Estimation of the repeatedly projected reduced density matrix under decoherence

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Decoherence is believed to deteriorate the ability of a purification scheme that is based on the idea of driving a system to a pure state by repeatedly measuring another system in interaction with the former and hinder for a pure state to be extracted asymptotically. Nevertheless, we find a way out of this difficulty by deriving an analytic expression of the reduced density matrix for a two-qubit system immersed in a bath. It is shown that we can still extract a pure state if the environment brings about only dephasing effects. In addition, for a dissipative environment, there is a possibility of obtaining a dominant pure state when we perform a finite number of measurements.

I. INTRODUCTION

The notion of measurement constitutes a key element in the quantum theory and there is a long history of debate on this subject [1]. It is, however, relatively recent that the following fact has been confirmed in a real laboratory that the measurement, which is usually described by a projection operator for simplicity but must be replaced, in a rigorous sense, with a generalized spectrum decomposition [2], does affect the dynamics of the quantum system drastically and in an essential way. The quantum Zeno effect [3], the effect caused by frequently repeated measurements and resulting in hindrance of the dynamical evolution of the quantum system [4] just measured, is one of such typical and well-known examples and has been discussed quite intensively after its first experimental observation [5]. It has also become clear that the action of measurement can have much more profound effects on the quantum systems than one naively expects or imagines from such phrases as "wave function collapse." Indeed, the effects have been shown to transfer to other quantum systems, not directly measured, but in interaction with the system measured, and the action of measurement can be utilized to drive even such quantum systems not directly touched to a pure state irrespective of their initial mixed states. This is the essence of the proposal of the repeated-measurement-based purification scheme [6], the applicability and effectiveness of which have been examined in various cases [7].

When one considers a possible implementation of such a theoretical scheme to experiments in the laboratories, it is necessary to examine its robustness against various imperfections to be encountered in the experiments, which may be expressed as a sort of decoherence from the environment. According to the analysis in Ref. [8], the projected dynamics of the quantum system suffers from the decoherence effect and it is shown that such an effect can deteriorate the ability

of the purification scheme to such an extent that no (nontrivial) pure state is able to be extracted in the limit of infinite number of projections. While this is somewhat an anticipated and discouraging result, the numerical estimations of the purity of the state to be extracted for a qubit system, which is coupled with another qubit that is subject to repeated interrogations and is immersed together with the latter in a common bosonic bath, show that there are parameter regions where states with high purities can be extracted [8]. On this basis we have studied the possibility to extract a quantum state with a higher purity under a dissipative environment by this purification scheme, even though it cannot survive in the limit of infinite number of measurements because of a kind of no-go theorem [8]. This paper is devoted to explore further such a possibility of extracting a dominant pure state at a finite number of measurements under a dissipative environment, as well as to point out its robustness against a dephasing environment, at infinite number of measurements.

It is desirable to obtain an explicit expression of the reduced density matrix for such investigations, though in the previous study [8] the ability of the purification scheme is discussed exclusively in terms of purity. Since solving analytically the projected dynamics with decoherence, that is the evolution described by a master equation supplemented with projections, is in general quite difficult, we confine ourselves to a simplest possible case, that is, two mutually interacting qubits, immersed in a common bosonic bath. We analyze the reduced density matrix of one of the two qubits when the state of the other qubit is regularly projected to one and the same state, according to the spirit of the purification scheme [6]. It is shown in the next section that if the bosonic environment causes only dephasing effects on the qubits, the purification scheme is robust enough to ensure one to extract a pure state, provided an appropriate state is confirmed on the other qubit at suitable intervals, just as in the ideal case. On the other hand, if the environment brings about a dissipative effect on the two-qubit system, it is quite difficult to solve the projected reduced dynamics in a compact form. However, in Sec. III it is shown that one can still estimate the behavior of the state analytically for a large, but finite number of measurements N. After an explicit expression of the

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projected density matrix for one of the qubits is derived in a closed form in Sec. III A, its asymptotic behavior for large N is presented in Sec. III B. Even under a dissipative environment, one can find a possibility of extracting a dominant pure state when the number of measurements N is large, but not extremely large, provided the dissipation is weak enough. However, it is to be noted that in the limit of infinite number of measurements, only a mixed state, except a trivial pure (ground) state, can survive, in accordance with the general theorem shown in [8]. We close the paper by giving the summary and discussions in Sec. IV. A few Appendixes are added to clarify some details that are not shown in the text.

II. PURIFICATION UNDER DEPHASING ENVIRONMENT

Consider a quantum system composed of two mutually interacting two-level systems (qubits), X and S, immersed in a common bosonic bath. Assume that the dynamics is given by the total Hamiltonian of the form

$$H_T = H_{XS} + H_{\text{bath}} + H_{\text{int}}, \tag{1}$$

where Hamiltonians for the two-qubit system and for the bosonic bath read as

$$H_{XS} = \frac{\omega}{2}\sigma_z^{(X)} + \frac{\Omega}{2}\sigma_z^{(S)} + g(\sigma_+^{(X)}\sigma_-^{(S)} + \sigma_-^{(X)}\sigma_+^{(S)}), \tag{2}$$

$$H_{\text{bath}} = \int d\boldsymbol{k}\,\omega_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}, \tag{3}$$

and the interaction Hamiltonian

$$H_{\text{int}} = \Sigma_z \int d\boldsymbol{k} [h(\boldsymbol{k}) a_{\boldsymbol{k}} + \text{H.c.}] \tag{4}$$

causes dephasing on the system. Here the notations are standard, $\sigma_\pm^{(i)} = (\sigma_x^{(i)} \pm i\sigma_y^{(i)})/2$ $(i=X,S)$, etc., and $\boldsymbol{\Sigma} = \boldsymbol{\sigma}^{(X)} + \boldsymbol{\sigma}^{(S)}$, and the interactions with the bath are assumed to be symmetric between X and S and are given by the same function $h(\boldsymbol{k})$, for simplicity. Since the above Hamiltonian H_{XS} is diagonalized,

$$H_{XS} = E_2|2\rangle\langle 2| + E_0|0\rangle\langle 0| + E_+|+\rangle\langle +| + E_-|-\rangle\langle -|, \tag{5}$$

where

$$|2\rangle = |\uparrow\rangle_X \otimes |\uparrow\rangle_S \equiv |\uparrow\uparrow\rangle,$$

$$|0\rangle = |\downarrow\downarrow\rangle,$$

$$|\pm\rangle = [\sqrt{1 \pm (\omega - \Omega)/2E_+}|\uparrow\downarrow\rangle \pm \sqrt{1 \mp (\omega - \Omega)/2E_+}|\downarrow\uparrow\rangle]/\sqrt{2},$$

$$E_2 = \frac{1}{2}(\omega + \Omega) = -E_0,$$

$$E_+ = \sqrt{(\omega - \Omega)^2/4 + g^2} = -E_-, \tag{6}$$

and the interaction is rewritten in terms of these eigenstates as

$$H_{\text{int}} = (|2\rangle\langle 2| - |0\rangle\langle 0|)\int d\boldsymbol{k}[2h(\boldsymbol{k})a_{\boldsymbol{k}} + \text{H.c.}], \tag{7}$$

the usual procedure [9] yields the master equation for the system

$$\dot{\rho}_{XS} = -i[\tilde{H}_{XS}, \rho_{XS}] + \gamma \left[(|2\rangle\langle 2| - |0\rangle\langle 0|)\rho_{XS}(|2\rangle\langle 2| - |0\rangle\langle 0|) \right.$$
$$\left. - \frac{1}{2}\{|2\rangle\langle 2| + |0\rangle\langle 0|, \rho_{XS}\} \right], \tag{8}$$

with the Hamiltonian $\tilde{H}_{XS}$, which is still diagonalized as in (6) but with an energy shift ΔE,

$$\tilde{E}_2 = E_2 + \Delta E, \quad \tilde{E}_0 = E_0 + \Delta E, \quad \tilde{E}_\pm = E_\pm. \tag{9}$$

The (dephasing) constant γ and the energy shift ΔE are given by the spectral density as usual [9]. A remarkable point is that the structure of the master equation is not affected by the temperature as in the dissipative case (see Sec. III), and the temperature only influences the values of the decay constant and of the energy shift. The master equation (8) is solved [10] in the following form [11]:

$$\rho_{XS}(t) = \sum_{i=0,\pm} K_i\, \rho_{XS}(0) K_i^\dagger, \tag{10}$$

where the three operators are given by

$$K_0 = e^{-i\tilde{E}_2 t - \gamma t/2}|2\rangle\langle 2| + e^{-i\tilde{E}_0 t - \gamma t/2}|0\rangle\langle 0|$$
$$+ e^{-iE_+ t}|+\rangle\langle +| + e^{-iE_- t}|-\rangle\langle -|,$$

$$K_+ = \sqrt{\cosh \gamma t - 1}(e^{-i\tilde{E}_2 t - \gamma t/2}|2\rangle\langle 2| + e^{-i\tilde{E}_0 t - \gamma t/2}|0\rangle\langle 0|),$$

$$K_- = \sqrt{\sinh \gamma t}(e^{-i\tilde{E}_2 t - \gamma t/2}|2\rangle\langle 2| - e^{-i\tilde{E}_0 t - \gamma t/2}|0\rangle\langle 0|). \tag{11}$$

Observe that these operators are all diagonal and the environment causes no transitions between different levels. Only the phases between different levels are affected by this type of interaction with the environment and this is the reason why it is called a dephasing interaction.

According to the spirit of the repeated-measurement-based purification scheme [6], we measure qubit X to confirm that it is in the same, particular state, while we do not touch the other qubit S, even though it is affected indirectly through its interaction with X. This process is repeated many times at regular intervals and we are interested in what happens to the state of qubit S, which also suffers the dephasing effect from the bath in addition to the projective actions on X. It has been shown in [8] that for a pure state to be finally extracted in this kind of process with projections and decoherence, it must be one of the simultaneous eigenstates of all the relevant projected operators and furthermore its eigenvalue for the projected dynamics must be the largest in magnitude, preferably close to unity. In our case of dephasing environment, it is easy to find such simultaneous eigenstates of the projected operators, as will be shown below.

Let us consider to measure the up state $|\uparrow\rangle_X$ regularly at $t = n\tau$ $(n = 1, 2, \ldots)$, since we already know that this type of measurement results in an optimal purification of qubit S in

the ideal case [7]. The relevant projected operators $V_{\uparrow(i)} = {}_X\langle\uparrow|K_i|\uparrow\rangle_X$ read as (the subscript S shall be suppressed in the following)

$$V_{\uparrow(0)} = {}_X\langle\uparrow|K_0|\uparrow\rangle_X$$

$$= e^{-i\tilde{E}_2\tau - \gamma\tau/2}|\uparrow\rangle\langle\uparrow| + \left(\cos E_+\tau + i\frac{\omega - \Omega}{2E_+}\sin E_+\tau\right)|\downarrow\rangle\langle\downarrow|,$$

$$V_{\uparrow(+)} = {}_X\langle\uparrow|K_+|\uparrow\rangle_X = \sqrt{\cosh\gamma\tau - 1}\, e^{-i\tilde{E}_2\tau - \gamma\tau/2}|\uparrow\rangle\langle\uparrow|,$$

$$V_{\uparrow(-)} = {}_X\langle\uparrow|K_-|\uparrow\rangle_X = \sqrt{\sinh\gamma\tau}\, e^{-i\tilde{E}_2\tau - \gamma\tau/2}|\uparrow\rangle\langle\uparrow|. \quad (12)$$

It is clear that both the up and down states (of qubit S) are the simultaneous eigenstates of all the above projected operators and the eigenvalues and eigenstates for the projected dynamics, which is given by the following map:

$$\rho(k\tau) = \sum_{i=0,\pm} V_{\uparrow(i)}\rho[(k-1)\tau]V_{\uparrow(i)}^\dagger, \quad (13)$$

are read from

$$\sum_{i=0,\pm} V_{\uparrow(i)}|\uparrow\rangle\langle\uparrow|V_{\uparrow(i)}^\dagger$$

$$= (|e^{-i\tilde{E}_2\tau - \gamma\tau/2}|^2 + |\sqrt{\cosh\gamma\tau - 1}\, e^{-i\tilde{E}_2\tau - \gamma\tau/2}|^2$$

$$+ |\sqrt{\sinh\gamma\tau}\, e^{-i\tilde{E}_2\tau - \gamma\tau/2}|^2)|\uparrow\rangle\langle\uparrow| = |\uparrow\rangle\langle\uparrow|, \quad (14)$$

$$\sum_{i=0,\pm} V_{\uparrow(i)}|\downarrow\rangle\langle\downarrow|V_{\uparrow(i)}^\dagger = \left|\cos E_+\tau + i\frac{\omega - \Omega}{2E_+}\sin E_+\tau\right|^2 |\downarrow\rangle\langle\downarrow|$$

$$\equiv |\xi|^2|\downarrow\rangle\langle\downarrow|, \quad (15)$$

$$\sum_{i=0,\pm} V_{\uparrow(i)}|\uparrow\rangle\langle\downarrow|V_{\uparrow(i)}^\dagger$$

$$= e^{-i\tilde{E}_2\tau - \gamma\tau/2}\left(\cos E_+\tau - i\frac{\omega - \Omega}{2E_+}\sin E_+\tau\right)|\uparrow\rangle\langle\downarrow|$$

$$= e^{-i\tilde{E}_2\tau - \gamma\tau/2}\xi^*|\uparrow\rangle\langle\downarrow|, \quad (16)$$

$$\sum_{i=0,\pm} V_{\uparrow(i)}|\downarrow\rangle\langle\uparrow|V_{\uparrow(i)}^\dagger$$

$$= e^{i\tilde{E}_2\tau - \gamma\tau/2}\left(\cos E_+\tau + i\frac{\omega - \Omega}{2E_+}\sin E_+\tau\right)|\downarrow\rangle\langle\uparrow|$$

$$= e^{i\tilde{E}_2\tau - \gamma\tau/2}\xi|\downarrow\rangle\langle\uparrow|. \quad (17)$$

Starting from a factorized initial state, $\rho_{XS}(0) = |\uparrow\rangle_X\langle\uparrow| \otimes \rho(0)$ [12], the density matrix is given by

$$\rho(N\tau) = \begin{pmatrix} \rho_{\uparrow\uparrow}(0) & (e^{-i\tilde{E}_2\tau - \gamma\tau/2}\xi^*)^N\rho_{\uparrow\downarrow}(0) \\ (e^{i\tilde{E}_2\tau - \gamma\tau/2}\xi)^N\rho_{\downarrow\uparrow}(0) & |\xi|^{2N}\rho_{\downarrow\downarrow}(0) \end{pmatrix}. \quad (18)$$

Observe that the up state $|\uparrow\rangle\langle\uparrow|$ is the eigenstate belonging to the eigenvalue unity and that there are no γ's left in its relation (14). It means that the dephasing has no effect on the ability of the purification scheme. The other eigenstates belong to eigenvalues smaller than unity if $\sin E_+\tau \neq 0$, just as in the ideal case with no decoherence. The above relations clearly show that we will be able to purify qubit S to the up state with no loss of probability, unless $\cos E_+\tau = \pm 1$, irrespective of its initial mixed state, when the qubit X is repeatedly confirmed to be in the up state. The repeated-measurement-based purification scheme is thus shown to be robust enough against the dephasing effect, at least when $|\uparrow\rangle_X$ is measured. On the other hand, it is easy to demonstrate that the only two possible pure states that can be extracted under dephasing environment are $|\uparrow\rangle$ and $|\downarrow\rangle$. Indeed, the projected operators ${}_X\langle\phi|K_\pm|\phi\rangle_X$, $\forall\,\phi$ are always diagonal in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, and the operator K_0 admits one of such two states as an eigenstate only when $|\uparrow\rangle_X$ or $|\downarrow\rangle_X$ is measured. It is worth stressing the possibility of extracting a pure state, i.e., $|\uparrow\rangle$, even under dephasing in certain conditions.

III. PURIFICATION UNDER DISSIPATIVE ENVIRONMENT

Consider next a dissipative environment and assume that it interacts with the two qubits, X and S, through the interaction Hamiltonian

$$H_{\text{int}} = \int dk\, h(k)(\Sigma_+ a_k + \Sigma_- a_k^\dagger), \quad \Sigma_\pm = \sigma_\pm^{(X)} + \sigma_\pm^{(S)}. \quad (19)$$

For the sake of simplicity, we shall set $\omega = \Omega$ [13], though generalization would be straightforward. The system Hamiltonian reads as

$$H_{XS} = \frac{\Omega}{2}\Sigma_3 + g(\sigma_+^{(X)}\sigma_-^{(S)} + \sigma_-^{(X)}\sigma_+^{(S)}) = \sum_{i=2,0,\pm} E_i|i\rangle\langle i|,$$
$$(20)$$

where eigenenergies and eigenstates are somewhat simplified,

$$|2\rangle = |\uparrow\uparrow\rangle,$$

$$|0\rangle = |\downarrow\downarrow\rangle,$$

$$|\pm\rangle = (|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)/\sqrt{2},$$

$$E_2 = \Omega = -E_0,$$

$$E_\pm = \pm g, \quad (21)$$

and the interaction Hamiltonian is rewritten as

$$H_{\text{int}} = \int dk\, \sqrt{2}h(k)[(|0\rangle\langle+| + |+\rangle\langle2|)a_k^\dagger + \text{H.c.}]. \quad (22)$$

When the bosonic bath is at temperature $T=0$, the master equation is derived under the usual conditions [9]

$$\dot{\rho}_{XS} = -i[\widetilde{H}_{XS}, \rho_{XS}] + \gamma \left(|+\rangle\langle 2|\rho_{XS}|2\rangle\langle +| - \frac{1}{2}\{|2\rangle\langle 2|, \rho_{XS}\} \right)$$

$$+ \gamma \left(|0\rangle\langle +|\rho_{XS}| +\rangle\langle 0| - \frac{1}{2}\{|+\rangle\langle +|, \rho_{XS}\} \right). \quad (23)$$

The decay constants $\gamma_{2\to+}$ and $\gamma_{+\to0}$ (we assume $\Omega > g$), given by the on-shell form factors, are assumed to be the same $\gamma \equiv \gamma_{2\to+} = \gamma_{+\to0}$, just for simplicity. The Hamiltonian $\widetilde{H}_{XS}$ is still diagonalized by $|i\rangle$ ($i = 2, 0, \pm$) in (21),

$$\widetilde{H}_{XS} = \sum_{i=2,0,\pm} \widetilde{E}_i |i\rangle\langle i|, \quad \widetilde{E}_2 = E_2 + \Delta E_2, \quad \widetilde{E}_+ = E_+ + \Delta E_+,$$

$$\widetilde{E}_0 = E_0, \quad \widetilde{E}_- = E_-. \quad (24)$$

We follow the recently developed technique [10] to obtain the solution of the master equation (23) in the following compact form:

$$\rho_{XS}(t) = e^{At}\rho_{XS}(0)e^{A^\dagger t} + (1 - e^{-\gamma t})B_0\rho_{XS}(0)B_0^\dagger$$

$$+ (1 - e^{-\gamma t} - \gamma t e^{-\gamma t})B_0 B_1 \rho_{XS}(0)B_1^\dagger B_0^\dagger$$

$$+ \gamma t e^{-\gamma t}B_1\rho_{XS}(0)B_1^\dagger, \quad (25)$$

where

$$e^{At} = e^{-i\widetilde{E}_2 t - \gamma t/2}|2\rangle\langle 2| + e^{-i\widetilde{E}_+ t - \gamma t/2}| +\rangle\langle +|$$

$$+ e^{-iE_0 t}|0\rangle\langle 0| + e^{-iE_- t}| -\rangle\langle -| \quad (26)$$

and

$$B_0 = |0\rangle\langle +|, \quad B_1 = | +\rangle\langle 2|. \quad (27)$$

Notice that one can think of a similar master equation that describes the dynamics of the two qubits X and S, in interaction with a common bosonic bath at temperature $T=0$ through the same coupling (19), but with a different mutual coupling from (2). For our purpose, however, it is desirable to make things as simple as possible, for, as will be seen below, even the above seemingly simplified dynamics (23) can bring us to quite involved expressions for the reduced density matrix of S when the state of X is periodically projected on one and the same state. Therefore, in this paper, we exclusively consider the dynamics (23) and endeavor to disclose the asymptotic behavior of the projected reduced density matrix for system S.

If the system X is to be repeatedly measured at $t = n\tau$ ($n = 1, 2, \dots$) to confirm that it is in the state $|\alpha\rangle_X = \alpha|\uparrow\rangle_X + |\downarrow\rangle_X$ [the normalization factor $(1 + |\alpha|^2)^{-1/2}$ is tentatively omitted here for notational simplicity], the relevant operators for the dynamics of qubit S read as follows:

$$_X\langle\alpha|e^{A\tau}|\alpha\rangle_X = e^{-i\widetilde{E}_2\tau - \gamma\tau/2}|\alpha|^2|\uparrow\rangle\langle\uparrow| + \frac{1}{2}e^{-i\widetilde{E}_+\tau - \gamma\tau/2}(\alpha^*|\downarrow\rangle + |\uparrow\rangle)(\alpha\langle\downarrow| + \langle\uparrow|) + e^{-iE_0\tau}|\downarrow\rangle\langle\downarrow| + \frac{1}{2}e^{-iE_-\tau}(\alpha^*|\downarrow\rangle - |\uparrow\rangle)(\alpha\langle\downarrow| - \langle\uparrow|)$$

$$= \begin{bmatrix} |\alpha|^2 e^{-i\widetilde{E}_2\tau - \gamma\tau/2} + \frac{1}{2}e^{-i\widetilde{E}_+\tau - \gamma\tau/2} + \frac{1}{2}e^{-iE_-\tau} & \frac{\alpha}{2}(e^{-i\widetilde{E}_+\tau - \gamma\tau/2} - e^{-iE_-\tau}) \\ \frac{\alpha^*}{2}(e^{-i\widetilde{E}_+\tau - \gamma\tau/2} - e^{-iE_-\tau}) & \frac{|\alpha|^2}{2}(e^{-i\widetilde{E}_+\tau - \gamma\tau/2} + e^{-iE_-\tau}) + e^{-iE_0\tau} \end{bmatrix} \equiv V, \quad (28)$$

$$_X\langle\alpha|B_0|\alpha\rangle_X = \frac{1}{\sqrt{2}}|\downarrow\rangle(\alpha\langle\downarrow| + \langle\uparrow|) = \begin{bmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{\alpha}{\sqrt{2}} \end{bmatrix} \equiv C_0, \quad (29)$$

$$_X\langle\alpha|B_0 B_1|\alpha\rangle_X = \alpha|\downarrow\rangle\langle\uparrow| = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \equiv C_1, \quad (30)$$

$$_X\langle\alpha|B_1|\alpha\rangle_X = \frac{\alpha}{\sqrt{2}}(\alpha^*|\downarrow\rangle + |\uparrow\rangle)\langle\uparrow| = \begin{bmatrix} \frac{\alpha}{\sqrt{2}} & 0 \\ \frac{|\alpha|^2}{\sqrt{2}} & 0 \end{bmatrix} \equiv C_2. \quad (31)$$

The projected reduced density matrix $\rho(\tau)$ of qubit S is given by the following map:

$$\rho(\tau) = V\rho(0)V^\dagger + (1 - e^{-\gamma\tau})C_0\rho(0)C_0^\dagger$$

$$+ (1 - e^{-\gamma\tau} - \gamma\tau e^{-\gamma\tau})C_1\rho(0)C_1^\dagger + \gamma\tau e^{-\gamma\tau}C_2\rho(0)C_2^\dagger. \quad (32)$$

A. Projected reduced density matrix

It is not difficult to confirm that the down state $|\downarrow\rangle$ can be a simultaneous eigenstate of all the above relevant operators if $\alpha = 0$. This corresponds to the case where qubit X is confirmed to be in the down state and qubit S is projected to the down state $|\downarrow\rangle\langle\downarrow|$ with probability 1. This fact has already been pointed out in Ref. [8]. Furthermore, one can easily see that for nonvanishing α, these relevant operators do not have a common eigenstate and therefore no pure state can be extracted in the limit of infinite number of measurements, ac-

cording to the general theorem [8]. Nevertheless we are interested in the asymptotic behavior of the projected reduced density matrix $\rho(N\tau)$ after N successive measurements, in the hope of finding a way out of, or bypassing such a "no-go" theorem.

In order to find the asymptotic form of the projected dynamics when the projection is repeated many times, observe first that the above operators C_0, C_1 which are expressed as (now the normalization constant is recovered)

$$C_0 = \frac{1}{\sqrt{2(1+|\alpha|^2)}}\begin{pmatrix}0\\1\end{pmatrix}(1,\alpha), \quad C_1 = \frac{\alpha}{1+|\alpha|^2}\begin{pmatrix}0\\1\end{pmatrix}(1,0)$$

(33)

in the up-down basis, enforce the system to be in a pure (down) state

$$\begin{pmatrix}0\\1\end{pmatrix}(0,1) = |\downarrow\rangle\langle\downarrow| \equiv \rho_{\downarrow\downarrow},$$

(34)

while the operator C_2, which can also be expressed as

$$C_2 = \frac{\alpha}{\sqrt{2(1+|\alpha|^2)}}\begin{pmatrix}1\\\alpha^*\end{pmatrix}(1,0),$$

(35)

drives the system to another pure state

$$\frac{1}{1+|\alpha|^2}\begin{pmatrix}1\\\alpha^*\end{pmatrix}(1,\alpha) \equiv |\alpha^*\rangle\langle\alpha^*| \equiv \rho_{\alpha^*},$$

(36)

irrespectively of the state the system had lived in just before projection. This means that in general the projected dynamics drives the system to a mixed state, which would be made more apparent if it is written in the following form (apart from normalization):

$$\rho(\tau) = V\rho(0)V^\dagger + F(\rho(0))\rho_{\downarrow\downarrow} + G(\rho(0))\rho_{\alpha^*},$$

(37)

where the positive constants F and G read as

$$F(\rho(0)) = \frac{(1-e^{-\gamma\tau})}{2(1+|\alpha|^2)}\langle\alpha^*|\rho(0)|\alpha^*\rangle$$

$$+ \frac{|\alpha|^2}{(1+|\alpha|^2)^2}(1-e^{-\gamma\tau}-\gamma\tau e^{-\gamma\tau})\langle\uparrow|\rho(0)|\uparrow\rangle \quad (38)$$

and

$$G(\rho(0)) = \frac{|\alpha|^2}{2(1+|\alpha|^2)}\gamma\tau e^{-\gamma\tau}\langle\uparrow|\rho(0)|\uparrow\rangle,$$

(39)

respectively.

It is not difficult to see that after N repetitions of the projected dynamics, the state of the system S, apart from the normalization, is driven to $[\rho_0 \equiv \rho(0)]$

$$\rho_N \equiv \rho(N\tau) = V^N\rho(0)V^{\dagger N} + \sum_{k=0}^{N-1}[F(\rho_k)V^{N-1-k}\rho_{\downarrow\downarrow}V^{\dagger N-1-k} + G(\rho_k)V^{N-1-k}\rho_{\alpha^*}V^{\dagger N-1-k}].$$

(40)

The coefficient $F(\rho_N)$ satisfies

$$F(\rho_N) = \frac{(1-e^{-\gamma\tau})}{2(1+|\alpha|^2)}\langle\alpha^*|\rho_N|\alpha^*\rangle + \frac{|\alpha|^2(1-e^{-\gamma\tau}-\gamma\tau e^{-\gamma\tau})}{(1+|\alpha|^2)^2}\langle\uparrow|\rho_N|\uparrow\rangle$$

$$= \frac{(1-e^{-\gamma\tau})}{2(1+|\alpha|^2)}\langle\alpha^*|V^N\rho(0)V^{\dagger N}|\alpha^*\rangle + \frac{|\alpha|^2(1-e^{-\gamma\tau}-\gamma\tau e^{-\gamma\tau})}{(1+|\alpha|^2)^2}\langle\uparrow|V^N\rho(0)V^{\dagger N}|\uparrow\rangle$$

$$+ \sum_{k=0}^{N-1}F(\rho_k)\left(\frac{(1-e^{-\gamma\tau})}{2(1+|\alpha|^2)}|\langle\alpha^*|V^{N-1-k}|\downarrow\rangle|^2 + \frac{|\alpha|^2(1-e^{-\gamma\tau}-\gamma\tau e^{-\gamma\tau})}{(1+|\alpha|^2)^2}|\langle\uparrow|V^{N-1-k}|\downarrow\rangle|^2\right)$$

$$+ \sum_{k=0}^{N-1}G(\rho_k)\left(\frac{(1-e^{-\gamma\tau})}{2(1+|\alpha|^2)}|\langle\alpha^*|V^{N-1-k}|\alpha^*\rangle|^2 + \frac{|\alpha|^2(1-e^{-\gamma\tau}-\gamma\tau e^{-\gamma\tau})}{(1+|\alpha|^2)^2}|\langle\uparrow|V^{N-1-k}|\alpha^*\rangle|^2\right).$$

(41)

Similarly, we have

$$G(\rho_N) = \frac{|\alpha|^2\gamma\tau e^{-\gamma\tau}}{2(1+|\alpha|^2)}\langle\uparrow|\rho_N|\uparrow\rangle$$

$$= \frac{|\alpha|^2\gamma\tau e^{-\gamma\tau}}{2(1+|\alpha|^2)}\langle\uparrow|V^N\rho(0)V^{\dagger N}|\uparrow\rangle + \sum_{k=0}^{N-1}F(\rho_k)\frac{|\alpha|^2\gamma\tau e^{-\gamma\tau}}{2(1+|\alpha|^2)}|\langle\uparrow|V^{N-1-k}|\downarrow\rangle|^2 + \sum_{k=0}^{N-1}G(\rho_k)\frac{|\alpha|^2\gamma\tau e^{-\gamma\tau}}{2(1+|\alpha|^2)}|\langle\uparrow|V^{N-1-k}|\alpha^*\rangle|^2.$$

(42)

It is now clear that these coefficients satisfy the recursion relation of the following form:

$$\begin{bmatrix} F(\rho_N) \\ G(\rho_N) \end{bmatrix} = \sum_{k=0}^{N-1} \mathcal{A}_{N-1-k} \begin{bmatrix} F(\rho_k) \\ G(\rho_k) \end{bmatrix} + b_N, \qquad (43)$$

where two-by-two matrices $\mathcal{A}_{N-1-k}$ and two-component column vectors b_N would be evident from the previous expressions. If a parameter x is introduced, its formal solution is easily found through an identity relation

$$\sum_{n=0}^{\infty} x^n \begin{bmatrix} F(\rho_n) \\ G(\rho_n) \end{bmatrix} = \left(1 - x \sum_{k=0}^{\infty} x^k \mathcal{A}_k \right)^{-1} \sum_{n=0}^{\infty} x^n b_n, \qquad (44)$$

that is, the coefficients $F(\rho_k)$ and $G(\rho_k)$ are given as those of the kth power x^k on the right-hand side. Plugging these values into (40), we obtain the explicit expression of the density matrix of qubit S at $N\tau$.

We assume the diagonalizability of V and introduce the eigenvectors of V,

$$V|u_i\rangle = \lambda_i |u_i\rangle, \quad \langle v_i|V = \lambda_i \langle v_i|, \quad \sum_{i=1}^{2} |u_i\rangle\langle v_i| = 1, \quad \langle v_i|u_j\rangle = \delta_{ij},$$

$$(45)$$

to expand it as

$$V = \sum_{i=1}^{2} \lambda_i |u_i\rangle\langle v_i|. \qquad (46)$$

The elements of the two-by-two matrix $\mathcal{A}_k$ and the column vector b_k are then expressed as

$$(\mathcal{A}_k)_{ij} = \sum_{a,b} (\Lambda_{ab})^k (\mathcal{C}^{ab})_{ij}, \quad (b_k)_i = \sum_{a,b} (\Lambda_{ab})^k (d^{ab})_i, \quad (47)$$

where

$$\Lambda_{ab} \equiv \lambda_a \lambda_b^* \qquad (48)$$

and the explicit expressions of $(\mathcal{C}^{ab})_{ij}$ and $(d^{ab})_i$ are found in Appendix A. The above identity relation (44) tells us that the coefficients $F(\rho_k)$ and $G(\rho_k)$ are explicitly given by

$$\begin{bmatrix} F(\rho_k) \\ G(\rho_k) \end{bmatrix} = \sum_{\substack{0 \le \ell, k_1, \ldots, k_\ell, n \le k \\ \ell + k_1 + \cdots + k_\ell + n = k}} \mathcal{A}_{k_1} \mathcal{A}_{k_2} \cdots \mathcal{A}_{k_\ell} b_n$$

$$= \sum_{\ell=0}^{k} \sum_{k_1=0}^{k-\ell} \sum_{k_2=0}^{k-\ell-k_1} \cdots \sum_{k_\ell=0}^{k-\ell-k_1-\cdots-k_{\ell-1}} \sum_{\substack{a_i, b_j \\ 1 \le i,j \le \ell+1}} (\Lambda_{a_1 b_1})^{k_1} \cdots (\Lambda_{a_\ell b_\ell})^{k_\ell} (\Lambda_{a_{\ell+1} b_{\ell+1}})^{k-\ell-k_1-\cdots-k_\ell} \mathcal{C}^{a_1 b_1} \cdots \mathcal{C}^{a_\ell b_\ell} d^{a_{\ell+1} b_{\ell+1}}. \quad (49)$$

Observe that the indices $k_1, \ldots, k_\ell$ only appear as the exponents of the eigenvalues. The summations over these indices are not difficult to perform and each summation results in a similar expression. Actually, since we can prove by induction (see Appendix B),

$$\sum_{k_1=0}^{k-\ell} \sum_{k_2=0}^{k-\ell-k_1} \cdots \sum_{k_\ell=0}^{k-\ell-k_1-\cdots-k_{\ell-1}} x_1^{k_1} \cdots x_\ell^{k_\ell} x_{\ell+1}^{k-\ell-k_1-\cdots-k_\ell} = \frac{x_{\ell+1}^k}{(x_{\ell+1}-x_\ell)(x_{\ell+1}-x_{\ell-1})\cdots(x_{\ell+1}-x_1)} + \frac{x_\ell^k}{(x_\ell-x_{\ell+1})(x_\ell-x_{\ell-1})\cdots(x_\ell-x_1)}$$

$$+ \cdots + \frac{x_1^k}{(x_1-x_{\ell+1})(x_1-x_\ell)\cdots(x_1-x_2)}$$

$$= \sum_{m=1}^{\ell+1} \frac{x_m^k}{\prod_{\substack{n \ne m \\ 1 \le n \le \ell+1}} (x_m - x_n)}, \qquad (50)$$

we arrive at

$$\begin{bmatrix} F(\rho_k) \\ G(\rho_k) \end{bmatrix} = \sum_{\ell=0}^{k} \sum_{m=1}^{\ell+1} \sum_{\substack{a_\alpha, b_\beta \\ 1 \le \alpha, \beta \le \ell+1}} \frac{(\Lambda_{a_m b_m})^k}{\prod_{\substack{n \ne m \\ 1 \le n \le \ell+1}} (\Lambda_{a_m b_m} - \Lambda_{a_n b_n})} \mathcal{C}^{a_1 b_1} \cdots \mathcal{C}^{a_\ell b_\ell} d^{a_{\ell+1} b_{\ell+1}}. \qquad (51)$$

Notice that ρ_N in (40) is expressed in terms of the eigenvectors $|u_i\rangle$ and $|v_j\rangle$ as

$$\rho_N = \sum_{i,j} (\Lambda_{ij})^N |u_i\rangle\langle v_i|\rho(0)|v_j\rangle\langle u_j| + \sum_{i,j}\sum_{k=0}^{N-1} (\Lambda_{ij})^{N-1-k}|u_i\rangle\langle u_j|(\langle v_i|\rho_{\downarrow\downarrow}|v_j\rangle,\langle v_i|\rho_{\alpha*}|v_j\rangle)\begin{bmatrix} F(\rho_k) \\ G(\rho_k) \end{bmatrix}. \tag{52}$$

Plugging the explicit forms of $F(\rho_k)$ and $G(\rho_k)$, (51) into (52), we are left with the following terms to be evaluated:

$$\sum_{k=0}^{N-1} (\Lambda_{ij})^{N-1-k}\begin{bmatrix} F(\rho_k) \\ G(\rho_k) \end{bmatrix} = \sum_{k=0}^{N-1} (\Lambda_{ij})^{N-1-k}\sum_{\ell=0}^{k}\sum_{m=1}^{\ell+1}\sum_{\substack{a_\alpha,b_\beta \\ 1\leq\alpha,\beta\leq\ell+1}} \frac{(\Lambda_{a_m b_m})^k}{\prod_{\substack{n\neq m \\ 1\leq n\leq\ell+1}}(\Lambda_{a_m b_m} - \Lambda_{a_n b_n})}\mathcal{C}^{a_1 b_1}\cdots\mathcal{C}^{a_\ell b_\ell}d^{a_{\ell+1}b_{\ell+1}}. \tag{53}$$

The summations over k and ℓ can be interchanged $\sum_{k=0}^{N-1}\sum_{\ell=0}^{k}=\sum_{\ell=0}^{N-1}\sum_{k=\ell}^{N-1}$ and that over k is performed to yield

$$= \sum_{\ell=0}^{N-1}\sum_{m=1}^{\ell+1}\sum_{\substack{a_\alpha,b_\beta \\ 1\leq\alpha,\beta\leq\ell+1}} \frac{1}{\prod_{\substack{n\neq m \\ 1\leq n\leq\ell+1}}(\Lambda_{a_m b_m} - \Lambda_{a_n b_n})}\frac{(\Lambda_{ij})^N(\Lambda_{a_m b_m}/\Lambda_{ij})^\ell - (\Lambda_{a_m b_m})^N}{\Lambda_{ij} - \Lambda_{a_m b_m}}\mathcal{C}^{a_1 b_1}\cdots\mathcal{C}^{a_\ell b_\ell}d^{a_{\ell+1}b_{\ell+1}}. \tag{54}$$

This is the explicit form of the second term on the right-hand side (rhs) of (52) and brings us to the exact expression of $\rho_N = \rho(N\tau)$.

B. Asymptotic behavior of the projected reduced density matrix

We are now in a position to evaluate the asymptotic form of ρ_N for large N. The particular form seen in (54) implies that the dominant contributions are due, in general, to those terms with the same set of indices $\{ij\}=\{a_1 b_1\}=\{a_2 b_2\}=\cdots=\{a_{\ell+1}b_{\ell+1}\}$, for in such a case there are order-N terms contributing constructively. This can be understood by looking at the original expression (49). See Appendix C for more explanations. The dominant contributions to the second term on the rhs of (52),

$$\sum_{i,j}\sum_{k=0}^{N-1} (\Lambda_{ij})^{N-1-k}|u_i\rangle\langle u_j|(\langle v_i|\rho_{\downarrow\downarrow}|v_j\rangle,\langle v_i|\rho_{\alpha*}|v_j\rangle)\begin{bmatrix} F(\rho_k) \\ G(\rho_k) \end{bmatrix}$$

$$= |u_1\rangle\langle u_1|(\langle v_1|\rho_{\downarrow\downarrow}|v_1\rangle,\langle v_1|\rho_{\alpha*}|v_1\rangle)\mathcal{M}_{11} + |u_2\rangle\langle u_2|(\langle v_2|\rho_{\downarrow\downarrow}|v_2\rangle,\langle v_2|\rho_{\alpha*}|v_2\rangle)\mathcal{M}_{22} + |u_1\rangle\langle u_2|(\langle v_1|\rho_{\downarrow\downarrow}|v_2\rangle,\langle v_1|\rho_{\alpha*}|v_2\rangle)\mathcal{M}_{12} + \text{H.c.}, \tag{55}$$

are found in the above (column) vectors $\mathcal{M}_{ij}$,

$$\mathcal{M}_{11} \sim \sum_{\ell=0}^{N-1}\Lambda_{11}^{N-1-\ell}\mathcal{A}_0^\ell b_0 + \sum_{\ell=0}^{N-1}\sum_{k=\ell+1}^{N-1}\sum_{k_1=0}^{k-\ell}\cdots\sum_{k_\ell=0}^{k-\ell-k_1-\cdots-k_{\ell-1}}\Lambda_{11}^{N-1-\ell}(\mathcal{C}^{11})^\ell d^{11}$$

$$= (\Lambda_{11} - \mathcal{A}_0)^{-1}(\Lambda_{11}^N - \mathcal{A}_0^N)b_0 + \sum_{k=1}^{N-1}\sum_{\ell=0}^{k-1}\binom{k}{\ell}\Lambda_{11}^{N-1-\ell}(\mathcal{C}^{11})^\ell d^{11}, \tag{56}$$

the second term of which is evaluated to be

$$\sum_{k=1}^{N-1}\sum_{\ell=0}^{k-1}\binom{k}{\ell}\Lambda_{11}^{N-1-\ell}(\mathcal{C}^{11})^\ell d^{11} = -\Lambda_{11}^N(\mathcal{C}^{11})^{-1}\left(1 + \frac{\mathcal{C}^{11}}{\Lambda_{11}}\right)\left[1 - \left(1 + \frac{\mathcal{C}^{11}}{\Lambda_{11}}\right)^{N-1}\right]d^{11} - \Lambda_{11}^{N-2}\mathcal{C}^{11}\left(1 - \frac{\mathcal{C}^{11}}{\Lambda_{11}}\right)^{-1}\left[1 - \left(\frac{\mathcal{C}^{11}}{\Lambda_{11}}\right)^{N-1}\right]d^{11}. \tag{57}$$

Similarly, we have

$$\mathcal{M}_{22} \sim \sum_{\ell=0}^{N-1} \Lambda_{22}^{N-1-\ell} \mathcal{A}_0^\ell b_0 + \sum_{\ell=0}^{N-1} \sum_{k=\ell+1}^{N-1} \sum_{k_1=0}^{k-\ell} \cdots \sum_{k_\ell=0}^{k-\ell-k_1-\cdots-k_{\ell-1}} \Lambda_{22}^{N-1-\ell} \left(\frac{\Lambda_{11}}{\Lambda_{22}}\right)^{k-\ell} (\mathcal{C}^{11})^\ell d^{11}$$

$$= (\Lambda_{22} - \mathcal{A}_0)^{-1} (\Lambda_{22}^N - \mathcal{A}_0^N) b_0 + \Lambda_{22}^{N-1} \left\{ \frac{1 + \dfrac{\mathcal{C}^{11}}{\Lambda_{11}}}{1 + \dfrac{\mathcal{C}^{11}}{\Lambda_{11}} - \dfrac{\Lambda_{22}}{\Lambda_{11}}} \left[\left(\frac{\Lambda_{11}}{\Lambda_{22}}\right)^{N-1} \left(1 + \frac{\mathcal{C}^{11}}{\Lambda_{11}}\right)^{N-1} - 1 \right] - \frac{\dfrac{\mathcal{C}^{11}}{\Lambda_{22}}}{1 - \dfrac{\mathcal{C}^{11}}{\Lambda_{22}}} \left[1 - \left(\frac{\mathcal{C}^{11}}{\Lambda_{22}}\right)^{N-1} \right] \right\} d^{11},$$

(58)

and

$$\mathcal{M}_{12} \sim \sum_{\ell=0}^{N-1} \Lambda_{12}^{N-1-\ell} \mathcal{A}_0^\ell b_0 + \sum_{\ell=0}^{N-1} \sum_{k=\ell+1}^{N-1} \sum_{k_1=0}^{k-\ell} \cdots \sum_{k_\ell=0}^{k-\ell-k_1-\cdots-k_{\ell-1}} \Lambda_{12}^{N-1-\ell} (\mathcal{C}^{12})^\ell d^{12}$$

$$= (\Lambda_{12} - \mathcal{A}_0)^{-1} (\Lambda_{12}^N - \mathcal{A}_0^N) b_0 - \Lambda_{12}^N (\mathcal{C}^{12})^{-1} \left(1 + \frac{\mathcal{C}^{12}}{\Lambda_{12}}\right) \left[1 - \left(1 + \frac{\mathcal{C}^{12}}{\Lambda_{12}}\right)^{N-1} \right] d^{12} - \Lambda_{12}^{N-2} \mathcal{C}^{12} \left(1 - \frac{\mathcal{C}^{12}}{\Lambda_{22}}\right)^{-1} \left[1 - \left(\frac{\mathcal{C}^{12}}{\Lambda_{12}}\right)^{N-1} \right] d^{12}.$$

(59)

For weak dissipation $\gamma\tau \ll 1$, the matrix elements of $\mathcal{C}^{ab}$ (and $\mathcal{A}_0$) are of the order of $\gamma\tau$ or higher [see (A1)], while the maximum (in magnitude) eigenvalue λ_1 is expected to be of order unity $\lambda_1 \sim O(1)$ with corrections of order $\gamma\tau$. In this case, the dominant contributions to $\mathcal{M}_{ij}$ are estimated to be

$$\mathcal{M}_{11} \sim (N-1)\Lambda_{11}^{N-1} d^{11}, \quad \mathcal{M}_{22} \sim \Lambda_{11}^{N-1} d^{11},$$

$$\mathcal{M}_{12} \sim (N-1)\Lambda_{12}^{N-1} d^{12}. \tag{60}$$

Therefore, if the above ρ_N (52) is suitably normalized, the state is approximated as a pure state

$$\rho_N \sim |u_1\rangle\langle u_1| + O(1/N, |\lambda_2/\lambda_1|^N), \tag{61}$$

for a large, but not extremely large N under which the above approximation is valid.

Notice that the validity of the above expression (60) and therefore that of the ensuing relation (61) are limited. Actually for a larger N, $N\gamma\tau \gg 1$, the approximation $(1+x)^N \sim 1 + Nx$ with $x = O(\gamma\tau)$, on which the expression (60) has been based, is no longer valid. Instead, since

$$(1+x)^N = e^{N\ln(1+x)} \sim e^{N(x - x^2/2 + \cdots)}, \tag{62}$$

we expect, for a larger N, say $1/\gamma\tau < N < 1/(\gamma\tau)^2$,

$$\mathcal{M}_{11} \sim \Lambda_{11}^{N-1} e^{(N-1)(\mathcal{C}^{11}/\Lambda_{11})} (\mathcal{C}^{11})^{-1} d^{11},$$

$$\mathcal{M}_{22} \sim \Lambda_{11}^{N-1} e^{(N-1)(\mathcal{C}^{11}/\Lambda_{11})} d^{11},$$

$$\mathcal{M}_{12} \sim \Lambda_{12}^{N-1} (\mathcal{C}^{12})^{-1} e^{(N-1)(\mathcal{C}^{12}/\Lambda_{12})} d^{12}. \tag{63}$$

Even though the coefficient $\mathcal{M}_{11}$ multiplying the term $|u_1\rangle\langle u_1|$ gives the major contribution, the others coefficients are of the same order as the former and never decrease as we

increase N. It is easily understood that a similar expression can be found for an even larger N. That is, we are unable to reach a pure state by increasing the number of measurements (projections) N, which is in accord with the theorem shown in [8].

A few comments are in order at this point. The pure state $|u_1\rangle\langle u_1|$ approximately extracted above in (61) for an intermediate N is not an eigenstate of the projected dynamics (37). Actually it is possible to prove that no pure state can be an eigenstate of such a positive dynamical map, provided that it includes more than two (in the case of two dimensions) projections that (are necessarily not orthogonal and) project to different pure states. Nevertheless, it is still possible to consider such a situation where effects of all but one such projections decrease more quickly than that of one particular projection and as a result for an appropriately large N the dynamical map effectively becomes single dimensional, i.e., an effective manifestation of purification. This is the case in (61). Second, in the strong-dissipation limit ($\gamma \to \infty$) many simplifications are expected to occur in the general form, but the essential points are intact. The density matrix ρ_N takes exactly the same form as the rhs of (52) supplemented with (55). It is easily seen that $G(\rho_k) = 0$, $\forall~k$ and the column vector appearing in (55), $\mathcal{C}^{a_1 b_1} \cdots \mathcal{C}^{a_\ell b_\ell} d^{a_{\ell+1} b_{\ell+1}}$, has only the upper component

$$\mathcal{C}^{a_1 b_1} \cdots \mathcal{C}^{a_\ell b_\ell} d^{a_{\ell+1} b_{\ell+1}} = \begin{bmatrix} (\mathcal{C}^{a_1 b_1})_{11} \cdots (\mathcal{C}^{a_\ell b_\ell})_{11} (d^{a_{\ell+1} b_{\ell+1}})_1 \\ 0 \end{bmatrix}. \tag{64}$$

The asymptotic form of the density matrix is easily evaluated

$$\rho_N \sim N|\lambda_1|^2 |u_1\rangle\langle u_1|(\langle v_1|\rho_{\downarrow\downarrow}|v_1\rangle, \langle v_1|\rho_{\alpha^*}|v_1\rangle)(\mathcal{C}^{11})^{N-2}d^{11} + N|\lambda_1|^2 |u_1\rangle\langle u_2|(\langle v_1|\rho_{\downarrow\downarrow}|v_2\rangle, \langle v_1|\rho_{\alpha^*}|v_2\rangle)(\mathcal{C}^{12})^{N-2}d^{12} + \text{H.c.}$$
$$+ (N-1)|\lambda_1|^2 |u_2\rangle\langle u_2|(\langle v_2|\rho_{\downarrow\downarrow}|v_2\rangle, \langle v_2|\rho_{\alpha^*}|v_2\rangle)(\mathcal{C}^{11})^{N-2}d^{11} + (N-1)|\lambda_1|^2(\rho_{\downarrow\downarrow}, \rho_{\alpha^*})(\mathcal{C}^{11})^{N-2}d^{11}$$
$$\sim N|\lambda_1|^2 [|u_1\rangle\langle u_1|\langle v_1|\rho_{\downarrow\downarrow}|v_1\rangle(\mathcal{C}^{11})_{11}^{N-2}(d^{11})_1 + |u_1\rangle\langle u_2|\langle v_1|\rho_{\downarrow\downarrow}|v_2\rangle(\mathcal{C}^{12})_{11}^{N-2}(d^{12})_1 + \text{H.c.}$$
$$+ |u_2\rangle\langle u_2|\langle v_2|\rho_{\downarrow\downarrow}|v_2\rangle(\mathcal{C}^{11})_{11}^{N-2}(d^{11})_1 + \rho_{\downarrow\downarrow}(\mathcal{C}^{11})_{11}^{N-2}(d^{11})_1]. \tag{65}$$

If we neglect differences between $(\mathcal{C}^{11})_{11}$ and $(\mathcal{C}^{12})_{11}$ and $(d^{11})_1$ and $(d^{12})_1$, the above expression tells nothing but that the state relaxes to the down state $\rho_{\downarrow\downarrow}=|\downarrow\rangle\langle\downarrow|$. On the other hand, if the matrix elements, $(\mathcal{C}^{11})_{11}$ and $(d^{11})_1$, are greater than the others, $(\mathcal{C}^{12})_{11}$ and $(d^{12})_1$, the formers overwhelm the latters as N increases and the density matrix will relax to a mixed state

$$\rho_N \rightarrow |u_1\rangle\langle u_1|\langle v_1|\rho_{\downarrow\downarrow}|v_1\rangle + |u_2\rangle\langle u_2|\langle v_2|\rho_{\downarrow\downarrow}|v_2\rangle + \rho_{\downarrow\downarrow}. \tag{66}$$

If the opposite case were possible, we would have been given another matrix

$$\rho_N \rightarrow \frac{|u_1\rangle\langle u_2| + |u_2\rangle\langle u_1|}{\langle u_2|u_1\rangle + \langle u_1|u_2\rangle}, \tag{67}$$

but it turns out that this is impossible, for this matrix can be shown to have a negative eigenvalue (the trace of the square of this matrix is shown to exceed 1!) and therefore it will never show up in our physical process. In all cases with strong dissipation, no nontrivial pure state is shown to be extracted, even at an intermediate stage with a finite number of measurements. Notice that these observations are actually consistent with the exact result; in the strong-dissipation limit $\gamma \rightarrow \infty$, the operator C_2 becomes irrelevant [see (32)] and since the operators C_0 and C_1 project qubit S to the down state, the simultaneous eigenstate should be the down state ($\alpha=0$), for which the operator V becomes diagonal. This means that $|u_{1(2)}\rangle \rightarrow |\downarrow(\uparrow)\rangle$ and $\rho_N \rightarrow \rho_{\downarrow\downarrow}$.

Finally, it is to be noticed that the dissipative environment also affects the yield, that is, the probability of obtaining the target pure state. Actually, the probability of obtaining the above pure state (61) is considered to be almost equal to the success probability, which is given by the normalization factor of ρ_N in (52), if the latter is dominated by the pure state as in (61), and this factor is of order $\sim (1+N\gamma\tau)\Lambda_{11}^N$ for an intermediate N, e.g., $1 \ll N < 1/\gamma\tau$. On the contrary, the dephasing environment causes no effect on the yield.

IV. SUMMARY AND DISCUSSIONS

We have shown in this paper that the repeated-measurement-based purification scheme is robust against a dephasing environment and the up state of qubit S, $|\uparrow\rangle\langle\uparrow|$, can be realized asymptotically without any loss of probability, when qubit X in (rotating-wave) interaction with the former is repeatedly confirmed to be in the up state, $|\uparrow\rangle_X$, just as in the ideal case. In order to reach the final pure state with fewer steps, we adjust parameters,

say, the interval between measurements τ, and minimize $|\cos E_+\tau + i(\omega-\Omega)(2E_+)^{-1}\sin E_+\tau|$,

$$\cos^2 E_+\tau + \frac{(\omega-\Omega)^2}{4E_+^2}\sin^2 E_+\tau$$
$$= 1 - \frac{4g^2}{(\omega-\Omega)^2 + 4g^2}\sin^2 E_+\tau \geq \frac{(\omega-\Omega)^2}{(\omega-\Omega)^2 + 4g^2}, \tag{68}$$

the equality of which is attained when $\cos E_+\tau=0$, realizing an optimal case. This condition is the same as that for the ideal case.

The reason why the dephasing does not affect the ability of this kind of purification scheme may be understood in the following way. The dephasing environment surely disturbs the phases of both qubits but causes no transitions between up and down states. On the other hand, our purification scheme is dependent on the probability of finding a quantum system in some definite state and the change in its phase has no relevance to this scheme. This is a naive interpretation of why the ability of this purification scheme is not affected by the presence of dephasing environment, when the state $|\uparrow\rangle$ is measured.

We have next considered the case of dissipative environment and examined the ability of the purification scheme, though it is already known [8] that no (nontrivial) pure state can be extracted in this case according to the general theorem. When the qubit X is repeatedly confirmed to be in a definite state, the target system S, in (rotating-wave) interaction with X, is forced to be in a definite state, while the surrounding environment constantly drives the system to the equilibrium. It seems that a kind of competition between two tendencies, one forced by the projection and the other relaxing to an equilibrium state, results in an approximate extraction of a dominant pure state at an intermediate stage with a large but not extremely large number of measurements. For a weak damping case, we are able to find an asymptotic expression of the state of qubit S, which shows that the dominant contribution is given by one of the eigenstates of the operator V (belonging to the largest eigenvalue in magnitude), which is a kind of projected evolution operator incorporating partly dissipative dynamics and reduces to the usual unitary operator supplemented with projection when there is no dissipation. Of course, if we repeat the measurement indefinitely, the dissipative dynamics would swiftly overwhelm the effect of projection (purification) and the system would never be driven to a (nontrivial) pure state, which is in accord with the general theorem.

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It is worth mentioning that in actual situations with various causes of decoherence, one must be careful about and aware of the presence of such nonideal elements and should not perform measurements indefinitely, for there is a possibility that an optimal result can be attained at a finite number of measurements, as in our simplified model, even though a general criterion seems to be quite difficult to be obtained at present.

APPENDIX A: ELEMENTS OF $\mathcal{C}^{ab}$ AND d^{ab}

Here we show the elements of $\mathcal{C}^{ab}$ and d^{ab} explicitly

$$(\mathcal{C}^{ab})_{11} = \frac{(1-e^{-\gamma\tau})}{2(1+|\alpha|^2)}\langle\alpha^*|u_a\rangle\langle v_a|\downarrow\rangle\langle\downarrow|v_b\rangle\langle u_b|\alpha^*\rangle$$
$$+ \frac{|\alpha|^2(1-e^{-\gamma\tau}-\gamma\tau e^{-\gamma\tau})}{(1+|\alpha|^2)^2}\langle\uparrow|u_a\rangle\langle v_a|\downarrow\rangle\langle\downarrow|v_b\rangle\langle u_b|\uparrow\rangle,$$

$$(\mathcal{C}^{ab})_{12} = \frac{(1-e^{-\gamma\tau})}{2(1+|\alpha|^2)}\langle\alpha^*|u_a\rangle\langle v_a|\alpha^*\rangle\langle\alpha^*|v_b\rangle\langle u_b|\alpha^*\rangle$$
$$+ \frac{|\alpha|^2(1-e^{-\gamma\tau}-\gamma\tau e^{-\gamma\tau})}{(1+|\alpha|^2)^2}\langle\uparrow|u_a\rangle\langle v_a|\alpha^*\rangle\langle\alpha^*|v_b\rangle\langle u_b|\uparrow\rangle,$$

$$(\mathcal{C}^{ab})_{21} = \frac{|\alpha|^2\gamma\tau e^{-\gamma\tau}}{2(1+|\alpha|^2)}\langle\uparrow|u_a\rangle\langle v_a|\downarrow\rangle\langle\downarrow|v_b\rangle\langle u_b|\uparrow\rangle,$$

$$(\mathcal{C}^{ab})_{22} = \frac{|\alpha|^2\gamma\tau e^{-\gamma\tau}}{2(1+|\alpha|^2)}\langle\uparrow|u_a\rangle\langle v_a|\alpha^*\rangle\langle\alpha^*|v_b\rangle\langle u_b|\uparrow\rangle, \quad (A1)$$

$$(d^{ab})_1 = \frac{(1-e^{-\gamma\tau})}{2(1+|\alpha|^2)}\langle\alpha^*|u_a\rangle\langle v_a|\rho(0)|v_b\rangle\langle u_b|\alpha^*\rangle$$
$$+ \frac{|\alpha|^2(1-e^{-\gamma\tau}-\gamma\tau e^{-\gamma\tau})}{(1+|\alpha|^2)^2}\langle\uparrow|u_a\rangle\langle v_a|\rho(0)|v_b\rangle\langle u_b|\uparrow\rangle,$$

$$(d^{ab})_2 = \frac{|\alpha|^2\gamma\tau e^{-\gamma\tau}}{2(1+|\alpha|^2)}\langle\uparrow|u_a\rangle\langle v_a|\rho(0)|v_b\rangle\langle u_b|\uparrow\rangle. \quad (A2)$$

APPENDIX B: DERIVATION OF THE RELATION (50)

It is not difficult to perform the first, say, two summations over k_ℓ and $k_{\ell-1}$ on the left-hand side of (50),

$$\sum_{k_1=0}^{k-\ell}\sum_{k_2=0}^{k-\ell-k_1}\cdots\sum_{k_\ell=0}^{k-\ell-k_1-\cdots-k_{\ell-1}} x_1^{k_1}\cdots x_\ell^{k_\ell}x_{\ell+1}^{k-\ell-k_1-\cdots-k_\ell}$$
$$=\sum_{k_1=0}^{k-\ell}\cdots\sum_{k_{\ell-1}=0}^{k-\ell-k_1-\cdots-k_{\ell-2}} x_1^{k_1}\cdots x_{\ell-1}^{k_{\ell-1}}\left(\frac{x_\ell^{k-(\ell-1)-k_1-\cdots-k_{\ell-1}}}{x_\ell-x_{\ell+1}}+\frac{x_{\ell+1}^{k-(\ell-1)-k_1-\cdots-k_{\ell-1}}}{x_{\ell+1}-x_\ell}\right)$$
$$=\sum_{k_1=0}^{k-\ell}\cdots\sum_{k_{\ell-2}=0}^{k-\ell-k_1-\cdots-k_{\ell-3}} x_1^{k_1}\cdots x_{\ell-2}^{k_{\ell-2}}\left(\frac{x_{\ell-1}^{k-(\ell-2)-k_1-\cdots-k_{\ell-2}}}{(x_{\ell-1}-x_{\ell+1})(x_{\ell-1}-x_\ell)}+\frac{x_\ell^{k-(\ell-2)-k_1-\cdots-k_{\ell-2}}}{(x_\ell-x_{\ell+1})(x_\ell-x_{\ell-1})}+\frac{x_{\ell+1}^{k-(\ell-2)-k_1-\cdots-k_{\ell-2}}}{(x_{\ell+1}-x_\ell)(x_{\ell+1}-x_{\ell-1})}\right). \quad (B1)$$

These expressions are already quite suggestive. Actually we easily perform the summation of the following form to obtain

$$\sum_{k_m=0}^{k-\ell-k_1-\cdots-k_{m-1}} x_m^{k_m}\left(\frac{x_{m+1}^{k-m-k_1-\cdots-k_m}}{(x_{m+1}-x_{\ell+1})(x_{m+1}-x_\ell)\cdots(x_{m+1}-x_{m+2})}+\cdots+\frac{x_{\ell+1}^{k-m-k_1-\cdots-k_m}}{(x_{\ell+1}-x_\ell)(x_{\ell+1}-x_{\ell-1})\cdots(x_{\ell+1}-x_{m+1})}\right)$$
$$=-x_m^{k-\ell-k_1-\cdots-k_{m-1}+1}\left(\frac{x_{m+1}^{\ell-m}}{(x_{m+1}-x_{\ell+1})\cdots(x_{m+1}-x_m)}+\cdots+\frac{x_{\ell+1}^{\ell-m}}{(x_{\ell+1}-x_\ell)\cdots(x_{\ell+1}-x_m)}\right)$$
$$+\frac{x_{m+1}^{k-(m-1)-k_1-\cdots-k_{m-1}}}{(x_{m+1}-x_{\ell+1})\cdots(x_{m+1}-x_m)}+\cdots+\frac{x_{\ell+1}^{k-(m-1)-k_1-\cdots-k_{m-1}}}{(x_{\ell+1}-x_\ell)\cdots(x_{\ell+1}-x_m)}. \quad (B2)$$

The quantity in the large parentheses in the first line of the rhs can be written as

$$\frac{x_{m+1}^{\ell-m}}{(x_{m+1}-x_{\ell+1})\cdots(x_{m+1}-x_m)}+\cdots+\frac{x_{\ell+1}^{\ell-m}}{(x_{\ell+1}-x_\ell)\cdots(x_{\ell+1}-x_m)}$$

$$=\frac{1}{\displaystyle\prod_{\ell+1\ge i>j\ge m}(x_i-x_j)}\left(x_{\ell+1}^{\ell-m}\prod_{\substack{i>j\\i,j\ne\ell+1}}(x_i-x_j)-x_\ell^{\ell-m}\prod_{\substack{i>j\\i,j\ne\ell}}(x_i-x_j)+\cdots+(-1)^{\ell-m}x_{m+1}^{\ell-m}\prod_{\substack{i>j\\i,j\ne m+1}}(x_i-x_j)\right). \qquad (B3)$$

Since there can be no singularities at $x_i=x_j$ ($i,j\ne m$) and the quantity in the last large parentheses is a polynomial of $(\ell-m)$th order all in $x_{\ell+1},\dots,x_{m+1}$, one can easily deduce its form as

$$f(x_m)(x_{\ell+1}-x_\ell)(x_{\ell+1}-x_{\ell-1})(x_{\ell+1}-x_{\ell-2})\cdots(x_{\ell+1}-x_{m+1})\times(x_\ell-x_{\ell-1})(x_\ell-x_{\ell-2})\cdots(x_\ell-x_{m+1})\times\cdots\times(x_{m+2}-x_{m+1}). \qquad (B4)$$

The remaining function $f(x_m)$ can be fixed, for example, by the behavior around $x_{\ell+1}=x_m$, to be $f(x_m)=(-1)^{\ell-m}x_m^{\ell-m}$. This means that the first term in (B2) is also expressed as

$$-x_m^{k-\ell-k_1-\cdots-k_{m-1}+1}\frac{f(x_m)}{\displaystyle\prod_{\ell+1\ge i>m}(x_i-x_m)}=\frac{x_m^{k-(m-1)-k_1-\cdots-k_{m-1}}}{(x_m-x_{\ell+1})(x_m-x_\ell)\cdots(x_m-x_{m+1})}. \qquad (B5)$$

The validity of (50) is now evident.

APPENDIX C: DOMINANT TERMS IN THE SUMMATIONS (54)

The fact that the dominant contributions in (54) are due, in general, to those terms with the same set of indices $\{ij\}=\{a_1b_1\}=\{a_2b_2\}=\cdots=\{a_{\ell+1}b_{\ell+1}\}$ is understood by observing that the expression (54) is nothing but the result of the summations of the terms with different powers of complex numbers and that no cancellations are possible among those terms with no relative phase, while strong cancellations are expected to occur among other terms with different phases. Indeed, (54) is nothing but the result of the summations of the form

$$\sum_{k=\ell}^{N-1}\sum_{k_1=0}^{k-\ell}\sum_{k_2=0}^{k-\ell-k_1}\cdots\sum_{k_\ell=0}^{k-\ell-k_1-\cdots-k_{\ell-1}}x^{N-1-k}x_1^{k_1}\cdots x_\ell^{k_\ell}x_{\ell+1}^{k-\ell-k_1-\cdots-k_\ell}$$

$$=\sum_{k=\ell}^{N-1}\sum_{k_1=0}^{k-\ell}\sum_{k_2=0}^{k-\ell-k_1}\cdots\sum_{k_\ell=0}^{k-\ell-k_1-\cdots-k_{\ell-1}}x^{N-1-\ell}(x_1/x_{\ell+1})^{k_1}\cdots(x_\ell/x_{\ell+1})^{k_\ell}(x_{\ell+1}/x)^{k-\ell}$$

$$=\sum_{k'=0}^{N-1-\ell}\sum_{k_1'=0}^{k'}\sum_{k_2'=0}^{k_1'}\cdots\sum_{k_\ell'=0}^{k_{\ell-1}'}x^{N-1-\ell}(x_1/x)^{k'}(x_2/x_1)^{k_1'}\cdots(x_{\ell+1}/x_\ell)^{k_\ell'}, \qquad (C1)$$

the absolute value of which is expected to be maximum when all complex variables $x,x_1,\dots,x_{\ell+1}$ are in phase. Since in the present case the absolute values of $\Lambda_{a_nb_n}$'s are bounded $|\Lambda_{a_nb_n}|\le|\Lambda_{11}|=|\lambda_1|^2$ (we assume that there is no degeneracy $|\lambda_1|>|\lambda_2|$ and that the eigenvalues λ_i have a nonvanishing relative phase), dominant contributions come from those terms with $x_1=\cdots=x_{\ell+1}=\Lambda_{11}$, when $x=\Lambda_{11}$ or $x=\Lambda_{22}$, and otherwise with $x_1=\cdots=x_{\ell+1}=x$. We can thus easily extract the dominant contributions. Notice that when $x=x_1=\cdots=x_{\ell+1}$, the above summations,

$$\sum_{k=\ell}^{N-1}\sum_{k_1=0}^{k-\ell}\cdots\sum_{k_\ell=0}^{k-\ell-k_1-\cdots-k_{\ell-1}}=\sum_{k=0}^{N-1-\ell}\sum_{k_1=0}^{N-1-\ell-k}\cdots\sum_{k_\ell=0}^{N-1-\ell-k-k_1-\cdots-k_{\ell-1}}=\sum_{\substack{0\le k,k_1,\dots,k_{\ell+1}\le N-1-\ell\\k+k_1+\cdots+k_{\ell+1}=N-1-\ell}}\cdots\sum, \qquad (C2)$$

express nothing but the number of distributions of $N-1-\ell$ identical balls into $\ell+2$ boxes allowing empty boxes, that is, $_NC_{\ell+1}$. Furthermore, the $k=\ell$ case is exceptional for in this case the summations over all a_i and b_j in (49) are trivially done.

[1] See, for example, B. d'Espagnat, *Conceptual Foundations of Quantum Mechanics* (Benjamin, New York, 1971); in *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, NJ, 1983); P. Busch, P. J. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement* (Springer-Verlag, Berlin, 1991); M. Namiki and S. Pascazio, Phys. Rep. **232**, 301 (1993).

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[12] This assumption is not restrictive, since we can always prepare the total system in this class of states by performing a measurement on X to confirm it in the state $|\uparrow\rangle_X$.

[13] The condition $\Omega = \omega$ is readily and naturally follows, if, for instance, the two qubits are realized by two spins with the same magnetic moment. Another example would be given by two identical atoms in a linear trap.