pmatrix}, \tag{11}$$

where the elements a, b, c_+ and c_- of V_{st} are determined by the four symplectic invariants $\det \mathbf{A}=a$ $\det \mathbf{B}=b$, $\det \mathbf{V}=(ab-c_+^2)(ab-c_-^2)$, and $\det \mathbf{C}=c_+c_-$. In a Markovian two-reservoir model the dynamics of the two-mode state are described by the quantum optical master equation of the damped quantum oscillator (interaction picture)

$$\dot{\rho} = \sum_{j=1}^{2} \frac{\gamma}{2}\{\bar{n}L[a_j^\dagger]\rho + (\bar{n}+1)L[a_j]\rho\}, \tag{12}$$

with mean bosonic occupation number $\bar{n}=(e^{\beta\hbar\omega_0}-1)^{-1}$, damping constant γ, and $L[o]\rho=2o\rho o^\dagger - o^\dagger o\rho - \rho o^\dagger o$. The time evolution of the matrix elements is then given by $a(t)=ae^{-\gamma t}+\Delta(t)$, $b(t)=be^{-\gamma t}+\Delta(t)$, and $c_\pm(t)=\pm c_\pm e^{-\gamma t}$ with $\Delta(t)=(2\bar{n}+1)(1-e^{-\gamma t})$ [17]. In the case of a two-mode squeezed vacuum state

$$|\psi_\xi\rangle = e^{\xi^*\hat{a}_1\hat{a}_2-\xi\hat{a}_1^\dagger\hat{a}_2^\dagger}|0\rangle_1|0\rangle_2 \tag{13}$$

with squeezing parameter ξ, the initial covariance matrix is already given in standard form with elements $a=b=\frac{1}{2}\cosh(2|\xi|)$ and $c_\pm=\pm\frac{1}{2}\sinh(2|\xi|)$. Applying the above separability criterion, a Markovian separability time can be derived from the separability function (3) which is reduced to $S(t)=e^{-2|\xi|}e^{-\gamma t}+\Delta(t)-1$. The time τ_1 after which the initial entanglement is lost is given by the condition $S(\tau_1)=0$ and reads

$$\tau_1 = \frac{1}{\gamma}\ln\left(1+\frac{1-e^{-2|\xi|}}{2\bar{n}}\right) \tag{14}$$

[7,16,17] for the two-reservoir model yielding $\tau_1\to\infty$ for $\bar{n}\to 0$. In the case of a single reservoir model where the corresponding master equation contains additional terms, the time evolution of the covariance matrix elements is slightly different and results in the separability time [14]

$$\tau_2 = \frac{1}{2\gamma}\ln\left(\frac{2\bar{n}+1-e^{-2|\xi|}}{2\bar{n}+1-e^{2|\xi|}}\right). \tag{15}$$

Thus a common reservoir extends the Markovian separability time. Furthermore the initial entanglement is partially preserved even for $\bar{n}>0$ if the initial squeezing exceeds a critical value $\xi_c=\frac{1}{2}\ln(2\bar{n}+1)$. These results change if the dynamics are described without the Born-Markov assumption.

III. NON-MARKOVIAN ENTANGLEMENT DYNAMICS

A. HPZ master equation of quantum Brownian motion

Non-Markovian effects are discussed here on the basis of the Caldeira-Leggett model of quantum Brownian motion [34,44,45] often referred to as independent-oscillator model [35,46]. It is a system plus reservoir model where the total Hamiltonian consists of three parts

$$H = H_s + H_b + H_{\mathrm{int}}, \tag{16}$$

with H_s as Hamiltonian of the subsystem which interacts via the Hamiltonian H_{int} with a bath that is described by a collection of a large number of harmonic oscillators $H_b=\Sigma_i\hbar\omega_i(b^\dagger b+1)$. In detail the Hamiltonian of the Caldeira-Leggett model is given by

$$H = \frac{p^2}{2m} + V(q) + \sum_{i=1}^{N}\left[\frac{p_i^2}{2m_i} + \frac{m_i\omega_i^2}{2}\left(x_i - \frac{c_i q}{m_i\omega_i^2}\right)^2\right], \tag{17}$$

where q and p are the Heisenberg-operators of coordinate and momenta of the Brownian oscillator moving in an harmonic potential $V(q)=\frac{1}{2}m\omega_0^2q^2$ and coupled to a bath of N-independent harmonic oscillators with variables x_i, p_i and frequencies ω_i. The bath is characterized by its spectral density

$$J(\omega) = \pi\sum_{i=1}^{N}\frac{c_i^2}{2m\omega_i}\delta(\omega-\omega_i) = \frac{\gamma\omega\Gamma^2}{\omega^2+\Gamma^2}. \tag{18}$$

The interaction is bilinear in the coordinates q and x_i of the subsystem and the bath, respectively, with coupling parameters c_i. The self-interaction term (proportional to q^2) in the Hamiltonian

$$H_{\mathrm{int}} = \sum_i\left[-c_ix_iq + \frac{c_i^2}{2m_i\omega_i^2}q^2\right] \tag{19}$$

renormalizes the oscillator potential to ensure that the observable frequency is close to bare oscillator frequency ω_0. From influence functional path integral techniques Hu, Paz, Zhang have derived the master equation [36]

$$\dot{\rho} = \frac{1}{i\hbar}[H_s,\rho] + \frac{m\delta\Omega^2(t)}{2i\hbar}[q^2,\rho] + \frac{\gamma_p(t)}{2i\hbar}[q,\{p,\rho\}] + \frac{D_{qp}(t)}{\hbar^2}[q,[p,\rho]] - \frac{D_p(t)}{\hbar^2}[q,[q,\rho]], \tag{20}$$

with $[\cdots]$ and $\{\cdots\}$ denoting commutator and anticommutator, respectively. This master equation is valid for arbitrary coupling and temperature. The non-Markovian character is contained in the time-dependent coefficients which read in expansion up to the second order in the system-bath coupling constant [36]

$$\gamma_p(t) = \frac{2}{\hbar m \omega_0} \int_0^t dt' L(t') \sin \omega_0 t', \tag{21}$$

$$\delta\Omega^2(t) = \frac{\gamma\Gamma}{m} - \frac{2}{\hbar m} \int_0^t dt' L(t') \cos \omega_0 t', \tag{22}$$

$$D_{qp}(t) = \frac{1}{m\omega_0} \int_0^t dt' K(t') \sin \omega_0 t', \tag{23}$$

$$D_p(t) = \int_0^t dt' K(t') \cos \omega_0 t', \tag{24}$$

where $L(t)=i\langle[\eta(t),\eta(0)]\rangle$ and $K(t)=\frac{1}{2}\langle\{\eta(t),\eta(0)\}\rangle$ are connected to the spectral density (18) by

$$L(t) = \frac{\hbar}{\pi} \int_0^\infty d\omega J(\omega) \sin \omega t, \tag{25}$$

$$K(t) = \frac{\hbar}{\pi} \int_0^\infty d\omega J(\omega) \coth\left(\frac{1}{2}\beta\hbar\omega\right) \cos \omega t. \tag{26}$$

$K(t)$ is the correlation function of the quantum noise term η resulting from averaging over the initial thermal bath distribution. The exact expressions of the HPZ coefficients are related to the Green's functions of the corresponding quantum Langevin equations [47,48]. The entanglement properties of the joint state of the oscillator and its environment have been studied in Ref. [49].

B. Two-reservoir model

The dynamics of two identical, not directly interacting modes (with coordinates and momenta q_j, p_j, $j=1,2$) in two uncorrelated reservoirs is modeled by the interaction Hamiltonian

$$H_{\text{int}} = -q_1 \sum_{i=1}^{\infty} c_i x_i^b - q_2 \sum_{i=1}^{\infty} c_i x_i^c \tag{27}$$

with $\langle x_i^b x_j^c + x_i^c x_j^b \rangle = 0 \ \forall i,j$. The master equation of the reduced density matrix is then given by the sum of the master equations of two single modes [30]:

$$\dot{\rho} = \sum_{j=1}^{2} \left\{ \left[\frac{p_j^2}{2i\hbar m} + \frac{m\gamma_q(t) q_j^2}{2i\hbar}, \rho \right] + \frac{\gamma_p(t)}{2i\hbar}[q_j,\{p_j,\rho\}] + \frac{D_{qp}(t)}{\hbar^2}[q_j,[p_j,\rho]] - \frac{D_p(t)}{\hbar^2}[q_j,[q_j,\rho]] \right\}. \tag{28}$$

The time-dependent coefficients are given by $\gamma_q(t)=\omega_0^2 + \delta\Omega^2(t) - \gamma\Gamma/m$ and Eqs. (21)–(24). The time evolution of a two-mode squeezed state in two uncorrelated non-Markovian channels has been studied recently in Ref. [30]. The authors derived a non-Markovian separability function which shows oscillations in case of an artificial out of resonance bath with $\Gamma \ll \omega_0$. In this two-reservoir model the initial entanglement is always completely lost and both modes are finally uncorrelated (even at zero temperature while $\tau_1 \to \infty$). This may not be the case in a common reservoir model as will be shown in the next sections.

C. Common-reservoir model

The dynamics of two identical, not directly interacting modes in a common reservoir is modeled by the interaction Hamiltonian

$$H_{\text{int}} = -q_1 \sum_{i=1}^{\infty} c_i q_i - q_2 \sum_{i=1}^{\infty} c_i q_i. \tag{29}$$

The corresponding master equation for a two-mode system with $H_s = H_{s1} + H_{s2}$ is then given by [50]

$$\dot{\rho} = \frac{1}{i\hbar}[H_s,\rho] + \frac{M\delta\tilde{\Omega}^2(t)}{i\hbar}[R^2,\rho] + \frac{\gamma_p(t)}{i\hbar}[R,\{P_R,\rho\}] + \frac{2D_{qp}(t)}{\hbar^2}[R,[P_R,\rho]] - \frac{2D_p(t)}{\hbar^2}[R,[R,\rho]] \tag{30}$$

with $\delta\tilde{\Omega}^2(t) = \delta\Omega^2(t) - \gamma\Gamma/m$. As one can see immediately, the master Eq. (30) contains only the normal coordinates $R = (q_1+q_2)/2$ and $P_R = p_1 + p_2$ but not the relative coordinate $x = q_1 - q_2$. This results from the specific coupling $\sim x_i(q_1 + q_2)$ and from the choice of two identical modes with $\omega_1 = \omega_2$. In this case, the motion of the relative coordinate is not accompanied by dissipation [51,52]. Additional assumptions are therefore necessary to describe the relaxation dynamics completely.

D. Modified common-reservoir model

The Hamiltonian (29) describes a bilinear coupling in the coordinates of the two modes and the coordinates of the bath variables. The subsystem is thus effectively coupled by the center-of-mass $R=\frac{1}{2}(q_1+q_2)$. The motion of the relative coordinate, however, is unitary. Therefore, in the case of an initial two-mode squeezed state only half of the entanglement is lost in the stationary state regardless of the coupling strength [33]. In order to be able to model a complete loss of the initial entanglement, an additional assumption about the dynamics of the relative coordinate is necessary if the two modes are identical. To be able to model the dissipative dynamics for arbitrary coupling we choose an interaction Hamiltonian similar to the original Caldeira-Leggett model substituting the variable q by R. A renormalization term proportional to R^2 ensures that the frequency shift is compensated in the stationary state. The modified interaction Hamiltonian thus reads

$$H_{\text{int}} = -\sum_{i=1}^{\infty} c_i x_i R + \sum_{i=1}^{\infty} \frac{c_i^2}{2m_i\omega_i^2} R^2 \tag{31}$$

to guarantee that the eigenvalue equation for the eigenfrequencies ν of the total system

$$[\nu^2 - \omega_1^2 + h(\nu)][\nu^2 - \omega_2^2 + h(\nu)] - h^2(\nu) = 0 \tag{32}$$

with $h(\nu) = \Sigma_i \frac{c_i^2 \nu^2}{4\omega_i^2(\omega_i^2 - \nu^2)}$ (where $m_i = 1 \ \forall i$), has just eigenvalues $\nu \geq 0$. The frequency shift in the stationary state is compensated and the observable and the bare oscillator frequency coincide. The Hamiltonian of the total system consists of the Hamiltonian of the two modes $H_s = H_{s1} + H_{s2}$, the Hamiltonian $H_b = \Sigma_i \hbar\omega_i (b^\dagger b + 1)$ of the bath modes as well as the interaction Hamiltonian (31). The system of coupled Heisenberg equations of motion can be solved equivalently to the case of a single Brownian oscillator and leads to two coupled quantum Langevin equations for the two modes $j=1,2$:

$$\ddot{q}_j = -\omega_j^2 q_j - \frac{\gamma(t)R(0)}{M} - \int_0^t dt' \frac{\gamma(t-t')\dot{R}(t')}{M} + \frac{\eta(t)}{M}, \tag{33}$$

with $\gamma(t) = \gamma\Gamma e^{-\Gamma t}$. If the two modes have different frequencies $\omega_1 \neq \omega_2$ the equations still remain coupled after transformation on coordinates x and R:

$$\ddot{x}(t) = -\Omega_+^2 x(t) - 2\Omega_-^2 R(t), \tag{34}$$

$$\ddot{R}(t) = -\Omega_+^2 R(t) - \frac{1}{2}\Omega_-^2 x(t) + \frac{1}{M}\Phi_R(t), \tag{35}$$

with $\Phi_R(t) = -\gamma(t)R(0) - \int_0^t dt' \gamma(t-t')\dot{R}(t') + \eta(t)$ and $\Omega_\pm^2 = (\omega_1^2 \pm \omega_2^2)/2$. The underlying process for both coordinates is dissipative. The stationary solution of this model has been given elsewhere [51]. In the case of two identical modes with $\omega_j = \omega_0$ for $j=1,2$ and total mass $M=2m$ the system of coupled Langevin Eqs. (33) reduces to a quantum Langevin equation of the center-of-mass motion

$$\ddot{R} = -\omega_0^2 R - \frac{\gamma(t)R(0)}{M} - \int_0^t dt' \frac{\gamma(t-t')\dot{R}(t')}{M} + \frac{\eta(t)}{M}, \tag{36}$$

with stationary correlations

$$\langle R^2 \rangle = \frac{\hbar}{\pi}\int_0^\infty d\omega \coth\left(\frac{1}{2}\beta\hbar\omega\right)\mathrm{Im}\{\tilde{\chi}(\omega)\}, \tag{37}$$

$$\langle P_R^2 \rangle = \frac{\hbar}{\pi}\int_0^\infty d\omega M^2\omega^2 \coth\left(\frac{1}{2}\beta\hbar\omega\right)\mathrm{Im}\{\tilde{\chi}(\omega)\}, \tag{38}$$

where $\tilde{\chi}(\omega) = [M\omega_0^2 - M\omega^2 - i\omega\tilde{\gamma}(\omega)]^{-1}$ with $\tilde{\gamma}(\omega) = \int_0^\infty dt \gamma(t) e^{i\omega t}$ is the susceptibility of the non-Markovian damped harmonic oscillator. The motion of the relative coordinate $x = (q_1 - q_2)$ is decoupled from the motion of R. In order to study the case of two identical modes with dissipation in the relative motion additional assumptions make sense. Otherwise the subspace of the relative motion is a decoherence free subspace. It seems plausible to assume a weakly dissipative dynamics of the relative coordinate given by the solution of the Born-Markovian master equation of the damped harmonic oscillator [12] in the form

$$\langle x^2(t) \rangle = \langle x^2(0) \rangle e^{-\gamma_p t} + \sigma_0^2(2\bar{n}+1)(1-e^{-\gamma_p t}), \tag{39}$$

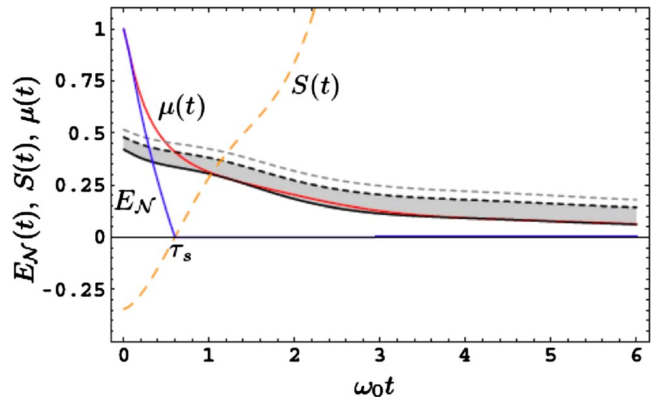

FIG. 1. (Color online) Time evolution of the logarithmic negativity $E_\mathcal{N}(t)$ (5) (blue line), separability function $S(t)$ (3) (yellow dashed line) and purity $\mu(t)$ (red line). The initial entanglement is definitely lost when $E_\mathcal{N}=0$ and $\mu(t)$ enters the separability region (8), respectively (gray shaded area). The area between the two dashed curves is the coexistence region (10). $E_\mathcal{N}=0$ for $t > \tau_s$ where $\tau_s < \tau_1, \tau_2$ (here $\tau_1 \approx \omega_0^{-1}$ and $\tau_2 \approx 2.2\omega_0^{-1}$) due to the effect of nonweak interaction strength $\gamma = 0.2\omega_0$. Parameters are $|\xi|=1$ and $\Gamma = 10\omega_0$, $T = 3.5\hbar\omega_0/k$.

$$\langle p_x^2(t) \rangle = \langle p_x^2(0) \rangle e^{-\gamma_p t} + \frac{\hbar^2}{4\sigma_0^2}(2\bar{n}+1)(1-e^{-\gamma_p t}), \tag{40}$$

where $\gamma_p = \lim_{t\to\infty}\gamma_p(t)$ is the stationary (Markovian) value of the coefficient (21) and $\sigma_0^2 = \frac{\hbar}{2m_x\omega_0}$ with reduced mass $m_x = M/4$. In the case of a squeezed two-mode state the initial values are given by $\langle x^2(0) \rangle = e^{-2|\xi|}$ and $\langle p_x^2(0) \rangle = \frac{1}{4}e^{2|\xi|}$. The dynamics of the normal coordinates are uncorrelated with $\langle \{x, R\}(t) \rangle = 0$. Of course, correlations in the original coordinates are still present. Instead of the assumptions (39) and (40) one could also suppose a quantum Langevin equation for the relative motion according to Eq. (36) with a small coupling constant $\gamma_x \ll \gamma$. However, this would lead to a more complex dynamical evolution without changing the main results for the stationary state. Therefore we restrict ourselves to the modified common-reservoir model with assumptions (39) and (40) for the relative motion.

IV. NUMERICAL ANALYSIS OF THE MODIFIED COMMON-RESERVOIR MODEL

In this section the non-Markovian entanglement dynamics of a squeezed two-mode state in the modified common reservoir model is numerically analyzed. The initial covariance matrix is given by Eq. (11) with elements $\langle q_j^2 \rangle = \langle p_j^2 \rangle = \frac{1}{2}\cosh(2|\xi|)$ for $j=1,2$ and $\langle \{q_1, q_2\} \rangle = -\langle \{p_1, p_2\} \rangle = \sinh(2|\xi|)$ (with $m=1$, $\omega_0=1$, and $\hbar=1$). The corresponding initial values of the variances of the normal coordinates therefore read $\langle R^2 \rangle = \frac{1}{4}e^{2|\xi|}$, $\langle P_R^2 \rangle = e^{-2|\xi|}$, $\langle x^2 \rangle = e^{-2|\xi|}$, and $\langle p_x^2 \rangle = \frac{1}{4}e^{2|\xi|}$ with minimal uncertainty $\langle R^2 \rangle \langle P_R^2 \rangle = \frac{1}{4}$ and $\langle x^2 \rangle \langle p_x^2 \rangle = \frac{1}{4}$.

Figures 1–4 give examples of the time evolution of a two-mode squeezed states in a common reservoir, in form of the logarithmic negativity $E_\mathcal{N}(t)$ (5) (blue line), separability function $S(t)$ (3) (yellow line) and purity $\mu(t)$ (red line).

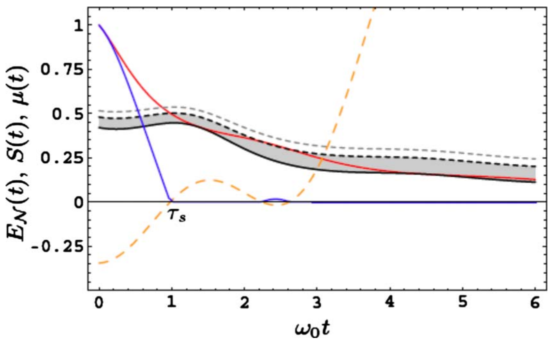

FIG. 2. (Color online) Non-Markovian effects (reducing cutoff frequency to $\Gamma \gtrsim \omega_0$) prolong the separability time (compared to $\Gamma \gg \omega_0$) and may lead to a Gaussian decay of the logarithmic negativity. Partial revivals of the logarithmic negativity can occur. Here $\Gamma = \omega_0$. Parameters ξ, γ, T and colors of the functions $E_{\mathcal{N}}(t)$, $S(t)$, $\mu(t)$ are chosen as in Fig. 1.

While for small coupling parameter γ and large cutoff frequency Γ the Markovian results would be reproduced, one can recognize deviations from the Markovian separability times τ_1 and τ_2 due to non-Markovian and strong coupling effects. The nonweak system-environment interaction accelerates decoherence with the consequence that the initial entanglement caused by the squeezing is lost faster (Fig. 1). In contrast to the Born-Markovian results in Ref. [14] the initial entanglement is also lost if the critical value ξ_c is exceeded. Specific non-Markovian effects (which are relevant if the bath correlation time scale Γ^{-1} is comparable to the separability time scale) are shown in Fig. 2: Non-Markovian influences prolong the separability time τ_s and may lead to a Gaussian decay of the logarithmic negativity instead of an exponential decay. Furthermore, Fig. 2 shows that partial revivals of quantum correlations can occur after an initial loss of entanglement. This can be seen from the oscillations of the logarithmic negativity between positive values (entanglement) and zero (separability). The same conclusions can be

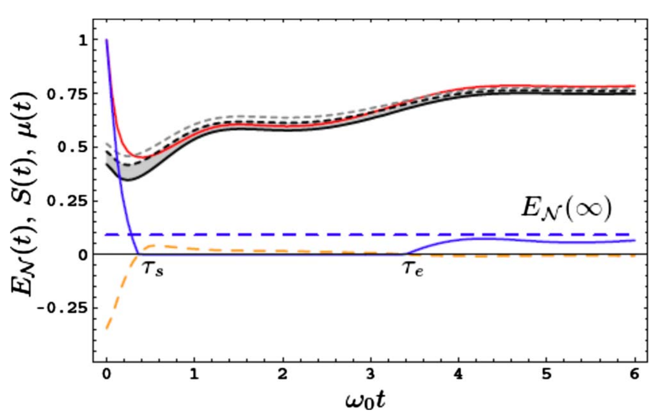

FIG. 3. (Color online) At low temperature and strong coupling, the two-mode state can get reentangled after a certain time τ_e with asymptotically constant quantum correlations. In this case the purity (red line) has left the gray shaded area (separability region) again and is above the gray dashed curve (entanglement region). Parameters here are $\gamma = 1.5\omega_0$, $\Gamma = 10\omega_0$, and $T = 10^{-3}\hbar\omega_0/k$ (with Markovian separability times $\tau_1, \tau_2 \to \infty$). Symbols and colors as in Fig. 1.

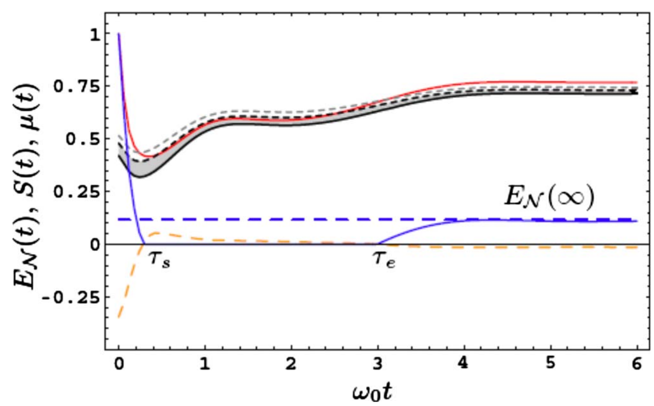

FIG. 4. (Color online) Increasing the coupling strength γ reduces the separability time τ_s as well as the reentangling time τ_e (compared to Fig. 3). Initial entanglement is destroyed faster while asymptotic quantum correlations are generated sooner. Parameters here $\gamma = 2\omega_0$, $\Gamma = 10\omega_0$, and $T = 10^{-3}\hbar\omega_0/k$. Symbols and colors as in Fig. 1.

drawn from the criterion for marginal and global purities. The two-mode state is separable if the total purity $\mu(t)$ is in the region determined by Eq. (9) (gray shaded area in Figs. 1–4).

The stationary state in Figs. 1 and 2 is separable in each case. However, it is possible that quantum correlations exist in the stationary state. This becomes obvious from Figs. 3 and 4, which show examples for low temperatures. Despite a loss of entanglement at short times $t < \tau_s$ quantum correlations between the two modes emerge again at later times and persist even in the steady state.

Thus, in this model there is not just a separability or disentangling time τ_s but also a reentangling time τ_e after which the two-mode state remains entangled asymptotically. However, it is difficult to find this characteristic time scale numerically due to the possible oscillations with partial revivals of the logarithmic negativity. Increasing the coupling strength γ reduces the separability time τ_s as well as the reentangling time τ_e as can be seen from a comparison between Figs. 3 and 4. The initial entanglement resulting from the squeezing is destroyed faster whereas environment-induced asymptotic quantum correlations are generated sooner. The reentanglement time scale is strongly affected by the system bath interaction strength γ but the initial squeezing with parameter ξ has little effect on the stationary quantum correlations, in contrast to the results in previous works [14,33]. Non-Markovian effects (induced by small cutoff frequency Γ) just extend the separability time scale τ_s but usually do not influence the reentangling time τ_e. Entanglement of the two modes can exist in the stationary state if the bath temperature is below a critical value, that depends on the coupling parameter γ and the cutoff frequency Γ.

In the following we derive a simple separability criterion for the stationary state. Since the normal coordinates $R = (q_1 + q_2)/2$ and $p_x = (p_1 - p_2)/2$ are EPR-like operators we can use product version (7) of the EPR-separability criterion in the form (for $\hbar = 1$)

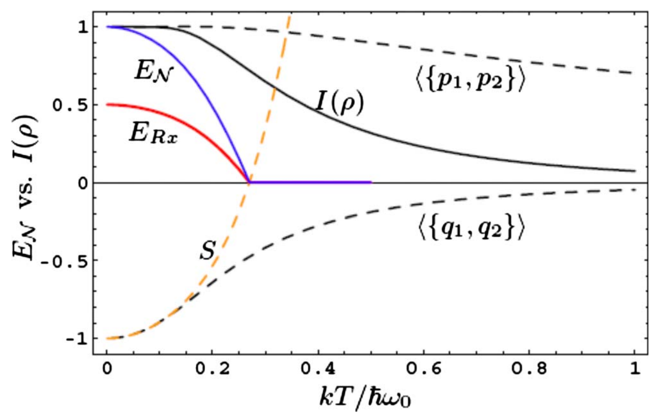

FIG. 5. (Color online) Separability criterion $S(\infty)$ (3), logarithmic negativity $E_{\mathcal{N}}$ (5) and $E_{Rx}=\max\{1-16\langle R^2\rangle\langle p_x^2\rangle,0\}$ compared to the mutual information $I(\rho)$ (43) and the stationary correlations $\langle\{p_1,p_2\}\rangle$, $\langle\{q_1,q_2\}\rangle$ (normalized by initial values) of two-mode state in dependence of the temperature of the common reservoir. ($\gamma=2\omega_0$, $\Gamma=10\omega_0$).

$$\langle R^2\rangle\langle p_x^2\rangle \geq \frac{1}{16} \Leftrightarrow \langle (q_1+q_2)^2\rangle\langle (p_1-p_2)^2\rangle \geq 1 \quad (41)$$

in order to derive a necessary condition for the existence of quantum correlations in the stationary state. Since in this case $\langle\{q_1,q_2\}\rangle<0$, $\langle\{p_1,p_2\}\rangle>0$ and the stationary covariance matrix of the modified QBM model is given in symmetric standard form, the product criterion (7) is equivalent to the PPT criterion [53] and thus detects the same set of entangled covariance matrices. Using the stationary correlations in Eq. (37) and from Eq. (39) this separability criterion can be rewritten in the form

$$\frac{\omega_0}{\pi}\int_0^\infty d\omega(2\bar{n}+1)\coth\left(\frac{1}{2}\beta\hbar\omega\right)\mathrm{Im}\{\tilde{\chi}(\omega)\} \geq \frac{1}{4}, \quad (42)$$

with mean occupation number $\bar{n}=(e^{\beta\hbar\omega_0}-1)^{-1}$ of the relative coordinate. Figure 5 compares this criterion in form of the function $E_{Rx}=\max\{1-16\langle R^2\rangle\langle p_x^2\rangle,0\}$ with the partial transposition separability criterion (3) and the logarithmic negativity (5) in dependence of the bath temperature. All these criteria give the same critical temperature. This is a specific feature of this model where the criterion (42) detects the same set of stationary entangled states as the logarithmic negativity does. However, the separability criterion (6) in the form $\langle R^2\rangle+\langle p_x^2\rangle\geq\frac{1}{2}$ for parameter $a=1$ might not detect all entangled states since it is not an optimal entanglement witness in this case. Using this criterion would result in a critical temperature that is below the threshold given by criterion (42). Furthermore, since the criterion (41) does not include all (time-dependent) elements of the covariance matrix (but those that are unequal to zero in the stationary state) it may not lead to the same separability or reentangling times. Nevertheless it can be applied to the stationary state where it becomes equivalent to the PPT criterion (3) and thus predicts the same critical temperature.

In order to illustrate the different nature of classical and quantum correlations the stationary correlations $\langle\{q_1,q_2\}\rangle$ and $\langle\{p_1,p_2\}\rangle$ as well as the mutual information of a two-mode Gaussian state are also plotted in Fig. 5. The mutual information is defined by

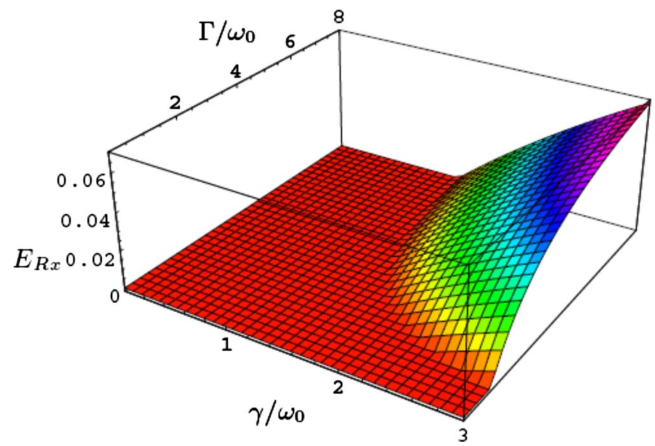

FIG. 6. (Color online) Stationary entanglement measured by the function $E_{Rx}=\max\{1-16\langle R^2\rangle\langle p_x^2\rangle,0\}$ (for $\hbar=1$) of a two-mode state in a common reservoir in dependence of coupling strength γ and cutoff frequency Γ at temperature $T=0.25\hbar\omega_0/k$.

$$I(\rho)=S_v(\rho_1)+S_v(\rho_2)-S_v(\rho), \quad (43)$$

where $S_v(\rho)$ is the von Neumann entropy of the total two-mode state and $S_v(\rho_1)$, $S_v(\rho_2)$ are the von Neumann entropies of the reduced states $\rho_1=\mathrm{Tr}_2[\rho]$ and $\rho_2=\mathrm{Tr}_1[\rho]$. The von Neumann entropy $S_v(\rho_j)=-\mathrm{Tr}[\rho_j \ln \rho_j]$ (for Boltzmann constant $k=1$) of the reduced systems can be calculated equivalently to that of a single-mode state and is given by

$$S_v(\rho_j)=\frac{1-\mu_j}{2\mu_j}\ln\frac{1+\mu_j}{1-\mu_j}-\ln\frac{2\mu_j}{1+\mu_j}, \quad (44)$$

with partial purities $\mu_j=\mathrm{Tr}[\rho_j^2]$. The total entropy can be derived from the symplectic eigenvalues $\nu_\pm$ of the two-mode state ρ characterized by its covariance matrix $\mathbf{V}$ and reads [54,55]

$$S_v(\rho)=f(\nu_-)+f(\nu_+), \quad (45)$$

with $f(\nu_\pm)=(\nu_\pm+\frac{1}{2})\ln(\nu_\pm+\frac{1}{2})-(\nu_\pm-\frac{1}{2})\ln(\nu_\pm-\frac{1}{2})$.

The mutual information is a measure of the amount of quantum and classical correlations [56] and decays more slowly with temperature than the logarithmic negativity does. Above a critical temperature (where the logarithmic negativity becomes zero) only classical correlations remain. The critical temperature depends on the spectral density of the bath and on the response function of the subsystem. The existence of stationary quantum correlations in dependence of the coupling constant γ and the cutoff frequency Γ is illustrated by Fig. 6. If the system-reservoir interaction constant becomes strong enough at a given temperature quantum correlations are still present in the stationary two-mode state.

V. SUMMARY AND CONCLUSIONS

In this paper we have studied the dynamics of two-mode squeezed states in a common thermal reservoir within an extended quantum Brownian motion model. Contrary to the two-reservoir model a common bath can provide an indirect coupling between the two subsystems and therefore a mechanism to correlate them. Thus, the coupling to the environment induces not only decoherence leading to separability but also can generate correlations resulting in asymptotic entanglement at low temperatures. The influence of non-Markovian and strong coupling effects on separability times scales was compared to previous results for Markovian dynamics. Non-Markovian effects increase the separability time but have little effect on the reentangling time. The reentangling time depends first of all on the system bath interaction strength. Increasing this coupling constant enforces entanglement generation and therefore reduces the reentangling time. However, it also enhances decoherence and thus weakens the initial quantum correlations reducing the separability time. For given parameters of the bath spectral density asymptotic entanglement is only present below a critical temperature. From the separability criterion for EPR-like operators a separability criterion was derived, which depends on the bath temperature and the response function of the open quantum system. Summarizing, the extended quantum Brownian motion model of a two-mode continuous variable system in a common reservoir provides an example of the case where environment-induced entanglement generation counteracts environment-induced decoherence. In view of the theory of quantum information with continuous variable systems these results may be relevant for experimental applications of correlated quantum channels.

ACKNOWLEDGMENTS

We would like to thank J. Eisert for valuable comments on the subject of this paper.

[1] S. L. Braunstein and A. K. Pati, *Quantum Information Theory with Continuous Variables* (Kluwer, Dordrecht, 2003).
[2] S. Braunstein and P. van Look, Rev. Mod. Phys. **77**, 513 (2005).
[3] N. J. Cerf, G. Leuchs, and E. S. Polzik, *Quantum Information with Continuous Variables of Atoms and Light* (Imperial College Press, London, 2007).
[4] A. Furusawa, J. L. Sorensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science **282**, 706 (1998).
[5] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).
[6] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).
[7] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[8] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[9] G. Giedke, M. M. Wolf, O. Kruger, R. F. Werner, and J. I. Cirac, Phys. Rev. Lett. **91**, 107901 (2003).
[10] Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, Phys. Rev. Lett. **68**, 3663 (1992).
[11] T. C. Zhang, K. W. Goh, C. W. Chou, P. Lodahl, and H. J. Kimble, Phys. Rev. A **67**, 033802 (2003).
[12] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2003).
[13] T. Dittrich, P. Hänggi, G.-L. Ingold, B. Kramer, G. Schön, and W. Zwerger, *Quantum Transport and Dissipation* (Wiley-VCH, Weinheim, 1998).
[14] J. S. Prauzner-Bechcicki, J. Phys. A **37**, L173 (2004).
[15] D. Braun, Phys. Rev. Lett. **89**, 277901 (2002).
[16] A. Serafini, F. Illuminati, M. G. A. Paris, and S. De Siena, Phys. Rev. A **69**, 022318 (2004).
[17] S. Olivares, M. G. A. Paris, and A. R. Rossi, Phys. Lett. A **319**, 32 (2003).
[18] J. J. Halliwell, J. Perez-Mercader, and W. H. Zurek, *Physical Origins of Time Asymmetry* (Cambridge University Press, Cambridge, 1994).
[19] M. S. Kim, J. Lee, D. Ahn, and P. L. Knight, Phys. Rev. A **65**, 040101(R) (2002).
[20] X. X. Yi, C. S. Yu, L. Zhou, and H. S. Song, Phys. Rev. A **68**, 052304 (2003).
[21] B. Bellomo, R. Lo Franco, and G. Compagno, Phys. Rev. Lett. **99**, 160502 (2007).
[22] C. Anastopoulos, S. Shresta, and B. L. Hu, e-print arXiv:quant-ph/0610007v2.
[23] F. Benatti and R. Floreanini, J. Phys. A **39**, 2689 (2006).
[24] F. Benatti and R. Floreanini, Int. J. Quantum Inf. **4**, 395 (2006).
[25] M. B. Plenio and S. F. Huelga, Phys. Rev. Lett. **88**, 197901 (2002).
[26] A. K. Rajagopal and R. W. Rendell, Phys. Rev. A **63**, 022116 (2001).
[27] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, Heidelberg, 1994).
[28] M. Ban, J. Phys. A **39**, 1927 (2006).
[29] H. McAneney, J. Lee, D. Ahn, and M. S. Kim, J. Mod. Opt. **52**, 935 (2005).
[30] S. Maniscalco, S. Olivares, and M. G. A. Paris, Phys. Rev. A **75**, 062119 (2007).
[31] J. H. An, M. Feng, and W. M. Zhang, e-print arXiv:0705.2472.
[32] K.-L. Liu and H.-S. Goan, Phys. Rev. A **76**, 022312 (2007).
[33] J. H. An and W. M. Zhang, Phys. Rev. A **76**, 042127 (2007).
[34] A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983).
[35] G. W. Ford, J. T. Lewis, and R. F. O'Connell, J. Stat. Phys. **53**, 439 (1988).
[36] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).
[37] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004).
[38] S. M. Tan, Phys. Rev. A **60**, 2752 (1999).
[39] M. D. Reid, Phys. Rev. A **40**, 913 (1989).
[40] V. Giovannetti, S. Mancini, D. Vitali, and P. Tombesi, Phys. Rev. A **67**, 022320 (2003).
[41] P. Hyllus and J. Eisert, New J. Phys. **8**, 51 (2006).
[42] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **92**, 087901 (2004).
[43] G. Adesso, A. Serafini, and F. Illuminati, Open Syst. Inf. Dyn. **12**, 189 (2005).

[44] A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983).
[45] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981).
[46] G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965).
[47] R. Karrlein and H. Grabert, Phys. Rev. E **55**, 153 (1997).
[48] F. Haake and R. Reibold, Phys. Rev. A **32**, 2462 (1985).
[49] J. Eisert and M. B. Plenio, Phys. Rev. Lett. **89**, 137902 (2002).
[50] C.-H. Chou, T. Yu, and B. L. Hu, Phys. Rev. E **77**, 011112 (2008).
[51] C. Hörhammer and H. Büttner, J. Phys. A **38**, 7325 (2005).
[52] O. S. Duarte and A. O. Caldeira, Phys. Rev. Lett. **97**, 250601 (2006).
[53] M. S. Kim, J. Lee, and W. J. Munro, Phys. Rev. A **66**, 030301(R) (2002).
[54] A. Serafini, F. Illuminati, and S. De Siena, J. Phys. B **37**, L21 (2004).
[55] A. S. Holevo and R. F. Werner, Phys. Rev. A **63**, 032312 (2001).
[56] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001).