Arbitrary control of multiple-qubit systems in the symmetric Dicke subspace

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We discuss a general physical mechanism for arbitrary control of the quantum states of multiple qubits in the symmetric Dicke subspace. The qubit-qubit coupling leads to unequal energy spacing in the symmetric Dicke subspace. This allows one to manipulate a prechosen transition with an external driving source, with other transitions remaining off-resonant. Any entangled state in the symmetric Dicke subspace can be created from the initial ground state by tuning the driving source. We illustrate the idea in cavity QED, but it should be applicable to other related systems.

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I. INTRODUCTION

The control of quantum systems is fundamental to quantum optics as any experimental investigation of nonclassical features relies on the ability to create and manipulate quantum states. Mathematically, one can design a unitary evolution operator to transform an initial state to the desired state. In physics, the main difficulty comes from the fact that the desired evolution operator is limited by the attainable Hamiltonian. In the context of cavity QED, methods have been presented to force a quantized electromagnetic field localized in a cavity from an initial ground state to any quantum state [1–3]. In these schemes the quantum state of one or more atoms are manipulated in a controllable way and the coherence of the atom is transferred to the cavity field. The atoms act as the source, "teaching" the cavity field to evolve to the desired state.

For a system composed of multiple particles, the superpositions of product states leads to entanglement. There are various types of multiparticle entanglement and the characterization of entanglement has not been completed yet. The control of quantum states of composite systems is a prerequisite for experimental study of entanglement properties [4,5]. Besides fundamental interest, the control of the time evolution of multiparticle systems is of importance for the implementation of quantum computers. The implementation of a quantum computational task corresponds to the performance of a unitary transformation on the quantum register, which is composed of multiple quantum bits (qubits) [6]. During the quantum logic operation, the qubits are generally in an entangled state. In essence, realizing a quantum computer is equivalent to controlling the time evolution of an N-qubit system. The Hilbert space increases exponentially as the number of qubits increases and the control of multiqubit systems is very complex. As far as we know, no realistic mechanism has been proposed for creating an arbitrary entangled state for N-qubit systems.

In this paper we design a general interaction Hamiltonian which can drive an N-qubit system to evolve from an initial product state to any superposition state in the symmetric Dicke subspace. The qubit-qubit coupling leads to unequal spacing of energy levels in the symmetric Dicke subspace. This allows one to arbitrarily manipulate a specific transition by using an external driving field, which is resonant with the prechosen transition but off-resonant with other transitions. Any symmetric entangled state can be obtained by appropriately adjusting the parameters of the external driving field. The well-known Greenberger-Horne-Zeilinger (GHZ) states [7] and W states [8] are special classes of symmetric states. Our idea provides a possibility for the creation of general symmetric entangled states. The idea can be realized in realistic physical systems.

The paper is organized as follows. In Sec. II, we describe the interaction Hamiltonian for the manipulation of a prechosen transition in the symmetric Dicke subspace. In Sec. III, we show that any symmetric entangled state can be created from the initial ground state by tuning the parameter of the external source. In Sec. IV, we discuss the physical realization of the scheme in cavity QED and analyze decoherence effects. We also estimate the probability that the atomic system undergoes unwanted transitions. A summary appears in Sec. V.

II. INTERACTION HAMILTONIAN

We consider an N-qubit system. The qubits have two states $|e\rangle$ and $|g\rangle$ with an energy-level difference ω_0. The Hamiltonian for the whole system is (assuming $\hbar=1$)

$$H = H_0 + H_1 + H_2, \tag{1}$$

where

$$H_0 = \omega_0 S_z, \tag{2}$$

$$H_1 = \lambda S^+ S^-, \tag{3}$$

$$H_2 = \varepsilon[e^{-i(\omega t-\theta)}S^+ + e^{i(\omega t-\theta)}S^-], \tag{4}$$

$$S_z = \frac{1}{2}\sum_{j=1}^{N}(|g_j\rangle\langle g_j| - |e_j\rangle\langle e_j|),$$

$$S^+ = \sum_{j=1}^{N}|e_j\rangle\langle g_j|,$$

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$$S^- = \sum_{j=1}^{N} |g_j\rangle\langle e_j|. \tag{5}$$

H_1 describes the qubit-qubit coupling with coupling strength λ, and H_2 describes the interaction between the qubit system and an external driving source with frequency ω. The parameters ω, ε, and θ are controllable via the external driving source. The operators S^+, S^-, and S_z obey the commutation relations of the angular-momentum operators. If the system is initially in a proper Dicke state $|J,-J+k\rangle$ [9], it would evolve within a subspace of the Dicke space spanned by $\{|J,-J\rangle, |J,-J+1\rangle, \ldots, |J,J\rangle\}$. We consider the case that the system is initially symmetric, thus the Hilbert space reduces to the symmetric Dicke subspace with $J=N/2$, formed by the Dicke states which are symmetric under the permutation of any two qubits. The state $|J,-J+k\rangle$ is a symmetric state with k particles being in the state $|e\rangle$, i.e.,

$$|J,-J+k\rangle = \binom{2J}{k}^{-1/2} \sum_j P_j(|e_1,e_2,\ldots,e_k,g_{k+1},g_{k+2},\ldots g_N\rangle), \tag{6}$$

where $\{P_j\}$ denotes the set of all distinct permutations of the qubits. In the symmetric Dicke subspace the collective operators S^+ and S^- act as

$$S^+|J,-J+k\rangle = \sqrt{(2J-k)(k+1)}|-J,-J+k+1\rangle,$$

$$S^-|J,-J+k\rangle = \sqrt{k(2J-k+1)}|J,-J+k-1\rangle. \tag{7}$$

The Hamiltonian H_1 does not induce a transition between different symmetric Dicke states, but shifts the energy level of the state $|J,-J+k\rangle$ by $k(2J-k+1)\lambda$. Due to the qubit-qubit coupling the energy-level spacing between $|J,-J+k+1\rangle$ and $|J,-J+k\rangle$ is $\omega_0+2(J-k)\lambda$, which is depending upon the excitation number of the state $|J,-J+k\rangle$. Thus the spacing of energy levels in the symmetric Dicke subspace becomes unequal. The detuning between the transition $|J,-J+k\rangle \to |J,-J+k+1\rangle$ and the classical source is $\omega_0+2(J-k)\lambda-\omega$.

Suppose that $\varepsilon \ll \lambda$ and $\omega=\omega_0+2(J-k)\lambda$. In this case only the transition $|J,-J+k\rangle \to |J,-J+k+1\rangle$ is resonant with the external driving source, while other transitions remain far off-resonance and can be neglected. Therefore the symmetric Dicke subspace further reduces to $\{|J,-J+k+1\rangle, |J,-J+k\rangle\}$ and the Hamiltonians H_0, H_1, and H_2 reduce to

$$\begin{aligned} H_0 = {} & (k-J)\omega_0|J,-J+k\rangle\langle J,-J+k| \\ & + (k-J+1)\omega_0|J,-J+k+1\rangle\langle J,-J+k+1|, \end{aligned} \tag{8}$$

$$\begin{aligned} H_1 = {} & \alpha_k|J,-J+k\rangle\langle J,-J+k| \\ & + \alpha_{k+1}|J,-J+k+1\rangle\langle J,-J+k+1|, \end{aligned} \tag{9}$$

$$\begin{aligned} H_2 = {} & \eta_k(e^{-i(\omega t-\theta)}|J,-J+k+1\rangle\langle J,-J+k| \\ & + e^{i(\omega t-\theta)}|J,-J+k\rangle\langle J,-J+k+1|), \end{aligned} \tag{10}$$

where

$$\alpha_k = k(2J-k+1)\lambda,$$

$$\eta_k = \sqrt{(2J-k)(k+1)}\varepsilon. \tag{11}$$

In the interaction picture with respect to H_0 we obtain the interaction Hamiltonian

$$H_I = H_1 + H_{2,I}, \tag{12}$$

where

$$\begin{aligned} H_{2,I} = {} & \eta_k[e^{i[2(k-J)\lambda t+\theta]}|J,-J+k+1\rangle\langle J,-J+k| \\ & + e^{-i[2(k-J)\lambda t+\theta]}|J,-J+k\rangle\langle J,-J+k+1|]. \end{aligned} \tag{13}$$

Taking advantage of the unequal energy spacing induced by the qubit-qubit coupling, we can selectively manipulate any transition in the symmetric Dicke subspace by tuning the frequency of the external field appropriately.

III. GENERATION OF ANY SYMMETRIC ENTANGLED STATE

The time evolution of this system is decided by Schrödinger's equation:

$$i\frac{d|\psi(t)\rangle}{dt} = H_I|\psi(t)\rangle. \tag{14}$$

We perform the unitary transformation

$$|\psi(t)\rangle = e^{-iH_1 t}|\psi'(t)\rangle. \tag{15}$$

Then we obtain

$$i\frac{d|\psi'(t)\rangle}{dt} = H'_{2,I}|\psi'(t)\rangle, \tag{16}$$

where

$$\begin{aligned} H'_{2,I} = {} & \eta_k(e^{i\theta}|J,-J+k+1\rangle\langle J,-J+k| \\ & + e^{-i\theta}|J,-J+k\rangle\langle J,-J+k+1|). \end{aligned} \tag{17}$$

The interaction induces the transition

$$\begin{aligned} |J,-J+k\rangle \to {} & \cos(\eta_{k+1}t)e^{-i\alpha_k t}|J,-J+k\rangle \\ & - ie^{i(\theta-\alpha_{k+1}t)}\sin(\eta_{k+1}t)|J,-J+k+1\rangle. \end{aligned} \tag{18}$$

Suppose that we desire to generate the superposition state

$$|\psi_d\rangle = \sum_{k=0}^{K} d_k|J,-J+k\rangle, \tag{19}$$

where d_k is a complex number, i.e., $d_k=|d_k|\,e^{i\varphi_k}$. Without loss of generality, we here assume that d_0 is real, i.e., $\varphi_0=0$. Assume that the qubit system is initially in the state $|J,-J\rangle$, i.e., all the qubits are initially in the ground state. We divide the time interval into K subintervals. The duration of the kth subinterval is t_k. During the kth subinterval, the frequency of the classical driving source is $\omega_k=\omega_0+2(J-k)\lambda$. The corresponding phase is θ_k. After the first subinterval the system evolves to

$$|\psi_1\rangle = \cos(\eta_1 t_1)|J,-J\rangle - ie^{i(\theta_1-\alpha_1 t_1)}\sin(\eta_1 t_1)|J,-J+1\rangle. \tag{20}$$

We adjust the duration t_1 to satisfy the following condition:

$$\cos(\eta_1 t_1) = d_0. \quad (21)$$

Then we obtain

$$|\psi_1\rangle = d_0|J,-J\rangle - i\sqrt{1-|d_0|^2}e^{i(\theta_1-\alpha_1 t_1)}|J,-J+1\rangle. \quad (22)$$

After the second subinterval the system evolves to

$$|\psi_2\rangle = d_0|J,-J\rangle - i\sqrt{1-|d_0|^2}e^{i(\theta_1-\alpha_1 t_1)}[\cos(\eta_2 t_2)e^{-i\alpha_1 t_2}|J,-J+1\rangle - ie^{i(\theta_2-\alpha_2 t_2)}\sin(\eta_2 t_2)|J,-J+2\rangle]. \quad (23)$$

Setting

$$\sqrt{1-|d_0|^2}\cos(\eta_2 t_2) = |d_1|, \quad (24)$$

we have

$$|\psi_2\rangle = d_0|J,-J\rangle - i|d_1|e^{i[\theta_1-\alpha_1(t_1+t_2)]}|J,-J+1\rangle - \sqrt{1-|d_0|^2-|d_1|^2}e^{i(\theta_1+\theta_2-\alpha_1 t_1-\alpha_2 t_2)}|J,-J+2\rangle. \quad (25)$$

After each subinterval the highest excitation number is increased by 1. The length of the kth subinterval t_k satisfies

$$\sqrt{1-\sum_{j=0}^{k-1}|d_j|^2}\cos(\eta_k t_k) = d_k. \quad (26)$$

This leads to the final state

$$|\psi_K\rangle = d_0|J,-J\rangle + \sum_{k=1}^{K} c_k|J,-J+k\rangle, \quad (27)$$

where

$$c_k = |d_k|e^{i\phi_k},$$

$$\phi_k = \sum_{j=1}^{k}\theta_j - \alpha_k\sum_{j=1}^{K-k+1}t_j - \sum_{j=1}^{k-1}\alpha_j t_j - k\pi/2. \quad (28)$$

Choose the phase of the driving source appropriately so that

$$\theta_k = \varphi_k - \sum_{j=1}^{k-1}\theta_j + \alpha_k\sum_{j=1}^{K-k+1}t_j + \sum_{j=1}^{k-1}\alpha_j t_j + k\pi/2. \quad (29)$$

Then we obtain $\phi_k=\varphi_k$ and $|\psi_K\rangle$ is just the desired state $|\psi_d\rangle$. In the following we propose an implementation of the idea with a cavity QED system, but it is not restricted in cavity QED.

IV. PHYSICAL IMPLEMENTATION IN CAVITY QED

We consider N two-level atoms interacting with a quantized cavity field and driven by a weak classical field. The Hamiltonian is

$$H = H_f + H_{a-q} + H_{a-c}, \quad (30)$$

where

$$H_f = \omega_0 S_z + \omega_c a^\dagger a, \quad (31)$$

$$H_{a-q} = g(a^\dagger S^- + aS^+), \quad (32)$$

$$H_{a-c} = \varepsilon[e^{-i(\omega t-\theta)}S^+ + e^{i(\omega t-\theta)}S^-], \quad (33)$$

$a^\dagger$ and a are the creation and annihilation operators for the cavity field, ω_0, ω_c, and ω are the frequencies for the atomic transition, cavity mode, and classical field, g is the coupling constant between the atoms and the cavity field, and ε and θ are the Rabi frequency and phase of the classical field. In the case $\delta_c=\omega_0-\omega_c \gg g\sqrt{\bar{n}+1}$, with $\bar{n}$ being the mean photon number of the cavity field, there is no energy exchange between the atomic system and the cavity. Then the Hamiltonian H_{a-q} can be replaced by the effective Hamiltonian [10]

$$H_e = \lambda_c\left[\sum_{j=1}^{N}(|e_j\rangle\langle e_j| - |g_j\rangle\langle g_j|)a^\dagger a + S^+S^-\right], \quad (34)$$

where $\lambda_c=g^2/\delta_c$. The first and second terms describe the photon-number dependent Stark shift, and the last term describes the dipole coupling among the atoms induced by the cavity mode. When the cavity mode is initially in the vacuum state $|0\rangle$ it will remain in the vacuum state throughout the procedure. Then the Hamiltonian H_f reduces to H_0 of Eq. (2) and H_e reduces to H_1 of Eq. (3). The parameters ω, ε, and θ are controllable by the classical field.

We now discuss the experimental implementation of the proposed scheme. In recent cavity QED experiments with long-living Rydberg atomic levels coupled to a cavity mode, the coupling constant is $g=2\pi\times25$ kHz [11,12]. The atomic radiative time and photon damping time are about $T_r=3\times10^{-2}$ s and $T_c=1\times10^{-3}$ s, respectively. We set $N=3$, $\delta=10$ g, and $\varepsilon=g/100$. Suppose that the desired state is

$$|\psi_d\rangle = \frac{1}{\sqrt{2}}(|3/2,-3/2\rangle + |3/2,-1/2\rangle). \quad (35)$$

Then the required atom-cavity-field interaction time is $t=\pi/(4\eta_1)=\pi/(4\sqrt{3}\varepsilon)\simeq0.29\times10^{-3}$ s. In this case the decay time for the superposition state $|\psi_d\rangle$ is $T_d=T_r/3=10^{-2}$ s. As the cavity mode is only virtually excited the effective decoherence rate due to the cavity decay is $\kappa=g^2/T_c\delta^2=10$ Hz. The infidelity induced by the decoherence is on the order of $t/T_d+t\kappa=3.19\times10^{-2}$.

We now consider the probability that the atomic system undergoes a transition to the state $|3/2,1/2\rangle$ via the off-resonant coupling. The detuning between the classical field and the transition $|3/2,-1/2\rangle\to|3/2,1/2\rangle$ is $2\lambda=0.2g$. The probability that the atomic system undergoes this transition is given by

$$P_{-1/2\to1/2} \sim \frac{1}{2}\frac{\eta_{k+1}^2}{\eta_{k+1}^2+\lambda^2}\sin^2(\sqrt{\eta_{k+1}^2+\lambda^2}t) \simeq 0.38\times10^{-2}. \quad (36)$$

With all of the above-mentioned nonideal situations being considered, the total error is about 3.57×10^{-2}.

V. SUMMARY

In conclusion, we have discussed a physical mechanism for arbitrary control of quantum states for a multiqubit in the symmetric Dicke subspace. As a consequence of qubit-qubit coupling, the energy levels are not equidistant in the symmetric Dicke subspace. If the driving field is tuned in resonance with a specific transition in the symmetric Dicke subspace, it would be off-resonant with other transitions. This allows one to selectively manipulate a prechosen transition. Any symmetric entangled state can be created by adjusting the driving field. The Hamiltonian can be realized in physical systems with qubit-qubit coupling available. We propose an implementation of the idea in cavity QED. The entanglement of symmetric Dicke states is robust against the loss of particles, as demonstrated in a recent experiment with photons [13]. The idea opens new perspectives for research of entanglement properties of general symmetric multiqubit states. The idea can also be used for preparation of entangled Dicke states for two atomic systems. Suppose that the first atomic system is first prepared in a superposition of Dicke states through the above-mentioned procedure. Then this atomic system is entangled with a light field via the exchange of excitations induced by atom-field interaction. The two atomic systems can be prepared in an entangled Dicke state by transferring the excitations of the light field to the second atomic system [14]. This provides a possibility for tests of quantum nonlocality with two entangled atomic systems. Apart from fundamental tests of quantum theory, entangled Dicke states are useful for quantum communication [15].

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