SO(4) symmetry in the relativistic hydrogen atom

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We show that the relativistic hydrogen atom possesses an SO(4) symmetry by introducing a kind of pseudospin vector operator. The same SO(4) symmetry is still preserved in the relativistic quantum system in presence of a U(1) monopolar vector potential as well as a non-Abelian vector potential. Lamb shift and SO(4) symmetry breaking are also discussed.

Symmetry principle is one of the cornerstones of modern physics. It has been playing a more and more significant role in theoretical physics since the early 20th century, when Einstein first put it as the primary feature of nature that constrains the allowable dynamical laws [1]. Einstein's profound change of attitude on symmetry principle has made great progress in the study of symmetry. In the latter half of the 20th century it has become the most dominant concept in the exploration and formulation of the fundamental laws of physics, and nowadays it serves as a guiding principle in the search for further unification theory. Dynamical symmetries are prevalent in many important physical models. For instance, in the nonrelativistic quantum mechanics, a three-dimensional harmonic oscillator has a U(3) symmetry, a three-dimensional hydrogen atom has an SO(4) symmetry [2], and the Haldane-Shastry model, which describes the one-dimensional long-range spin-interaction chain, has a Yangian symmetry [3].

It is well-known that the nonrelativistic quantum mechanics is an approximate theory of the relativistic one. This gives rise to a fundamental open question: Suppose a nonrelativistic quantum system possesses a certain dynamical symmetry, when its corresponding relativistic quantum mechanical version is taken into account, will the same symmetry still reside in the system? The harmonic oscillator and hydrogen atom are the two simplest prototype models in quantum physics. Dirac himself has introduced a kind of relativistic version for the quantum mechanical harmonic oscillator with the Hamiltonian $H = \vec{\alpha} \cdot (\vec{p} - i\beta M \omega \vec{r}) + \beta M$, which is now known as the Dirac oscillator [2]. However, so far the full symmetry of the Dirac oscillator has not yet been clear. Very recently, Ginocchio has made remarkable progress [4] by showing that U(3) symmetry does reside in a kind of relativistic harmonic oscillator, whose Dirac Hamiltonian reads $H = \vec{\alpha} \cdot \vec{p} + \beta M + (1+\beta)M\omega^2 r^2/2$, where $\vec{\alpha}$ and β are the Dirac matrices, $\vec{p}$ is the three-dimensional linear momentum, $\vec{r}$ is the spatial coordinate, r its magnitude, M is the mass, ω is the frequency, and the velocity of light and the Planck constant have been set equal to unity, $c = \hbar = 1$. This fact also suggests that Ginocchio's version of the relativistic quantum harmonic oscillator may be a more natural extension than the Dirac oscillator.

Whether the relativistic hydrogen atom (RHA) has an

SO(4) symmetry is still open. The purpose of this Brief Report is threefold. First, we show that there is indeed an SO(4) symmetry in the usual relativistic hydrogen atom by introducing a kind of pseudospin vector operator. Second, we illustrate that the same SO(4) symmetry is still preserved in the relativistic quantum system in presence of a U(1) monopolar vector potential. Third, we find that the relativistic hydrogen atom still possesses an SO(4) symmetry if some kind of appropriate non-Abelian vector potentials are presented. This reflects that the hydrogen atom (or the Kepler system) is a highly symmetric system, whatever in the levels of classical mechanics, quantum mechanics, or even the relativistic quantum theory.

SO(4) symmetry in the usual RHA. The Dirac Hamiltonian for a relativistic hydrogen atom reads

$$H_{rha} = \vec{\alpha} \cdot \vec{p} + \beta M - \frac{a}{r}, \qquad (1)$$

where $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$, $\vec{\sigma}$ is the vector of Pauli matrices, **1** is the 2×2 identity matrix, $a = e^2$ the fine structure constant, and e the electric charge. Its energy spectra is given by the Sommerfeld formula [5]

$$\frac{E}{M} = \left(1 + \frac{a^2}{(n - |\kappa| + \sqrt{\kappa^2 - a^2})^2} \right)^{-1/2},$$

$$|\kappa| = (j + 1/2) = 1, 2, 3, \dots, \quad n = 1, 2, 3, \dots, \qquad (2)$$

where n is the radial quantum number, $\kappa = \pm(j+1/2)$ are eigenvalues of the Dirac's operator $K = \beta(\vec{\Sigma} \cdot \vec{L} + 1)$ with $K^2 = \vec{J}^2 + 1/4$, $\vec{J} = \vec{L} + \vec{S}$ is the total angular momentum, $\vec{L} = \vec{r} \times \vec{p}$ the orbital angular momentum, and $\vec{S} = \vec{\Sigma}/2$ the spin-1/2 angular momentum. In the nonrelativistic limit, the Hamiltonian (1) reduces to the usual nonrelativistic hydrogen atom $H_{nrha} = p^2/2M - a/r$, and the Sommerfeld formula reduces correspondingly to the Bohr formula $E_n = -Ma^2/2n^2 \approx E - Mc^2$.

The Hamiltonian H_{nrha} commutes with $\vec{L}$ and the well-known Pauli-Runge-Lentz vector [6,7], which form a dynamical symmetry group of SO(4). Evidently, the Bohr's energy formula depends only on the principal quantum number n, and it has n^2-fold degeneracies due to the SO(4) symmetry. However, Sommerfeld's energy formula is j-dependent, for fixed n and j, the energy has only $2(2j+1)$-fold degeneracies for $n \neq |\kappa|$ and $(2j+1)$-fold degeneracies for $n = |\kappa|$,

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this does not apparently support that the RHA still possesses an SO(4) symmetry, since any reduction or elimination of degeneracy usually implies broken symmetry. Nevertheless, the Hilbert space of H_{rha} is larger than that of H_{nrha} by considering the additional intrinsic spin space. Therefore it is still possible to restore an SO(4) symmetry for H_{rha} through combining properly the operators of $\vec{r}$, $\vec{p}$, and $\vec{\Sigma}$.

If this is the case, the question is, what are the six relativistic generators? What we need to do first is to find out six linear independent operators (i.e., six integrals of motion for the RHA) that all commute with H_{rha}, and then arrange them to be six generators of the SO(4) group. Up to now, people have known five of them. The first three are three components of the total angular momentum operator $\vec{J}$. The fourth is the Dirac's operator K mentioned above. The fifth integral of motion was discovered by Johnson and Lippmann [8] in 1950, which is now called the Johnson-Lippmann (JL) operator. Such a famous discovery has stirred a great furor, and many people have been attracted in this problem [9–11]. The JL operator reads [11]

$$D = \gamma^5 \vec{\alpha} \cdot \hat{r} - \frac{i}{Ma} K \gamma^5 (H_{rha} - M\beta), \quad (3)$$

with its square is

$$D^2 = 1 + \left(\frac{H_{rha}^2}{M^2} - 1 \right) \frac{K^2}{a^2}, \quad (4)$$

where $\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{\alpha} = \gamma^5 \vec{\Sigma}$, and $\hat{r} = \vec{r}/r$ is a unit vector. The physical significance of the JL operator in the nonrelativistic limit is nothing but a projection of the Pauli-Runge-Lentz vector on the spin angular momentum vector [2,11].

The commutation relations among H_{rha} and these five conserved quantities are $[H_{rha}, \vec{J}] = [H_{rha}, K] = [H_{rha}, D] = 0$, $[\vec{J}, K] = [\vec{J}, D] = 0$, and remarkably $\{K, D\} = KD + DK = 0$, namely K and D are anticommutative. As usual, the simultaneous eigenfunctions of $\{H_{rha}, J^2, J_3\}$ are twofold Krammer's degeneracies, i.e.,

$$|\psi_{njm_j}^+(\vec{r})\rangle = \frac{1}{\sqrt{\mathcal{N}}} \begin{pmatrix} f(r) \phi_{jm_j}^A \\ ig(r) \phi_{jm_j}^B \end{pmatrix},$$

$$|\psi_{njm_j}^-(\vec{r})\rangle = \frac{1}{\sqrt{\mathcal{N}}} \begin{pmatrix} f(r) \phi_{jm_j}^B \\ ig(r) \phi_{jm_j}^A \end{pmatrix}, \quad (5)$$

with $H_{rha}|\psi_{njm_j}^{\pm}\rangle = E|\psi_{njm_j}^{\pm}\rangle$, $\vec{J}^2|\psi_{njm_j}^{\pm}\rangle = j(j+1)|\psi_{njm_j}^{\pm}\rangle$, $J_3|\psi_{njm_j}^{\pm}\rangle = m_j|\psi_{njm_j}^{\pm}\rangle$, and m_j runs from $-j$ to j. Here $\mathcal{N} = \int_0^{+\infty} dr[f^2(r) + g^2(r)]$ is the normalized coefficient, $f(r)$ and $g(r)$ are real functions, $\phi_{jm_j}^A = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l+m+1} Y_{lm}(\theta, \varphi) \\ \sqrt{l-m} Y_{l,m+1}(\theta, \varphi) \end{pmatrix}$, $\phi_{jm_j}^B = \frac{1}{\sqrt{2l+3}} \begin{pmatrix} -\sqrt{l-m+1} Y_{l+1,m}(\theta, \varphi) \\ \sqrt{l+2+m} Y_{l+1,m+1}(\theta, \varphi) \end{pmatrix}$, $Y_{lm}(\theta, \varphi)$ is the spherical harmonics, and $(\vec{\sigma} \cdot \hat{r}) \phi_{jm_j}^A = -\phi_{jm_j}^B$, $(\vec{\sigma} \cdot \hat{r}) \phi_{jm_j}^B = -\phi_{jm_j}^A$. The eigenstates $|\psi_{njm_j}^{\pm}(\vec{r})\rangle$ are distinguished by the Dirac operator as $K|\psi_{njm_j}^{\pm}(\vec{r})\rangle = \pm |\kappa| |\psi_{njm_j}^{\pm}(\vec{r})\rangle$. The existence of the JL operator is the direct reason that causes the twofold Krammer degen-

eracies [8]. Later on we shall show this fact from the viewpoint of the pseudospin operators.

The anticommutativity between operators K and D motivates us to introduce the pseudospin vector operator $\vec{T} = (T_1, T_2, T_3) = (\tau_1, \tau_2, \tau_3)/2$, where

$$\tau_1 = \frac{D}{\sqrt{D^2}},$$

$$\tau_2 = \frac{iDK}{\sqrt{D^2 K^2}},$$

$$\tau_3 = \frac{K}{\sqrt{K^2}}. \quad (6)$$

The operators τ_3 and τ_1 are defined by rescaling K and D such that $\tau_3^2 = \tau_1^2 = 1$. The operator τ_2 is defined by the commutator $[\tau_3, \tau_1] = 2i\tau_2$, or $\tau_2 = i\tau_1 \tau_3$. The vector operator $\vec{\tau}$ plays a similar role as the Pauli matrices vector $\vec{\sigma}$. If $|\psi_{njm_j}^+(\vec{r})\rangle$ is an eigenstate of H_{rha} and τ_3, then $\tau_1|\psi_{njm_j}^+(\vec{r})\rangle$ is also an eigenstate of H_{rha} and τ_3 because of $\tau_1 \tau_3 \tau_1 = -\tau_3$. This is the reason causing the twofold Krammer degeneracies.

It is easy to show that $[T_i, T_j] = i\epsilon_{ijk} T_k$ and $T^2 = \frac{1}{2}(\frac{1}{2} + 1)$. The vector operator $\vec{T}$ has a property like spin-1/2, yet it is not a spin because it contains $\vec{r}$ and $\vec{p}$, consequently we call it a pseudospin-1/2 vector operator. One may have $[J_i, T_j] = 0$, in other words, $\vec{J}$ and $\vec{T}$ are two independent angular momentum vectors that commute with the Hamiltonian H_{hra}. Therefore after making the following simple linear combinations:

$$\vec{I} = \vec{J} + \vec{T},$$

$$\vec{R} = \vec{J} - \vec{T}, \quad (7)$$

one arrives at an SO(4) algebraic relation: $[I_i, I_j] = i\epsilon_{ijk} I_k$, $[I_i, R_j] = i\epsilon_{ijk} R_k$, $[R_i, R_j] = i\epsilon_{ijk} I_k$. Since $[H_{rha}, \vec{I}] = [H_{rha}, \vec{R}] = 0$, this ends the finding of SO(4) dynamical symmetry in the usual RHA.

Full energy spectra have been derived by the well-known ladder-operator procedure [2,5]. The symmetry involved in the system is helpful for us to obtain the energy spectra. For example, in Eq. (4), since D^2 is positively defined, its minimal eigenvalue is zero, then one can get precisely the ground state energy of the hydrogen atom [11] as $E|_{n=\kappa=1} = M(1-a^2)^{1/2}$. Furthermore, with the aid of the pseudospin-1/2 vector operator, one can establish an elegant formula between energy spectra E and the integrals of functions $f(r)$ and $g(r)$. More explicitly, let us denote $|\pm\rangle \equiv |\psi_{njm_j}^{\pm}\rangle$, which are eigenstates of T_3 with eigenvalues equal to $\pm 1/2$. $T_+ = T_1 + iT_2$ is the raising operator $T_+|-\rangle = |+\rangle$, hence $\langle +|T_+|-\rangle = 1$. It is easy to check that $T_+|-\rangle = \frac{D}{\sqrt{D^2}}|-\rangle$ $= \frac{1}{\sqrt{D^2}} \frac{1}{\sqrt{\mathcal{N}}} \begin{pmatrix} -f(r)\phi_{jm_j}^A + \frac{1}{Ma}(E+M)(j+\frac{1}{2})g(r)\phi_{jm_j}^A \\ -\frac{i}{Ma}(E-M)(j+\frac{1}{2})f(r)\phi_{jm_j}^B - ig(r)\phi_{jm_j}^B \end{pmatrix}$. Thus $\langle +|T_+|-\rangle = \frac{-1}{\sqrt{D^2}}(1 - \frac{j+1/2}{Na} \int_0^{+\infty} dr[2f(r)g(r)]) = 1$. Let $\int_0^{+\infty} dr[2f(r)g(r)]/\mathcal{N} = b$, we then obtain an equation $1 + \sqrt{D^2} = (j+1/2)b/a$. By using Eq. (4), we obtain the desired formula

$$\frac{E^2}{M^2} = b^2 - \frac{2a}{j+\frac{1}{2}}b + 1. \qquad (8)$$

The exact solutions of $f(r)$ and $g(r)$ are [12,13]:

$$f(r) = \sqrt{M+E}[-\tilde{n}F(1-\tilde{n}, 2\nu+1, \rho) + (Ma\lambda + \kappa)F(-\tilde{n}, 2\nu$$
$$+ 1, \rho)]\rho^{\nu-1}e^{-\rho/2},$$

$$g(r) = \sqrt{M-E}[-\tilde{n}F(1-\tilde{n}, 2\nu+1, \rho) - (Ma\lambda + \kappa)F(-\tilde{n}, 2\nu$$
$$+ 1, \rho)]\rho^{\nu-1}e^{-\rho/2}, \qquad (9)$$

where $\tilde{n} = n - |\kappa|$, $\nu = \sqrt{K^2 - a^2}$, $\rho = 2r/Ma\lambda$, and $\lambda = 1/\sqrt{M^2 - E^2}$. Substituting Eq. (9) into Eq. (8), one can verify directly the correctness of Eq. (8).

SO(4) symmetry in the RHA with a U(1) monopole. In order to find the reason for the existence of the smallest electric charge, Dirac published a paper in 1931 which started the subject of magnetic monopoles [14]. After 1931, the theory of monopoles has been studied extensively in many literatures, such as [15,16]. The Dirac Hamiltonian of the U(1)-monopolar RHA reads

$$H'_{rha} = \vec{\alpha} \cdot \vec{\pi} + m\beta - \frac{a}{r}, \qquad (10)$$

where $\vec{\pi} = \vec{p} - e\vec{A}$, $\vec{A}$ is the vector potential of a U(1) Wu-Yang monopole with strength g [16], and $\vec{B} = \vec{\nabla} \times \vec{A} = g\frac{\vec{r}}{r^3}$ is the magnetic field satisfying the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$. Similarly, one finds that H'_{rha} possesses an SO(4) symmetry, the corresponding generators are $\vec{I}' = \vec{J}' + \vec{T}'$, $\vec{R}' = \vec{J}' - \vec{T}'$, where $\vec{J}' = \vec{L}' + \vec{S}$, $\vec{L}' = \vec{r} \times \vec{\pi} - q\hat{r}$ is the monopole-dependent orbital angular momentum vector, $q = eg = \frac{1}{2}$integer is the magnetic charge, $\vec{T}' = (D'/\sqrt{D'^2}, iD'K'/\sqrt{D'^2K'^2}, K'/\sqrt{K'^2})/2$, $K' = \beta[\vec{\Sigma} \cdot (\vec{r} \times \vec{\pi}) + 1] = \beta[\vec{\Sigma} \cdot \vec{L}' + q\vec{\Sigma} \cdot \hat{r} + 1]$ is the monopole-dependent Dirac's operator, $K'^2 = \vec{J}'^2 + 1/4 - q^2$, $D' = \gamma^5 \vec{\alpha} \cdot \hat{r} - (iK'/Ma)\gamma^5(H'_{rha} - M\beta)$ is the monopole-dependent JL operator, and $D'^2 = 1 + (H'^2_{rha}/M^2 - 1)(K'^2/a^2)$. It is worthy to mention that D' and D'^2 keep the same structures of D and D^2.

The SO(4) symmetry of H'_{rha} can be checked directly. Correspondingly, its energy spectra is given by the monopolar Sommerfeld formula

$$\frac{E'}{M} = \left(1 + \frac{a^2}{(n' - |\kappa'| + \sqrt{\kappa'^2 - a^2})^2}\right)^{-1/2}, \qquad (11)$$

where $\kappa' = \pm\sqrt{(j'+1/2+q)(j'+1/2-q)}$ are eigenvalues of the Dirac's operator K', and $n' = 0 + |\kappa'|$, $1 + |\kappa'|$, $2 + |\kappa'|$, ... When $q = 0$, all the extended monopole-dependent operators and relations reduce to the usual ones in RHA. Furthermore, in the nonrelativistic limit, the Hamiltonian (10) reduces to the nonrelativistic monopolar-hydrogen atom $H'_{nrha} = \vec{\pi}^2/2M + q^2/2Mr^2 - a/r$. Such a quantum system still has an SO(4) symmetry due to the monopole-dependent orbital angular momentum vector and the monopole-dependent Pauli-Runge-Lentz vector [17].

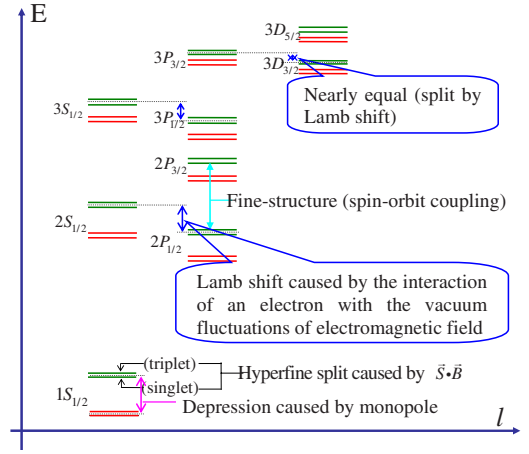*SO(4) symmetry in the RHA with a non-Abelian vector*

FIG. 1. (Color online) Low-lying energy levels of H_{rha} and H'_{rha} with $q = 1/2$ (not drawn to scale). The green lines (upper pairs) denote the energy spectra E in Eq. (2), while the red lines (lower pairs) denote the energy spectra E' in Eq. (11). The energy levels of H'_{rha} are depressed below those of H_{rha}, and such depressions fade away as n, j tend to infinite. For $n = 2$, $j = 1/2$, the depression caused by the monopole ($\approx 10^{14}$ Hz) is much bigger than that of the Lamb shift ($\approx 10^9$ Hz).

potential. Let us add a non-Abelian vector potential $\vec{\mathcal{A}} = iW(r)\vec{\Sigma} \times \vec{r}$ to RHA, where $W(r)$ is an arbitrary real function of r. Then the Dirac Hamiltonian reads

$$H''_{rha} = \vec{\alpha} \cdot (\vec{p} - e\vec{\mathcal{A}}) + \beta m - \frac{a}{r}. \qquad (12)$$

Similarly, H''_{rha} possesses an SO(4) symmetry with the corresponding generators $\vec{I}'' = \vec{J} + \vec{T}''$, $\vec{R}'' = \vec{J} - \vec{T}''$. Here $\vec{J}$ is the usual total angular momentum operator, $\vec{T}'' = (D''/\sqrt{D''^2}, iD''K/\sqrt{D''^2K^2}, K/\sqrt{K^2})/2$, $D'' = \gamma^5\vec{\alpha} \cdot \hat{r} - (iK/Ma)\gamma^5(H''_{rha} - M\beta)$ is the JL operator in presence of the non-Abelian vector potential, and $D''^2 = 1 + (H''^2_{rha}/M^2 - 1)(K^2/a^2)$. Also the operators D'' and D''^2 share the same forms of D and D^2. The non-Abelian vector potential satisfies the Coulomb gauge $\vec{\nabla} \cdot \vec{\mathcal{A}} = 0$, from $\mathcal{B}_i = (1/2)\epsilon_{ijk}(\partial_j\mathcal{A}_k - \partial_k\mathcal{A}_j + [\mathcal{A}_j, \mathcal{A}_k])$ one has the "magnetic" field as $\vec{\mathcal{B}} = i[2W(r) - rW'(r)]\vec{\Sigma} + i[2W^2(r) + W'(r)/r](\vec{\Sigma} \cdot \vec{r})\vec{r}$. Interestingly, if we choose $W(r) = -1/r^2$, then the "magnetic" field $\vec{\mathcal{B}} = (-4i/r^2)\vec{\Sigma}$ is proportional to the spin vector.

Lamb shift and SO(4) symmetry breaking. The spectrum formula (2) indicates that the levels having the same n and j values should be degenerate, for instance, the $2S_{1/2}$ and $2P_{1/2}$ levels should share the same energy. However, In 1947 Lamb and Retherford made an elaborate experiment and it showed that the $2P_{1/2}$ energy level was depressed about 1.057×10^9 Hz below the $2S_{1/2}$ energy level [18]. This effect is now called the Lamb shift. It cannot be explained in the framework of the ordinary quantum mechanics and gives rise to the birth of quantum electrodynamics (QED). In fact, the successful calculation of these small quantum corrections to the Dirac energy levels was one of the remarkable achieve-

ments of quantum field theory. From the viewpoint of QED, Lamb shift is caused by the vacuum polarization and vertex corrections [19], and the effective potential reads

$$\Delta V_{Lamb} \approx \frac{4a^2}{3M^2}\left(\ln\frac{M}{\mu} - \frac{1}{5}\right)\delta^3(\vec{r}) + \frac{a^2(\vec{\Sigma}\cdot\vec{L})}{4\pi M^2 r^3}. \quad (13)$$

The second term represents the interaction between spin and orbital angular momentum. Obviously, the corrected Hamiltonian $\mathcal{H}_{rha} = H_{rha} + \Delta V_{Lamb}$ commutes only with $\vec{J}$ and K, thus the SO(4) symmetry is broken.

In Fig. 1, we have plotted some low-lying energy levels of H_{rha} and H'_{rha} with $q = 1/2$ (not drawn to scale). The green lines (upper pairs) denote the energy spectra E in Eq. (2), while the red lines (lower pairs) denote the energy spectra E' in Eq. (11). One finds that $E'_{n'j'}$ are depressed below E_{nj}. For instance, let $\Delta^q_{nj} = E_{nj} - E'_{n'j'}$ denote the depressions, then for $q = 1/2$ one has the depression for the ground state as $\Delta^{1/2}_{1,1/2}$

$= 1.098 \times 10^{15}$ Hz, and that of the first excited state as $\Delta^{1/2}_{2,1/2} = 1.225 \times 10^{14}$ Hz. Actually, such depressions fade away as n, j become larger, e.g., the depression of the levels $3D_{5/2}$ $(1.294 \times 10^{12}$ Hz) is only about $1/1000$ of that of $1S_{1/2}$ $(1.098 \times 10^{15}$ Hz). For fixed n, j, the depressions caused by the monopole are much bigger than those of the Lamb shift.

In conclusion, we have shown that the relativistic hydrogen atom possesses an SO(4) symmetry by introducing a kind of pseudospin vector operator. The same SO(4) symmetry is still preserved in the relativistic quantum system in presence of a U(1) monopolar vector potential as well as a non-Abelian vector potential. When the effect of Lamb shift is taken into account, the SO(4) symmetry in the quantum system is broken.

[1] D. J. Gross, Proc. Natl. Acad. Sci. U.S.A. **93**, 14256 (1996).

[2] O. L. De Lange and R. E. Raab, *Operator Methods in Quantum Mechanics* (Clarendon, Oxford, 1991).

[3] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988); B. S. Shastry, *ibid.* **60**, 639 (1988).

[4] J. N. Ginocchio, Phys. Rev. Lett. **95**, 252501 (2005).

[5] H. Katsura and H. Aoki, J. Math. Phys. **47**, 032301 (2006).

[6] W. Pauli, Z. Phys. **36**, 336 (1926); V. Fock, *ibid.* **98**, 145 (1935); V. Bargman, *ibid.* **99**, 168 (1936).

[7] M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330 (1966).

[8] For the Johnson-Lippmann operator, one may refer to O. L. De Lange and R. E. Raab, *Operator Methods in Quantum Mechanics* (Ref. [2]).

[9] L. C. Biedenharn, Phys. Rev. **126**, 845 (1962); J. E. Mandula, *ibid.* **185**, 1774 (1969).

[10] C. Fronsdal, Phys. Rev. D **3**, 1299 (1971).

[11] T. T. Khachidze and A. A. Khelashvili, Am. J. Phys. **74**, 628 (2006).

[12] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Plenum, New York, 1977).

[13] B. A. Robson and S. H. Sutanto, Int. J. Theor. Phys. **40**, 1475 (2001).

[14] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **133**, 60 (1931).

[15] I. Tamm, Z. Phys. **71**, 141 (1931); M. Fierz, Helv. Phys. Acta **17**, 27 (1944).

[16] T. T. Wu and C. N. Yang, Nucl. Phys. B **107**, 365 (1976).

[17] J. L. Chen, M. L. Ge, and K. Xue, J. Phys. A **32**, 945 (1999).

[18] W. E. Lamb, Jr. and R. C. Retherford, Phys. Rev. **72**, 241 (1947).

[19] M. Kaku, *Quantum Field Theory, A Modern Introduction* (Oxford University Press, New York, 1993).