Efficiency of higher-dimensional Hilbert spaces for the violation of Bell inequalities

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We have determined numerically the maximum quantum violation of over 100 tight bipartite Bell inequalities with two-outcome measurements by each party on systems of up to four-dimensional Hilbert spaces. We have found several cases, including the ones where each party has only four measurement choices, where two-dimensional systems, i.e., qubits, are not sufficient to achieve maximum violation. In a significant proportion of those cases when qubits are sufficient, one or both parties have to make trivial, degenerate "measurements" in order to achieve maximum violation. The quantum state corresponding to the maximum violation in most cases is not the maximally entangled one. We also obtain the result that bipartite quantum correlations can always be reproduced by measurements and states which require only real numbers if there is no restriction on the size of the local Hilbert spaces. Therefore in order to achieve maximum quantum violation on any bipartite Bell inequality (with any number of settings and outcomes), there is no need to consider complex Hilbert spaces.

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I. INTRODUCTION

One of the most astonishing features of quantum-mechanics is its nonlocal nature. Separated observers sharing an entangled state and performing measurements on them may induce nonlocal correlations which violate Bell inequalities [1,2]. In contrast, separable states satisfy all the possible Bell inequalities with any measurement settings.

A general setting concerning Bell inequalities is that measurements are made on a system that is decomposed into N subsystems. On each of these subsystems one out of m_i, $i=1,\ldots,N$ observables is measured, producing k_i, $i=1,\ldots,N$ outcomes each. In almost all the cases investigated up to now in order to maximally violate them the dimension of the quantum states did not have to be larger than the number of outcomes of the respective parties. A notable exception to it is the bipartite $k_A=3$ and $k_B=2$ Bell inequality in Ref. [3], which is maximally violated by maximally entangled qutrits. Thus for maximum violation this inequality needs Hilbert spaces of dimension larger than the number of outcomes on Bob's side (but equal to the number of outcomes on Alice's side). In this respect Gill went further and raised the question [4] whether there exist Bell inequalities with d outcomes on both sides which are maximally violated by higher than d-dimensional quantum states. Indeed, this question has already been answered independently by Refs. [5,6] in the case of two parties and in Ref. [7] in the tripartite case proving the existence of such Bell inequalities for $d=2$. In particular in Ref. [5] an explicit correlation Bell inequality was presented with binary outcomes [5], and with measurement settings $m_A=8$ and $m_B=4$, which requires states of dimension larger than two to obtain maximal violation.

These examples to Gill's problem can be interpreted via the very recent concept of dimension witnesses [6]. Dimension witnesses allow the dimension of the Hilbert space to be measured. A d-dimensional Hilbert space witness is an inequality whose violation by correlations of measurement results on members of composite systems signals that the Hilbert space dimension of the component systems is larger than d. From a Bell inequality whose maximum violation may not be achieved by qubits a two-dimensional witness follows. Whenever the violation exceeds the maximum value obtainable with qubits, one can be sure that the dimension of the Hilbert spaces of the systems measured is larger than two.

In the present numerical investigation of many two-outcome two-party Bell inequalities our aim is twofold. First, we wish to find further examples of two-dimensional, or even higher dimensional Hilbert space witnesses. Including marginal probabilities in the Bell inequalities this can be achieved with a smaller number of measurement settings than the $m_A=8$ and $m_B=4$ case of Ref. [5]. Then we also show that any bipartite Bell inequality can be violated with settings and states in the real Hilbert space in the same extent as with settings and states in the complex Hilbert space.

Actually, we believe that these results are not only of academic interest: On one hand, higher dimensional systems have been produced in the laboratory in a number of schemes, subjected to Bell-type tests as well. In particular in Ref. [8] the experimental violation of a spin-1 Bell inequality has been presented using four-photon states, while in Refs. [9,11] Bell-type tests based on the inequality of Collins *et al.* [10] have been performed for orbital angular momentum and energy-time entangled photons producing qutrits, respectively. Also, two-photon interference experiments have demonstrated time-bin entanglement up to $d=20$ dimensionality [12]. On the other hand, in quantum information theory the dimension of the Hilbert space can be seen as a resource. The fact that higher dimensions may allow stronger correlations, and the possibility of testing dimensionality by dimension witnesses may prove to be important, and may allow practical applications in quantum information protocols, and specifically in quantum cryptography [13]. The relevance of these matters in quantum key distribution has been pointed out in Ref. [6]. The security of most known protocols may depend crucially on the ability to test Hilbert space dimen-

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sion, an example is shown in Ref. [14], although this problem may be overcome by device-independent quantum cryptography [15].

In particular, in this paper we consider tight bipartite two-outcome Bell inequalities corresponding to the facets of the local polytope [16] with up to five settings 2–89 of Ref. [17], and the 31 cases with up to four settings considered by Brunner and Gisin [18]. We note that there is some overlap between the two lists. We used projective measurements in all cases, since for binary outcomes it has been shown [19] that general positive operator-valued measure (POVM) measurements are never relevant. The tools used in the numerical exploration are gathered in Sec. II, then in Sec. III we give a list of tables presenting the numbers corresponding to the maximum quantum violations in cases of real and complex qubits (three-dimensional spaces) and real qutrits, taking into account degenerate measurements as well. For all but two inequalities we considered such component spaces were sufficient for maximum violation. In one case complex qutrits, and in one case real ququarts (four-dimensional spaces) were necessary to achieve the maximum violation. For both cases the gain was marginal, not much larger than numerical uncertainty. The numbers obtained are discussed in Sec. III, and some conclusions are given in Sec. IV. Finally, in the Appendix we provide a proof on the equivalence of real and complex Hilbert spaces in reproducing bipartite quantum correlations if there is no constraint on the size of the component Hilbert spaces.

II. METHOD

The quantum value of the expression in the Bell inequality is an expectation value of a Hermitian operator. The maximum expectation value of such an operator is its largest eigenvalue. Therefore to find the maximum quantum violation we have to find those measurement operators for both Alice and Bob whose combination as it appears in the inequality gives the largest possible eigenvalue [20]. This way the parameters to be optimized are those of the measurement operators, no parameter of the vector enters the problem. The vector can be determined as the eigenvector belonging to the maximum eigenvalue.

As the outcome of each measurement has to be either 0 or 1, the measurement operators to be considered are projectors in the component Hilbert spaces of Alice and Bob. In the case of two-dimensional Hilbert spaces each nondegenerate measurement operator projects to a one-dimensional subspace, which may be defined by a unit vector $|m\rangle$ of irrelevant phase as $|m\rangle\langle m|$. Such a vector can be characterized by two parameters, it is convenient to use the two angles on the Bloch sphere. As it turned out to be essential, we also considered trivial, degenerate measurement operators as well. Such a measurement, represented by the zero and the unit operator always brings the result 0 and 1, respectively, therefore it need not be performed at all. If some of the measurements to achieve the maximum quantum value are degenerate, then another Bell expression with less measurement settings will have the same maximum quantum value. We can get that by replacing the operators of those measurements by their definite outcomes.

We performed the optimization with all combinations of nondegenerate, zero, and unit operators. For three-dimensional spaces a nondegenerate measurement operator is either a one- or a two-dimensional projector. A unit vector of irrelevant phase is again sufficient to define either a one- and a two-dimensional projector as $|m\rangle\langle m|$ and $I-|m\rangle\langle m|$, respectively. Four real parameters, for example, the two polar angles and the phases of two components (one component may be chosen real) are needed to characterize such a three-dimensional complex vector. Although we have considered only nondegenerate operators, as each of them may be either a one- or a two-dimensional projector, many optimization runs are necessary to cover all combinations. In the case of four-dimensional component spaces we confined ourselves to two-dimensional projection operators. To make the optimization of the many parameters involved for all combinations of the dimensions of the operators would have taken too much computer time. A two-dimensional projector in a four-dimensional complex space requires eight real parameters to define.

We may reduce the number of parameters involved by using the fact that both Alice and Bob may choose their bases freely. With an appropriate unitary operation we may transform one of the operators, say the first one, into a diagonal form. This eliminates all parameters of that operator. Then we may apply another unitary operator that does not affect the matrix of the first operator to simplify the matrix of the second operator as much as possible. If there exists further transformation that leaves the first two matrices unchanged, it may be used to reduce the number of parameters of the third operator, and so on. Following this recipe, for qubit spaces the vector characterizing the first (nondegenerate) operator will be one of the basis vectors (no parameter), while the one corresponding to the second operator may be transformed to have both components real (one parameter).

In a three-dimensional Hilbert space the components of a unit vector may be parametrized as $(\cos\varphi\sin\vartheta e^{i\alpha},\sin\varphi\sin\vartheta e^{i\beta},\cos\vartheta)$, the third component is chosen real (four parameters). The vector corresponding to the first measurement operator may be transformed to (0,0,1) (no parameter). This form is invariant to a unitary transformation of the u_{12} type (operation within the subspace spanned by the first two basis vectors). With such an operation we may eliminate the second component of the second vector, and we also make its first component real, leaving the form $(\sin\vartheta_2,0,\cos\vartheta_2)$ (one parameter). After this we still have the freedom to eliminate the phase of the second component of the third vector.

In the case of four-dimensional Hilbert spaces, the first measurement operator may be diagonalized to have the form $\mathrm{diag}(1,1,0,0)$. Then we may apply a further transformation of the form $u_{12}u_{34}$ to simplify the second operator. We can obviously diagonalize the two 2×2 blocks in the upper left and the lower right corners. Then using the fact that the matrix corresponds to a two-dimensional projector, it can be shown that the rest of the transformed matrix must also have a special form, which with a further allowed operation may be simplified to the two-parameter form of $(\mathbb{1}+\mathcal{H})/2$, where $\mathbb{1}$ is the unit matrix, and

$$\mathcal{H} = \begin{pmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & \cos\psi & 0 & \sin\psi \\ \sin\phi & 0 & -\cos\phi & 0 \\ 0 & \sin\psi & 0 & -\cos\psi \end{pmatrix}.$$

This has been shown in Ref. [21]. The first two matrices leave no further room to simplify the third and any further operators, it will take eight parameters to characterize each of them. We have chosen those parameters by using the fact that the matrix of the most general two-dimensional projector in the four-dimensional space may be produced by applying the most general transformation of the form $u_{12}u_{34}$ to the two-parameter matrix above. Each of the two-dimensional unitary operators u_{12} and u_{34} have four parameters. However, an overall phase is irrelevant, and it also turns out that the effect of the transformation to the special form will only depend on the difference of two phase angles in the operators, which makes it possible to eliminate one more parameter, leaving altogether just the necessary number of $2+(2\times 4-2)=8$ parameters.

We determined the maximum violation with both complex and real Hilbert spaces. A measurement operator in the real space needs just half as many real parameters to characterize as in a complex space of the same number of dimensions. The parameters we used were the same as in the complex space with all phase angles taken to be zero. For optimization we applied an uphill simplex method [22], namely the routine AMOEBA from Ref. [23]. As such a method climbs to a local maximum, to find the global one we restarted the method from random positions many times, at least 10 000 times for the 4×4 dimensional Hilbert spaces. We still cannot be sure that we have found all global optima, especially for the largest cases with five measurement settings for each party. Nevertheless, the results calculated with spaces of different dimensions are fully consistent with each other. Either with complex or real spaces, a higher dimensional calculation has always given at least as large a violation as the lower dimensional ones. When we managed to find a larger value, some optimization runs still ended up with the lower dimensional result. From properties of the optimum in the higher dimensional case, namely the number of terms in the Schmidt decomposition of the eigenvector, and the relation of the subspace defined by the Schmidt decomposition to the measurement operators may reveal if it actually corresponds to a lower dimensional case. The four-dimensional calculations can and do reproduce all the lower dimensional results we considered, including the two-dimensional cases with degenerate operators. When the Schmidt decomposition shows that the eigenvector occupies only two-dimensional subspaces of Alice and Bob's component spaces, and there are measurement operators that project to exactly those subspaces, or to their complementary space, then the outcome of those measurements performed on the eigenstate will always be definite (one or two, respectively). Such measurements on the eigenstate behave like degenerate ones. Actually, we realized from such analysis that in most cases when we found a larger violation with ququarts than with qubits, the higher dimensionality was not essential, the same violation could be achieved with qubits by choosing the operators above degenerate. In their recent paper Brunner and Gisin also concluded that for one of their cases they needed degenerate [18] measurements. The four-dimensional calculation reproduces the three-dimensional results too, and may even reveal which measurement operators should be two and which ones should be one-dimensional projectors for maximum violation. In the former case the two-dimensional subspace onto which the measurement operator projects lies within the three-dimensional subspace defined by the three-term Schmidt decomposition, while in the latter case their intersection is one-dimensional, and the vectors from the subspaces orthogonal to the intersection are orthogonal to each other.

III. DISCUSSION OF THE RESULTS

We calculated the maximum violation of the tight bipartite Bell inequalities A_2-A_{89} listed in Ref. [17]. (A_1 is a trivial one, which cannot be violated). These inequalities are the part involving at most five measurement settings per party of a huge list of inequalities obtained with the method described in Ref. [24]. We also included the 31 known tight inequalities with up to four measurement settings per party considered recently by Brunner and Gisin [18]. We adopted the notation used in that paper. Out of the 26 inequalities of 4422 type (four measurement settings for Alice, four for Bob, with two outcomes for Alice and two for Bob), 20 were newly introduced there, while I^1_{4422} was presented in [25], I^2_{4422} in [27], A_5, A_6, AII_1, and AII_2 in [28], while AS_1 and AS_2 in [29]. The only 2222 one is the Clauser-Horne-Shimony-Holt (CHSH) inequality [2]. The Bell inequality found in Ref. [30] is the only 3322 type. Actually, more than 600 such inequalities were introduced there, but later they all proved to be equivalent [25,26]. The three 4322 cases were introduced in Ref [25]. The two lists we considered have some overlap, we marked those cases in our tables. The quantum violations shown in the tables are the differences of the quantum values (the maximum eigenvalues we got) and the classical limits. For all but one inequality this means the quantum value itself, as the classical limit is zero, while for I^7_{4422} it is one.

In Table I we listed all those cases for which we could not find a stronger violation in any of our calculations than the maximum violation we achieved with real qubits, performing only nondegenerate measurements. In all tables we marked with an asterisk the cases when maximum violation was achieved with the maximally entangled state. For most instances this is not so, which has also been noted in Ref. [18]. Table II contains the inequalities when we got the maximum violation with measurements on complex qubits. For the cases in these tables we got the same values for maximum violation with complex qutrits and complex ququarts than with complex qubits, and real qutrits did as well as real qubits. However, with real ququarts we could always achieve the same amount of violation as with complex qubits. It is generally true that if a bipartite Bell inequality with arbitrary outputs per party can be violated by a certain amount with projective measurements in n-dimensional Hilbert spaces, than they can be violated by at least as much with projective

TABLE I. Maximum quantum violation of Bell inequalities calculated with real qubit component spaces, with nondegenerate measurements. Higher dimensional spaces have given no larger violation for these cases. Entries when maximum violation is achieved by the maximally entangled state are marked by asterisks.

Case	Type	Qubit Real	Case	Type	Qubit Real
CHSH(A_2)	2222	0.207107*	A_{27}	5522	0.648307
$I_{3322}(A_3)$	3322	0.250000*	A_{28}	5522	0.640314*
I_{4322}^3	4322	0.436492*	A_{30}	5522	0.569821
I_{4422}^2	4422	0.621371	A_{31}	5522	0.573817
A_5	4422	0.435334	A_{35}	5522	0.624908
AS_1	4422	0.541241*	A_{40}	5522	0.607864
AS_2	4422	0.878493*	A_{42}	5522	0.619865
AII_1	4422	0.605554	A_{43}	5522	0.610765
AII_2	4422	0.500000*	A_{51}	5522	0.660781
I_{4422}^5	4422	0.436492*	A_{52}	5522	0.621861
I_{4422}^9	4422	0.461684	A_{53}	5522	0.638610
I_{4422}^{10}	4422	0.613946	A_{54}	5522	0.593681
I_{4422}^{11}	4422	0.638354	A_{57}	5522	0.660344
I_{4422}^{12}	4422	0.618814	A_{58}	5522	0.648890
I_{4422}^{17}	4422	0.671409	A_{72}	5522	0.696282
A_{10}	5422	0.415390	A_{74}	5522	0.689069
A_{22}	5422	0.623457	A_{77}	5522	0.665558
A_{24}	5522	0.604799	A_{78}	5522	0.892702
A_{25}	5522	0.603379			

measurements in $2n$-dimensional real Hilbert spaces. This property is an immediate consequence of an even more general statement, which is provided in the Appendix. It is an open question whether Lemma A.1 could be somehow generalized so that this statement would be true for any multipartite Bell inequalities as well. From the construction it follows, and we have demonstrated in the Appendix, that the Schmidt decomposition of the state in the four-dimensional real space has four terms, the Schmidt coefficients are pairwise equal, and the ratio of the pairs equals the ratio of the Schmidt factors from the qubit case with the same violation.

There are surprisingly many inequalities that can be violated more, sometimes very significantly more by allowing measurements to be degenerate, than by confining ourselves only to nontrivial ones. Tables III and IV show the cases when we got the maximum violation with real and complex qubits, respectively, taking one or more measurements of Alice, or Bob, or of both of them degenerate, i.e., either unity or zero. As we have already mentioned, the four-dimensional calculations can always reproduce these values even by confining ourselves to rank two measurements (two-dimensional projectors) by operators that project onto the subspace that the eigenvector occupies, or onto the orthogonal one. However, when a complex qubit result is reproduced with real ququarts, the eigenvector requires the whole component spaces (four terms in Schmidt decomposition), therefore the effect of degenerate operators cannot be simulated with rank two operators this way.

TABLE II. Maximum quantum violation is reached with complex qubits, no degenerate measurements.

Case	Type	Qubit Real	Qubit Complex
I_{4422}^6	4422	0.414214*	0.449490*
I_{4422}^7	4422	0.441730	0.454837
A_8	5422	0.555704*	0.591650*
A_9	5422	0.451695	0.465243
A_{11}	5422	0.445211	0.456108
A_{12}	5422	0.452098	0.487709
A_{15}	5422	0.447760	0.449628
A_{19}	5422	0.588932	0.622630
A_{20}	5422	0.564956	0.602240
A_{23}	5522	0.528521	0.546073
A_{26}	5522	0.486495	0.527555
A_{29}	5522	0.456259	0.492064
A_{32}	5522	0.396861	0.413553
A_{33}	5522	0.561909	0.622631
A_{36}	5522	0.419088	0.438868
A_{37}	5522	0.456106	0.486887
A_{38}	5522	0.428958	0.469913
A_{39}	5522	0.612269	0.617203
A_{41}	5522	0.419234	0.478563
A_{47}	5522	0.402679	0.460854
A_{48}	5522	0.431439	0.454841
A_{49}	5522	0.454198	0.466694
A_{50}	5522	0.500000*	0.518290
A_{79}	5522	0.606128	0.624315
A_{81}	5522	0.662368	0.669010
A_{83}	5522	0.696038	0.696166
A_{85}	5522	0.610060	0.641141
A_{86}	5522	0.780438	0.800443

Brunner and Gisin [18] calculated the maximum quantum violation by applying degenerate measurements only for their I_{4422}^4 inequality. They did that after realizing that this inequality cannot be violated by the maximally entangled state without such measurements. They state $(1/\sqrt{2}-1/2)$ as the value of maximum quantum violation, which they achieved by taking two measurement operators of both parties degenerate. We found twice as large maximum violation by taking two measurement operators of only one party degenerate (see Table III). We also found that a very small violation may be achieved by using only true two-outcome measurement. The violating state is far from the maximally entangled state, it has Schmidt coefficients of 0.9158 and 0.4016.

So far we have only shown cases for which maximum violation could be achieved in qubit spaces. The existence of Bell inequalities for which this is not the case has been proved in Refs. [5–7]. Particularly, in Ref. [5] we were able to give concrete examples of correlation Bell inequalities (i.e., inequalities without local marginals) whose maximal violation is not achieved by qubits. In the present list we

TABLE III. Maximum quantum violation is reached with real qubits, with some measurement operators degenerate.

Case	Type	Qubit Real Nondegenerate operators	Qubit Complex Nondegenerate operators	Qubit Real Degenerate operator allowed
I^1_{4322}	4322	0.154701	0.236068	0.414214*
$I^2_{4322}(A_4)$	4322	0.231613	0.259587	0.299038*
A_6	4422	0.222941	0.232051*	0.299038*
I^3_{4422}	4422	0.238042	0.238042	0.414214*
I^4_{4422}	4422	0.055979	0.055979	0.414214*
I^{13}_{4422}	4422	0.249466	0.250000*	0.434855
I^{14}_{4422}	4422	0.407621	0.410296	0.479410*
I^{15}_{4422}	4422	0.238273	0.250000*	0.434855
I^{16}_{4422}	4422	0.240659	0.240659	0.414214*
A_{17}	5422	0.221946	0.221946	0.375447
A_{18}	5422	0.210377	0.212229	0.384355
A_{34}	5522	0.461083	0.513972	0.535012
A_{44}	5522	0.500000*	0.533925	0.536494
A_{55}	5522	0.451941	0.486823	0.621320*
A_{56}	5522	0.675426*	0.675426*	0.689312*
A_{59}	5522	0.430220	0.430220	0.448826
A_{63}	5522	0.327627	0.327627	0.479410*
A_{69}	5522	0.330388	0.330388	0.609610
A_{70}	5522	0.465198	0.465198	0.605223
A_{71}	5522	0.418729	0.418729	0.449016
A_{73}	5522	0.800326	0.852797	0.883138
A_{75}	5522	0.572736	0.587052	0.605151
A_{80}	5522	0.136376	0.174354	0.375447
A_{82}	5522	0.314943	0.314943	0.454573
A_{84}	5522	0.605340	0.619437	0.623457
A_{88}	5522	0.076842	0.076842	0.414214*

found numerically quite a few such cases, now for Bell expressions with marginals. In all such cases except for two, real qutrit spaces were enough for maximum violation, see Table V. For most of them, in two dimensions larger violation can be achieved by allowing degenerate operators than by not allowing them (no entry in the appropriate place when it is not so). With qutrits we can do even better. However, for most entries in the list the increase is quite small, no more than a couple of percents, sometimes even much less, which means these cases may have no practical and experimental relevance. For a few cases the gain is more than 10%. We find the largest increase (about 0.1, or 18%) for I^{18}_{4422}. It is interesting to note that there exist Bell inequalities that can be violated more with real qutrits than with complex qubits, and there are also examples for the opposite (at least without allowing degenerate measurements for qutrits, which we

TABLE IV. Maximum quantum violation is reached with complex qubits, with some measurement operators degenerate.

Case	Type	Qubit Real Nondegenerate operators	Qubit Complex Nondegenerate operators	Qubit Real Degenerate operator allowed	Qubit Complex Degenerate operator allowed
A_{16}	5422	0.416036	0.416036	0.446167	0.457107*
A_{45}	5522	0.482065	0.509936	0.534037	0.537239
A_{61}	5522	0.307654	0.307654	0.395168	0.401925
A_{62}	5522	0.219048	0.231812	0.395168	0.401925
A_{66}	5522	0.345116	0.360817	0.452098	0.487709

TABLE V. Maximum quantum violation is reached with real qutrits.

Case	Type	Qubit Real Nondegenerate operators	Qubit Complex Nondegenerate operators	Qubit Real Degenerate operator allowed	Qubit Complex Degenerate operator allowed	Qutrit Real
$I^1_{4422}(A_7)$	4422	0.197048	0.197048	0.250000*	0.250000*	0.287868
I^8_{4422}	4422	0.420651	0.420651	0.484313*	0.484313*	0.487768
I^{18}_{4422}	4422	0.181236	0.181236	0.543599	0.543599	0.642967
I^{19}_{4422}	4422	0.369700	0.430724*	0.443587	0.443587	0.497171
I^{20}_{4422}	4422	0.305645	0.305645	0.434324	0.434324	0.449669
A_{13}	5422	0.397412	0.403098	0.414214*	0.414214*	0.419982
A_{14}	5422	0.449958	0.453901	0.452465		0.464584
A_{46}	5522	0.446602	0.449849			0.458105
A_{60}	5522	0.252968	0.252968	0.375447	0.375447	0.390611
A_{64}	5522	0.375234	0.375234	0.375447	0.375447	0.390089
A_{65}	5522	0.208545	0.208545	0.347759*	0.353146	0.355021
A_{67}	5522	0.395696	0.395696			0.396289
A_{68}	5522	0.385731	0.385731			0.395718
A_{76}	5522	0.404741	0.415397	0.447555	0.447555	0.489863
A_{89}	5522	0.131420	0.131420	0.250000*	0.250000*	0.288932

have not tried). For all cases in Table V each party has at least four measurements, in the smallest ones each of them has just four. We will show in a forthcoming paper that for correlation type inequalities to get larger violation with higher-dimensional spaces than with qubits, one of the parties must have at least four measurements, and then the other one must have at least seven measurements. All 4422, 5422, and 6422 correlation type Bell inequalities can maximally be violated by qubits.

We found one single inequality in the list that we could violate more with complex qutrits than with real ones or with qubits. The maximum violation of A_{21} (5422) with real qubits (no degenerate measurement) is 0.099 090, with complex qubit (no degenerate measurement) 0.125 000, with real and complex qubit (degenerate measurement allowed) 0.299 038 (maximally entangled state), with real qutrit 0.316 523, and with complex qutrit 0.317 496. The last improvement is very small, but it does not seem to be due to numerical error.

For A_{87} (5522) we found that one needed ququarts to get maximum violation, but the improvement was even less convincing. The best qubit value is 0.756 199 (both with real and complex qubit), while the maximum we got with both real and complex ququarts is 0.756 247. From a more detailed analysis of the solution we could not see a way to reduce it to a lower dimensional space. It turned out that this violation could be achieved by taking two measurement operators equal. Therefore we calculated the maximum violation with qubits of the 5422 inequality we got by uniting these two measurements, and we found 0.755 931, a slightly smaller value than for the original inequality. The difference from the ququart value is still extremely small, but at least it seems to be more than numerical error.

In our calculations the maximum number of dimensions for the component spaces was four. Moreover, we allowed degenerate measurements only for qubit spaces, and confined ourselves to rank two measurements in four dimensions. For some cases on the list it is possible that without these restrictions one could find a larger maximum quantum violation. We may also have missed the true global optimum in higher dimensions for some examples due to limitations of the optimization algorithm we have chosen. However, we are confident that this has not happened in qubit spaces. Then the optimization involves much less parameters, and the highest eigenvalue has been found in a sizeable proportion of the optimization runs, altogether many times for each inequality we considered. We also got the same results many more times from higher dimensional runs. Whenever we got a larger eigenvalue, it belonged to an eigenstate with a Schmidt decomposition of more than two terms. Therefore our result that there are inequalities among the ones we considered in this paper that can be more violated in higher dimensional spaces than by qubits is valid.

IV. SUMMARY

Let us briefly summarize the main results achieved in this work. We investigated numerically the maximum quantum values for tight bipartite two-outcome two-party Bell inequalities with the local Hilbert spaces restricted to $d=2$, 3, and 4 dimensions. We found that some of them with at least four measurement settings on each side require qutrits, and one with five settings on each side requires ququarts to be maximally violated. By interpreting these results via the concept of dimension witnesses [6,7], we found several inequalities which act as a two-dimensional Hilbert space witness, and we also found a three-dimensional Hilbert space witness. Let us stress that although these witnesses are results of heuristic numerical investigations, we are confident about their validity. On the other hand, in analogy to the terminology *dimension* witnesses one may inquire whether *reality* witnesses could be constructed, which would be able to distinguish complex Hilbert spaces from real ones. Actually, the existence of such kind of a witness has been quested by Gisin in Ref. [29]. However, according to our results presented in the Appendix, we may safely say that no reality witness can be constructed for the case of two parties since by doubling the size of the local complex Hilbert space of each party one may reconstruct all the joint probabilities with local real Hilbert spaces as well. Although the question remains also open for the multipartite case, numerical study supports us in believing that our Lemma holds for the most general case as well.

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APPENDIX: ON THE EQUIVALENCE OF REAL AND COMPLEX HILBERT SPACES IN REPRODUCING BIPARTITE QUANTUM CORRELATIONS

Here the following main result is shown.

Lemma A.1. Joint probabilities between two separated observers which has quantum origin can always be reproduced by measurements and states which require only real numbers.

This fact, which is interesting by its own, has some striking consequences, an immediate one is that the maximum quantum violation of any bipartite Bell inequality (with any number of settings and outcomes) can be achieved in the real Hilbert space as well.

To set the scene, we assume that two separated observers, Alice and Bob, may perform one of a finite number of measurements, and that each measurement has a certain number of outcomes. We label outcomes corresponding to different measurements distinctly, so that each outcome a and b is uniquely associated to a single measurement of Alice and Bob, respectively. Let S_A and S_B be n-dimensional complex Hilbert spaces of the two parties, respectively, and $|V\rangle$ be any vector in the tensor product space $S_A \otimes S_B$. Let $P_a(P_b)$ be projection operator associated with outcome $a(b)$ of $S_A(S_B)$.

In the light of the above definitions, we say that the joint probabilities p_{ab} admit a quantum representation [31] if there exists a quantum state ρ on the composite Hilbert space, a set of projectors $P_a \otimes \mathbb{1}$ of Alice's and a set of projectors $\mathbb{1} \otimes P_b$ of Bob's system, such that

$$p_{ab} = \mathrm{Tr}(P_a P_b \rho). \tag{A1}$$

Note that since we do not impose any limitation on the dimension of the local Hilbert spaces, we may consider projection operators instead of the more general POVM measurements. The Bell expression consists of a linear combination of probabilities (A1). The projectors belonging to different outcomes of a measurement are orthogonal to each other, and they sum up to unity.

First we prove the following correspondence between joint distributions arising from projection measurements in complex n-dimensional local Hilbert spaces and projection measurements in real $2n$-dimensional local Hilbert spaces.

Lemma A.2. There exist projection operators P'_a and P'_b of the $2n$-dimensional real spaces S'_A and S'_B, respectively, and $|V'\rangle \in S'_A \otimes S'_B$ such that the corresponding expectation values are equal, i.e.,

$$\langle V|P_a \otimes P_b|V\rangle = \langle V'|P'_a \otimes P'_b|V'\rangle, \tag{A2}$$

where the state $|V\rangle$ and operators P_a, P_b are defined above, and $|V'\rangle$, P'_a, and P'_b depend only on $|V\rangle$, P_a, and P_b, respectively.

Proof. Let us use a matrix representation. Let us choose orthonormal bases in each component space, and let the basis in the product space be the basis consisting of the products of the basis vectors of the component spaces. Hence we can write

$$|V\rangle = \sum V_{ij}|v_i^A\rangle|v_j^B\rangle, \tag{A3}$$

and

$$A_{ij} = \langle v_i^A|P_a|v_j^A\rangle, \tag{A4}$$

$$B_{ij} = \langle v_i^B|P_b|v_j^B\rangle, \tag{A5}$$

where the basis vectors $\{|v_i^A\rangle\}_{i=1}^n$ and $\{|v_j^B\rangle\}_{j=1}^n$ span, respectively, Alice and Bob's local state spaces. This way the vectors of the product space will be represented by matrices of two indices. Then the expectation value above can be expressed as

$$\sum_{i,j,k,l} V_{ij}^* A_{ik} B_{jl} V_{kl} = \mathrm{Tr}(AVB^T V^\dagger),$$

where A, B, and V are the matrix representations of P_a, P_b, and $|V\rangle$, respectively. The value of the expression is a real number, as it gives the expectation value of a Hermitian operator in the product space.

Let us consider the following mapping [32]. Let us replace each component $v_i = v_i^R + i v_i^I$ of the n-dimensional complex vector with the two-element real block of (v_i^R, v_i^I), and each component $A_{ij} = A_{ij}^R + i A_{ij}^I$ of a two-index matrix with the 2×2 block of

$$\begin{pmatrix} A_{ij}^R & -A_{ij}^I \\ A_{ij}^I & A_{ij}^R \end{pmatrix}.$$

One can prove that the image of the product of either a matrix and a vector, or two matrices will be equal to the corresponding product of the images. For $n=1$ this is easy to show. For $n>1$ the multiplication in the $2n$-dimensional space may be done block-by-block, yielding the correct result. The mapping also conserves the linear combinations of both vectors and matrices. When transposing matrices one has to be careful. The image of the transpose of a matrix will be the transpose of the image of the complex conjugate of the matrix. The complex conjugation is needed to get the 2×2 blocks right (they are not transposed in the image of the transpose). Hermitic conjugation is preserved by the mapping. It is also easy to see that the trace operation on the image will give a real number, which is twice the real part of the value calculated for the original complex matrix (in each block the real part of the diagonal matrix element will occur twice, while the imaginary part will be off-diagonal). Given these rules in hand it is clear that the image of a projector is also a projector, the images of orthogonal projectors are orthogonal projectors, and if matrices sum up to unity, their images will do so too. Therefore the images of a set of measurement operators will satisfy the properties required.

Let $|V'\rangle$ be the vector in $S_A'\otimes S_B'$ whose matrix V' is constructed with the above rule for two-index matrices from the matrix V of $|V\rangle$, and then multiplied by $1/\sqrt{2}$ to get it properly normalized. We note that the mapping rule to be applied in the product space is not the same as the one applied in the component spaces. That rule would actually give just $2n^2$ components instead of the $(2n)^2$ ones. Let there be the matrix of P_a' and P_b', i.e., A' and B' the image of A and B^*, respectively. Then $A'V'B'^TV'^\dagger$ will be the image of $(1/2)AVB^TV^\dagger$, the factor of 1/2 is occurring due to the $1/\sqrt{2}$ normalization factor in the construction of V' from V. As the trace of $AVB^TV^\dagger$, which is the expectation value in the complex space, is real, its value is one-half of the trace of its image, i.e., it is equal to the trace of $A'V'B'^TV'^\dagger$, which is the expectation value in the real space. ■

Note that for an arbitrary mixed state $\rho=\Sigma\lambda_i|V_i\rangle\langle V_i|$ the expectation value $\mathrm{Tr}(P_aP_b\rho)$ is the convex sum of the expectations (A2) with coefficients λ_i, which entails the main result Lemma A.1 we wanted to show.

Aside from its conceptual interest, we mention two interesting situations where this fact may prove to be useful beyond justifying our numerical experience that real ququarts could yield at least the same amount of violation as complex qubits. On one hand, in the inequality presented by Bechmann-Pasquinucci and Gisin in Ref. [3] having three and two measurement outcomes per Alice and Bob, respectively, the maximum quantum violation can be achieved with projective measurements sharing a maximally entangled state of dimension three. However, numerical evidence suggests that using measurement settings which require real numbers, the optimum quantum violation could not be reached. It has arisen as a natural question [29] whether a higher value could be achieved by using only real numbers but allowing one to occupy larger Hilbert spaces. Our result gives the answer in negative for this question regarding this particular Bell inequality and also proves conclusively that all bipartite Bell inequalities can be maximally violated by quantum states and measurement settings which need in an appropriate basis only real numbers. This latter problem for the general multipartite case was posed by Gisin (see also problem 32, fundamental questions number 11 in Ref. [33]).

On the other hand, in Ref. [31] a hierarchy of conditions has been formulated through a semidefinite program [34]. This approach can be used, for instance, to obtain upper bounds on the quantum violation of arbitrary Bell inequalities. In this case, however, the matrix Γ in question, which should satisfy the positive semidefinite constraint is in general Hermitian. Our results, however, entail that this matrix needs to be in fact real valued, i.e., it must be a symmetric matrix. This stronger condition thus may provide us with a tighter upper bound on any bipartite Bell inequality than the one which originally required the weaker Hermitian condition.

Now let us illustrate with a simple example, consisting of a qubit at each party, the method of how to obtain the projection operators and the respective states from the original complex valued ones. In this case the state of two qubits can be written in an appropriate basis as $|V\rangle=\alpha|v_1^a\rangle|v_1^b\rangle+\beta|v_2^a\rangle|v_2^b\rangle$, where the α and β Schmidt coefficients are non-negative numbers, their square adding up to 1. Thus the matrix V in Eq. (A3) takes the following simple form, $\mathrm{diag}(\alpha,\beta)$ whereas a nondegenerate projector on the state space of Alice and Bob can be written as $P_\nu=(\mathbb{1}\pm\vec{\nu}\vec{\sigma})/2,\ \nu\in a,b$. Applying the mapping rule, discussed above, we obtain the following real valued 4×4 matrices, $V=(1/\sqrt{2})\mathrm{diag}(\alpha,\alpha,\beta,\beta)$, implying the entangled state (with nonzero α and β) in the four-dimensional state space, $|V\rangle=(\alpha|00\rangle+\alpha|11\rangle+\beta|22\rangle+\beta|33\rangle)/\sqrt{2}$ and the corresponding projection operators $P_\nu'=(\mathbb{1}\pm\vec{\nu}\vec{\sigma}')/2,\ \nu\in a,b$, where $\sigma_x'=\sigma_x\otimes\mathbb{1}$, $\sigma_y'=-\sigma_y\otimes\sigma_y$, and $\sigma_z'=\sigma_z\otimes\mathbb{1}$.

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