Subluminal wave bullets: Exact localized subluminal solutions to the wave equations

Michel Zamboni-Rached*

Centro de Ciências Naturais e Humanas, Universidade Federal do ABC, Santo André, SP, Brazil and DMO-FEEC, State University at Campinas, Campinas, SP, Brazil

Erasmo Recami†

Facoltà di Ingegneria, Università Statale di Bergamo, Bergamo, Italy and INFN, Sezione di Milano, Milan, Italy

(Received 19 September 2007; published 11 March 2008)

In this work it is shown how to obtain, in a simple way, localized (nondiffractive) subluminal pulses as exact analytic solutions to the wave equations. These ideal subluminal solutions, which propagate without distortion in any homogeneous linear media, are herein obtained for arbitrarily chosen frequencies and bandwidths, avoiding in particular any recourse to the noncausal (backward moving) components that so frequently plague the previously known localized waves. Such solutions are suitable superpositions of—zeroth order, in general—Bessel beams, which can be performed either by integrating with respect to (w.r.t.) the angular frequency ω, or by integrating w.r.t. the longitudinal wave number k_z: Both methods are expounded in this paper. The first one appears to be powerful enough; we study the second method as well, however, since it allows us to deal even with the limiting case of zero-speed solutions (and furnishes a way, in terms of continuous spectra, for obtaining the so-called "frozen waves," so promising also from the point of view of applications). We briefly treat the case, moreover, of nonaxially symmetric solutions, in terms of higher-order Bessel beams. Finally, some attention is paid to the known role of special relativity, and to the fact that the localized waves are expected to be transformed one into the other by suitable Lorentz transformations. In this work we fix our attention especially on acoustics and optics: However, results of the present kind are valid whenever an essential role is played by a wave equation (such as electromagnetism, seismology, geophysics, gravitation, elementary particle physics, etc.).

DOI: 10.1103/PhysRevA.77.033824 PACS number(s): 42.25.Bs, 42.25.Fx, 41.20.Jb, 46.40.Cd

I. INTRODUCTION

For more than 10 years, the so-called (nondiffracting) "localized waves" (LW), which are solutions to the wave equations (scalar, vectorial, spinorial, etc.), are in fashion, both in theory and in experiment. In particular, rather well known are the ones with luminal or superluminal peak velocity [1], such as the so-called X-shaped waves (see [2,3] and references therein; for a review, see, e.g., Ref. [4]), which are supersonic in acoustics [5], and superluminal in electromagnetism (see [6] and references therein).

Since Bateman [7] and later on Courant and Hilbert [8], it was already known that luminal LWs exist, which are as well solutions to the wave equations. More recently, some attention [9–13] started to be paid to subluminal solutions too. Let us recall that all the LWs propagate without distortion—and in a self-reconstructive way [14–16]—in a homogeneous linear medium (apart from local variations): In the sense that their square magnitude keeps its shape during propagation, while local variations are shown only by its real, or imaginary, parts.

As in the superluminal case, the (more orthodox, in a sense) subluminal LWs can be obtained by suitable superpositions of Bessel beams. They have been until now almost neglected, however, for the mathematical difficulties met in getting analytic expressions for them, difficulties associated with the fact that the superposition integral runs over a finite interval. We want here to readdress the question of such subluminal LWs, showing, by contrast, that one can indeed arrive at exact (analytic) solutions, both in the case of integration over the Bessel beams' angular frequency ω, and in the case of integration over their longitudinal wave number k_z. The first approach, herein investigated in detail, is enough to get the majority of the desired results; we study also the second one, however, since it allows treating the limiting case of zero-speed solutions, which have been called "frozen waves" (and also furnishes a second method—based on a continuous spectrum—for obtaining such waves, so promising also from the point of view of applications). Moreover, we shall briefly deal with nonaxially symmetric solutions, in terms of higher-order Bessel beams. At last, some attention is paid to the known role of special relativity, and to the fact that the localized waves are expected to be transformed one into the other by Lorentz transformations.

As already claimed, the present paper is devoted to the exact, analytic solutions, i.e., to ideal solutions. In another article, we shall go on to the corresponding pulses with finite energy, or truncated, sometimes having recourse—in those cases, only—to approximations. We shall fix our attention especially on acoustics and optics: However, our results are valid whenever an essential role is played by a wave equation (such as electromagnetism, seismology, geophysics, gravitation, elementary particle physics, etc.).

Let us recall that, in the past, too much attention was not even paid to 1983 Brittingham's paper [17], wherein he

*mzamboni@ufabc.edu.br; mzamboni@dmo.fee.unicamp.br
†recami@mi.infn.it

showed the possibility of obtaining pulse-type solutions to the Maxwell equations, which propagate in free space as a kind of speed-c "soliton." This lack of attention was partially due to the fact that Brittingham had been able neither to get correct finite-energy expressions for such "wavelets," nor to make suggestions about their practical production. Two years later, however, Sezginer [18] was able to show how to obtain quasinondiffracting luminal pulses endowed with a finite energy. Finite-energy pulses no longer travel undistorted for an infinite distance, but they can nevertheless propagate without deformation for a long field depth, much larger than the one achieved by ordinary pulses, such as the Gaussian ones: cf., e.g., Refs. [19–24] and references therein.

Only after 1985 the general theory of all LWs started to be extensively developed [25–31,2,6,3] both in the case of beams, and in the case of pulses. For reviews, see for instance, Refs. [4,32,23,21,24] and references therein. For the propagation of LWs in bounded regions (such as guides), see Refs. [33–36] and references therein. For the focusing of LWs, see Refs. [37,38] and references therein. As to the construction of LWs propagating in dispersive media, cf. Refs. [39–47]; and, for lossy media, see Ref. [16] and references therein. At last, for finite energy, or truncated, solutions see Refs. [48–50,24,3,34].

By now, the LWs have been experimentally produced [5,51,52], and are being applied in fields ranging from ultrasound scanning [53,54,11] to optics (for the production, e.g., of new type of tweezers [55]). All these works demonstrated that nondiffracting pulses can travel with an arbitrary peak speed v, that is, with $0<v<\infty$; while Brittingham and Sezginer had confined themselves to the luminal case ($v=c$) only. As already commented, the superluminal and luminal LWs have been, and are being, intensively studied, while the subluminal ones have been neglected: Almost all the papers dealing with them had until now recourse to the paraxial [56] approximation [57], or to numerical simulations [12], due to the above-mentioned mathematical difficulty in obtaining exact analytic expressions for subluminal pulses. Actually, only *one* analytic solution is known [9–11,28,57,58] biased by the physically inconvenient facts that its frequency spectrum is very large, it does not even possess a well-defined central frequency, and, even more, that backward-traveling [26,24] components (ordinarily called "noncausal," since they should be entering the antenna or generator) are *a priori* needed for constructing it. The aim of the present paper is to show how subluminal localized exact solutions can be constructed with any spectra, in any frequency bands, and for any bandwidths; and, moreover, *without* employing [3,23] any backward-traveling components.

II. FIRST METHOD FOR CONSTRUCTING PHYSICALLY ACCEPTABLE, SUBLUMINAL LOCALIZED PULSES

Axially symmetric solutions to the scalar wave equation are known to be superpositions of zero-order Bessel beams over the angular frequency ω and the longitudinal wave number k_z, i.e., in cylindrical coordinates,

$$\Psi(\rho,z,t)=\int_0^{\infty}d\omega\int_{-\omega/c}^{\omega/c}dk_z\bar{S}(\omega,k_z)J_0\left(\rho\sqrt{\frac{\omega^2}{c^2}-k_z^2}\right)e^{ik_zz}e^{-i\omega t}, \tag{1}$$

where $k_\rho^2\equiv\omega^2/c^2-k_z^2$ is the transverse wave number; quantity k_ρ^2 must be positive since evanescent waves cannot come into play.

The condition characterizing a nondiffracting wave is the existence [24,59] of a linear relation between longitudinal wave number k_z and frequency ω for all the Bessel beams entering superposition (1); that is to say, the chosen spectrum must entail [3,21] for each Bessel beam a linear relationship of the type[1]

$$\omega=vk_z+b, \tag{2}$$

where, as we will see, the constant v is the wave velocity (i.e., the pulse peak velocity) and $b\geq 0$. Requirement (2) can be regarded also as a specific space-time coupling, implied by the chosen spectrum $\bar{S}$. Equation (2) must be obeyed by the spectra of any one of the three possible types (subluminal, luminal or superluminal) of nondiffracting pulses. Let us mention that with the choice in Eq. (2) the pulse regains its initial shape after the space interval $\Delta z_1=2\pi v/b$; the more general case can be, however, considered [3,47] when b assumes any values $b_m=mb$ (with m an integer), and the periodicity space interval becomes $\Delta z_m=\Delta z_1/m$. We are referring ourselves, of course, to the real (or imaginary) part of the pulse, since its modulus is known to be endowed with rigid motion.

In the subluminal case, of interest here, the only exact solution known until now, represented by Eq. (10) below, is the one found by Mackinnon [9]. Indeed, by taking into explicit account that the transverse wave number k_ρ of each Bessel beam entering Eq. (1) must be real, it can be easily shown (as first noticed by Salo *et al.* for the analogous acoustic solutions [12]) that in the subluminal case b cannot vanish, but must be larger than zero: $b>0$. Then, by using conditions (2) and $b>0$, the subluminal localized pulses can be expressed as integrals over the frequency only,

$$\Psi(\rho,z,t)=\exp\left(-ib\frac{z}{v}\right)\int_{\omega_-}^{\omega_+}d\omega S(\omega)J_0(\rho k_\rho)\exp\left(i\omega\frac{\zeta}{v}\right), \tag{3}$$

[1]More generally, as shown in Ref. [3], the chosen spectrum must call into play, in the plane ω,k_z, if not exactly the line (2), at least a region in the proximity of a straight line of such a type. It is interesting that in the latter case one obtains solutions endowed with finite energy, but possessing a finite "depth of field," that is, nondiffracting only until a certain finite distance.

where now

$$k_\rho = \frac{1}{v}\sqrt{2b\omega - b^2 - (1 - v^2/c^2)\omega^2} \tag{4}$$

with

$$\zeta \equiv z - vt \tag{5}$$

and with

$$\omega_- = \frac{b}{1 + v/c},$$

$$\omega_+ = \frac{b}{1 - v/c}. \tag{6}$$

As anticipated, the Bessel beam superposition in the subluminal case results to be an integration over a finite interval of ω, which does clearly show that the backward-traveling (noncausal) components correspond to the interval $\omega_- < \omega < b$. It could be noticed that Eq. (3) does not represent the most general exact solution, which on the contrary is a sum [47] of such solutions for the various possible values of b just mentioned above: That is, for the values $b_m = mb$ and spatial periodicity $\Delta z_m = \Delta z_1/m$; but we can confine ourselves to solution (3) without any real loss of generality, since the actual problem is evaluating in analytic form the integral entering Eq. (3). For any mathematical and physical details, see Ref. [47].

Now, if one adopts the change of variables

$$\omega \equiv \frac{b}{1 - v^2/c^2}\left(1 + \frac{v}{c}s\right), \tag{7}$$

Eq. (3) becomes [12]

$$\Psi(\rho,z,t) = \frac{b}{c}\frac{v}{1 - v^2/c^2}\exp\left(-i\frac{b}{v}z\right)\exp\left(i\frac{b}{v}\frac{1}{1 - v^2/c^2}\zeta\right) \times \int_{-1}^{1} ds S(s) J_0\left(\frac{b}{c}\frac{\rho}{\sqrt{1 - v^2/c^2}}\sqrt{1 - s^2}\right) \times \exp\left(i\frac{b}{c}\frac{1}{1 - v^2/c^2}\zeta s\right). \tag{8}$$

In the following we shall adhere to some symbols standard in special relativity [since the whole topic of subluminal, luminal, and superluminal LWs is strictly connected [4,6,60] with the principles and structure of special relativity (cf. [61,62] and references therein), as we shall mention in the conclusions]; namely,

$$\beta \equiv \frac{v}{c}, \quad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}. \tag{9}$$

As already mentioned, Eq. (8) has until now yielded *one* analytic solution for $S(s)$=const, only [for instance, $S(s)=1$]; which means nothing but $S(\omega)$=const: In this case, one obtains indeed the Mackinnon solution [9,28,11,50]

$$\Psi(\rho,\zeta,\eta) = 2\frac{b}{c}v\gamma^2 \exp\left(i\frac{b}{c}\beta\gamma^2\eta\right)\mathrm{sinc}\left[\sqrt{\frac{b^2}{c^2}\gamma^2(\rho^2 + \gamma^2\zeta^2)}\right], \tag{10}$$

which however, for its above-mentioned drawbacks, is endowed with little physical and practical interest. In Eq. (9) the sinc function has the ordinary definition

$$\mathrm{sinc}\, x \equiv (\sin x)/x$$

and

$$\eta \equiv z - Vt, \quad \text{with } V \equiv \frac{c^2}{v}, \tag{11}$$

where V and v are related by the de Broglie relation. [Notice that Ψ in Eq. (10), and in the following ones, is eventually a function (besides of ρ) of z,t via ζ and η, both functions of z and t.]

We can, however, construct by a very simple method subluminal pulses corresponding to whatever spectrum, and devoid of backward-moving (i.e., "entering") components, by also taking advantage of the fact that in our Eq. (8) the integration interval is finite: That is, by transforming it in a good, instead of an evil. Let us first observe that Eq. (8) does not admit only the already known analytic solution corresponding to $S(s)$=const, and more in general to $S(\omega)$=const, but it will also yield an exact, analytic solution for any exponential spectra of the type

$$S(\omega) = \exp\left(\frac{i2n\pi\omega}{\Omega}\right), \tag{12}$$

with n any integer number, which means for any spectra of the type $S(s)=\exp(in\pi/\beta)\exp(in\pi s)$, as can be easily seen by checking the product of the various exponentials entering the integrand. In Eq. (12) we have set

$$\Omega \equiv \omega_+ - \omega_-.$$

The solution is written in this more general case as

$$\Psi(\rho,\zeta,\eta) = 2b\beta\gamma^2 \exp\left(i\frac{b}{c}\beta\gamma^2\eta\right)\exp\left(in\frac{\pi}{\beta}\right) \times \mathrm{sinc}\left[\sqrt{\frac{b^2}{c^2}\gamma^2\rho^2 + \left(\frac{b}{c}\gamma^2\zeta + n\pi\right)^2}\right]. \tag{13}$$

Let us explicitly notice that also in Eq. (13) quantity η is defined as in Eqs. (11), where V and v obey the de Broglie relation $vV = c^2$, the subluminal quantity v being the velocity of the pulse envelope, and V playing the role (in the envelope's interior) of a superluminal phase velocity.

The next step, as anticipated, consists just in taking *advantage* of the finiteness of the integration limits for expanding any arbitrary spectra $S(\omega)$ in a Fourier series in the interval $\omega_- \le \omega \le \omega_+$,

$$S(\omega) = \sum_{n=-\infty}^{\infty} A_n \exp\left(+ in\frac{2\pi}{\Omega}\omega\right), \tag{14}$$

where (we can go back, now, from the s to the ω variable),

$$A_n = \frac{1}{\Omega}\int_{\omega_-}^{\omega_+} d\omega S(\omega)\exp\left(- in\frac{2\pi}{\Omega}\omega\right) \tag{15}$$

quantity Ω being defined as above.

Then, on remembering the special solution (13), we can infer from expansion (14) that, for any arbitrary spectral function $S(\omega)$, a rather general, axially symmetric, analytic solution for the subluminal case can be written as

$$\Psi(\rho,\zeta,\eta) = 2b\beta\gamma^2 \exp\left(i\frac{b}{c}\beta\gamma^2\eta\right) \sum_{n=-\infty}^{\infty} A_n \exp\left(in\frac{\pi}{\beta}\right) \times \text{sinc}\left[\sqrt{\frac{b^2}{c^2}\gamma^2\rho^2 + \left(\frac{b}{c}\gamma^2\zeta + n\pi\right)^2}\right], \tag{16}$$

in which the coefficients A_n are still given by Eq. (15). Let us repeat that our solution is expressed in terms of the particular Eq. (13), which is a Mackinnon-type solution.

The present approach presents many advantages. We can easily choose spectra localized within the prefixed frequency interval (optical waves, microwaves, etc.) and endowed with the desired bandwidth. Moreover, as already said, spectra can now be chosen such that they have zero value in the region $\omega_- \leq \omega \leq b$, which is responsible for the backward-traveling components of the subluminal pulse.

Let us stress that, even when the adopted spectrum $S(\omega)$ does not possess a known Fourier series [so that the coefficients A_n cannot be exactly evaluated via Eq. (15)], one can calculate approximately such coefficients without meeting any problem, since our general solutions (16) will still be exact solutions.

Let us set forth in the following some examples.

Examples. In general, optical pulses generated in the laboratory possess a spectrum centered on some frequency value, ω_0, called the carrier frequency. The pulses can be, for instance, ultrashort, when $\Delta\omega/\omega_0 \geq 1$; or quasimonochromatic, when $\Delta\omega/\omega_0 \ll 1$, where $\Delta\omega$ is the spectrum bandwidth.

These kinds of spectra can be mathematically represented by a Gaussian function, or functions with similar behavior.

First two examples. Let us first consider a Gaussian spectrum

$$S(\omega) = \frac{a}{\sqrt{\pi}}\exp[- a^2(\omega - \omega_0)^2] \tag{17}$$

whose values are negligible outside the frequency interval $\omega_- < \omega < \omega_+$ over which the Bessel beams superposition in Eq. (3) is made, it being $\omega_- = b/(1+\beta)$ and $\omega_+ = b/(1-\beta)$. Of course, relation (2) must still be satisfied, and with $b>0$, for getting an ideal subluminal localized solution. Notice that, with spectrum (17), the bandwidth [actually, the full width at half-maximum (FWHM)] results to be $\Delta\omega = 2/a$. Let us emphasize that, once v and b have been fixed, the values of a and ω_0 can then be selected in order to kill the backward-traveling components, that exist, as we know, for $\omega < b$.

The Fourier expansion in Eq. (14), which yields, with the above spectral function (17), the coefficients

$$A_n \simeq \frac{1}{\Omega}\exp\left(- in\frac{2\pi}{\Omega}\omega_0\right)\exp\left(\frac{- n^2\pi^2}{a^2\Omega^2}\right), \tag{18}$$

constitutes an excellent representation of the Gaussian spectrum (17) in the interval $\omega_- < \omega < \omega_+$ (provided that, as we requested, our Gaussian spectrum does get negligible values outside the frequency interval $\omega_- < \omega < \omega_+$).

In other words, on choosing a pulse velocity $v < c$ and a value for the parameter b, a subluminal pulse with the above frequency spectrum (17) can be written as Eq. (16), with the coefficients A_n given by Eq. (18): The evaluation of such coefficients A_n being rather simple. Let us repeat that, even if the values of the A_n are obtained via a (rather good) approximation, we based ourselves on the *exact* solution equation (16).

One can, for instance, obtain exact solutions representing subluminal pulses for optical frequencies. Let us get the subluminal pulse with velocity $v = 0.99c$, angular carrier frequency $\omega_0 = 2.4 \times 10^{15}$ Hz (that is, $\lambda_0 = 0.785$ μm), and bandwidth (FWHM) $\Delta\omega = \omega_0/20 = 1.2 \times 10^{14}$ Hz, which is an optical pulse of 24 fs (that is the FWHM of the pulse intensity). For a complete pulse characterization, one must choose the value of the frequency b: Let it be $b = 3 \times 10^{13}$ Hz; as a consequence one has $\omega_- = 1.507 \times 10^{13}$ Hz and $\omega_+ = 3 \times 10^{15}$ Hz. (This is exactly a case in which the considered pulse is not plagued by the presence of backward-traveling components, since the chosen spectrum forwards totally negligible values for $\omega < b$.) The construction of the pulse does already result satisfactory when considering approximately 51 terms ($-25 \leq n \leq 25$) in the series entering Eq. (16).

Figure 1 shows our pulse, plotted by considering the mentioned 51 terms. Namely, Fig. 1(a) depicts the orthogonal projection of the pulse intensity; Fig. 1(b) shows the three-dimensional intensity pattern of the *real part* of the pulse, which reveals the carrier wave oscillations.

Let us stress that the ball-like shape[2] for the field intensity should be typically associated with all the subluminal LWs, while the typical superluminal ones are known to be X shaped [2,6,60], as predicted, since long, by special relativity in its "nonrestricted" version: See Refs. [61,62,6,4] and references therein.

A second spectrum $S(\omega)$ would be, for instance, the "inverted parabola" one, centered at the frequency ω_0, that is,

[2]It can be noted that each term of the series in Eq. (16) corresponds to an ellipsoid or, more specifically, to a spheroid, for each velocity v.

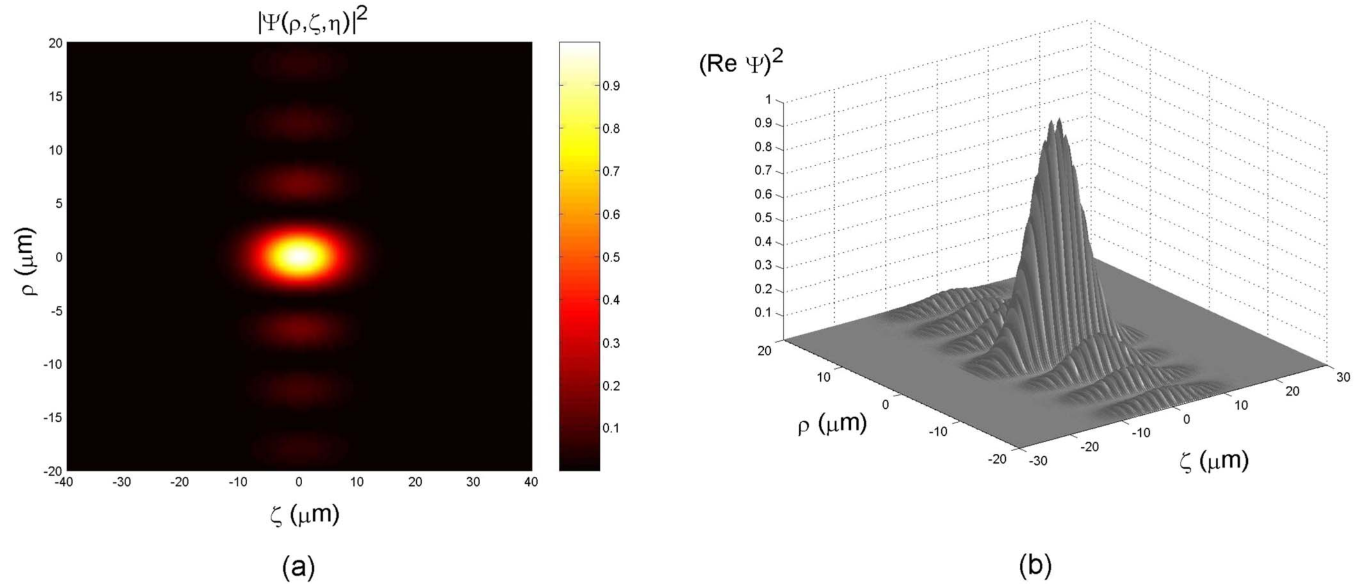

FIG. 1. (Color online) (a) The intensity orthogonal projection for the pulse corresponding to Eqs. (17) and (18) in the case of an optical frequency (see the text); (b) the three-dimensional intensity pattern of the real part of the same pulse, which reveals the carrier wave oscillations.

$$S(\omega) = \begin{cases} \dfrac{-4[\omega-(\omega_0-\Delta\omega/2)][\omega-(\omega_0+\Delta\omega/2)]}{\Delta\omega^2} & \text{for } \omega_0-\Delta\omega/2 \le \omega \le \omega_0+\Delta\omega/2, \\ 0 & \text{otherwise,} \end{cases} \tag{19}$$

where $\Delta\omega$, the distance between the two zeros of the parabola, can be regarded as the spectrum bandwidth. One can expand $S(\omega)$, given in Eq. (19), in the Fourier series (14), for $\omega_- \le \omega \le \omega_+$, with coefficients A_n that—even if straightforwardly calculable—results to be complicated, so that we skip reporting them here explicitly. Let us here only mention that spectrum (19) may be easily used to get, for instance, an ultrashort (femtasecond) optical nondiffracting pulse, with satisfactory results even when considering very few terms in expansion (14).

Third example. As a third, interesting example, let us consider the very simple case when—within the integration limits ω_-, ω_+—the complex exponential spectrum (12) is replaced by the real function (still linear in ω)

$$S(\omega) = \frac{a}{1-\exp[-a(\omega_+-\omega_-)]}\exp[a(\omega-\omega_+)], \tag{20}$$

with a a positive number (for $a=0$ one goes back to the Mackinnon case). Spectrum (20) is exponentially concentrated in the proximity of ω_+, where it reaches its maximum value; and becomes more and more concentrated (on the left-hand side of ω_+, of course) as the arbitrarily chosen value of a increases, the frequency bandwidth being $\Delta\omega=1/a$. Let us recall that, on their turn, quantities ω_+ and ω_- depend on the pulse velocity v and on the arbitrary parameter b.

By performing the integration as in the case of spectrum (12), instead of solution (13) in the present case one eventually gets the solution

$$\Psi(\rho,\zeta,\eta) = \frac{2ab\beta\gamma^2\exp(ab\gamma^2)\exp(-a\omega_+)}{1-\exp[-a(\omega_+-\omega_-)]}\exp\left(i\frac{b}{c}\beta\gamma^2\eta\right) \times \text{sinc}\left(\frac{b}{c}\gamma^2\sqrt{\gamma^{-2}\rho^2-(av+i\zeta)^2}\right). \tag{21}$$

After Mckinnon's, Eq. (21) appears to be the simplest closed-form solution, since both of them do not need any recourse to series expansions. In a sense, our solution (21) might be regarded as the subluminal analogous of the (superluminal) X-wave solution; a difference being that the standard X-shaped solution has a spectrum starting with 0, where it assumes its maximum value, while in the present case the spectrum starts at ω_- and gets increasing afterwards until ω_+. More important is to observe that the Gaussian spectrum has *a priori* two advantages with respect to (w.r.t.) Eq. (20): It may be more easily centered around any value ω_0 of ω, and, when increasing its concentration in the surrounding of ω_0, the spot transverse width does not increase indefinitely, but tends to the spot width of a Bessel beam with $\omega=\omega_0$ and $k_z=(\omega_0-b)/v$, at variance with what happens for spectrum (20); however, solution (21) is noticeable, since it is really the simplest one.

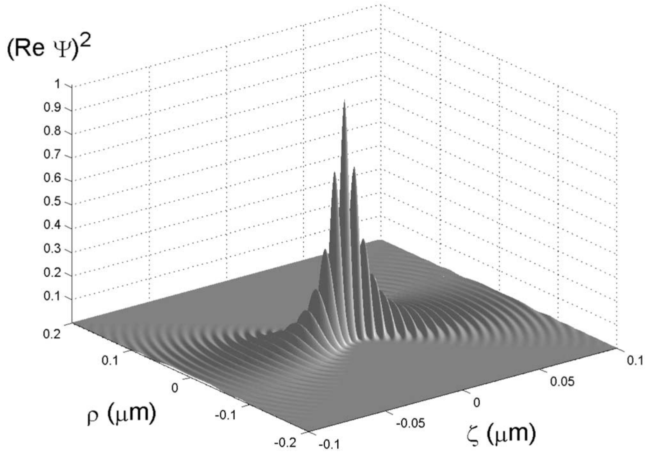

FIG. 2. The intensity of the real part of the subluminal pulse corresponding to spectrum (20), with $v=0.99c$, $b=3\times10^{13}$ Hz (which result in $\omega_-=1.5\times10^{13}$ Hz and $\omega_-=3\times10^{15}$ Hz), $\Delta\omega/\omega_+=1/100$ (i.e., $a=100$).

Figure 2 shows the intensity of the real part of the subluminal pulse corresponding to this spectrum, with $v=0.99c$, $b=3\times10^{13}$ Hz (which result in $\omega_-=1.5\times10^{13}$ Hz and $\omega_+=3\times10^{15}$ Hz), $\Delta\omega/\omega_+=1/100$ (i.e., $a=100$). This is an optical pulse of 0.2 ps.

III. SECOND METHOD FOR CONSTRUCTING SUBLUMINAL LOCALIZED PULSES

The previous method appears to be very efficient for finding out analytic subluminal LWs, but it loses its validity in the limiting case $v\to0$, since for $v=0$ it is $\omega_-\equiv\omega_+$ and the integral in Eq. (3) degenerates, furnishing a null value. By contrast, we are interested also in the $v=0$ case, since it corresponds to some of the most interesting, and potentially useful, LWs: Namely, to the so-called "frozen waves," which are stationary solutions to the wave equations, possessing a static envelope.

Before going on, let us recall that the theory of frozen waves was developed in Refs. [49,63,16], by having recourse to discrete superpositions in order to bypass the need of numerical simulations. In the case of continuous superpositions, numerical simulations were performed in Ref. [64]. However, the method presented in this section does allow us to find analytical exact solutions (without any further need, now, of numerical simulations) even for frozen waves consisting in continuous superpositions. Actually, we are going to see that the present method works whatever is the chosen field-intensity shape, also in regions with size of the order of the wavelength.

It is possible to obtain such results by starting again from Eq. (1), with constraint (2), but going on—this time—to integrals over k_z, instead of over ω. It is enough to write relation (2) in the form

$$k_z=\frac{1}{v}(\omega-b) \qquad (2')$$

for expressing the exact solutions (1) as

$$\Psi(\rho,z,t)=\exp(-ibt)\int_{k_z\,\min}^{k_z\,\max}dk_zS(k_z)J_0(\rho k_\rho)\exp(i\zeta k_z), \qquad (22)$$

with

$$k_z\,\min=\frac{-b}{c}\frac{1}{1+\beta},$$

$$k_z\,\max=\frac{b}{c}\frac{1}{1-\beta}, \qquad (23)$$

and with

$$k_\rho^2=-\frac{k_z^2}{\gamma^2}+2\frac{b}{c}\beta k_z+\frac{b^2}{c^2}, \qquad (24)$$

where quantity ζ is still defined according to Eq. (5), always with $v<c$.

One can show that the unique exact solution previously known [9] may be rewritten in the form of Eq. (22) with $S(k_z)=\text{const}$. Then, on following the same procedure exploited in our first method (preceding section), one can find out exact solutions corresponding to

$$S(k_z)=\exp\left(\frac{i2n\pi k_z}{K}\right), \qquad (25)$$

where

$$K\equiv k_z\,\max-k_z\,\min,$$

by performing the change of variable [analogous, in its finality, to the one in Eq. (7)]

$$k_z\equiv\frac{b}{c}\gamma^2(s+\beta). \qquad (26)$$

At the end, the exact subluminal solution corresponding to the spectrum (25) results to be

$$\Psi(\rho,\zeta,\eta)=2\frac{b}{c}\gamma^2\exp\left(i\frac{b}{c}\beta\gamma^2\eta\right)\exp(in\pi\beta)\times\text{sinc}\left[\sqrt{\frac{b^2}{c^2}\gamma^2\rho^2+\left(\frac{b}{c}\gamma^2\zeta+n\pi\right)^2}\right]. \qquad (27)$$

We can again observe that any spectra $S(k_z)$ can be expanded, in the interval $k_z\,\min<k_z<k_z\,\max$, in a Fourier series

$$S(k_z)=\sum_{n=-\infty}^{\infty}A_n\exp\left(+in\frac{2\pi}{K}k_z\right), \qquad (28)$$

with coefficients now given by

$$A_n=\frac{1}{K}\int_{k_z\,\min}^{k_z\,\max}dk_zS(k_z)\exp\left(-in\frac{2\pi}{K}k_z\right), \qquad (29)$$

quantity K having been defined above.

At the very end of the whole procedure, the general exact solution representing a subluminal LW, for any spectra $S(k_z)$, can be eventually written as

$$\Psi(\rho,\zeta,\eta) = 2\frac{b}{c}\gamma^2 \exp\left(i\frac{b}{c}\beta\gamma^2\eta\right)\sum_{n=-\infty}^{\infty} A_n \exp(in\pi\beta) \times \text{sinc}\left[\sqrt{\frac{b^2}{c^2}\gamma^2\rho^2 + \left(\frac{b}{c}\gamma^2\zeta + n\pi\right)^2}\right], \quad (30)$$

whose coefficients are expressed in Eq. (29), and where quantity η is defined as above, in Eq. (11).

Interesting examples could be easily worked out, as we did at the end of the preceding section.

IV. STATIONARY SOLUTIONS WITH ZERO-SPEED ENVELOPES ("FROZEN WAVES")

Here, we shall refer to the (second) method, expounded in the preceding section. Our solution (30), for the case of envelopes *at rest*, that is, in the case $v=0$ (which implies $\zeta = z$), becomes

$$\Psi(\rho,z,t) = 2\frac{b}{c}\exp(-ibt) \times \sum_{n=-\infty}^{\infty} A_n \,\text{sinc}\left[\sqrt{\frac{b^2}{c^2}\rho^2 + \left(\frac{b}{c}z + n\pi\right)^2}\right], \quad (31)$$

with coefficients A_n given by Eq. (29) with $v=0$, so that its integration limits simplify into $-b/c$ and b/c, respectively; thus, one obtains

$$A_n = \frac{c}{2b}\int_{-b/c}^{b/c} dk_z S(k_z)\exp\left(-in\frac{c\pi}{b}k_z\right). \quad (29')$$

Equation (31) is an exact solution, corresponding to stationary beams with a *static* intensity envelope. Let us observe, however, that even in this case one has energy propagation, as it can be easily verified from the power flux $\boldsymbol{S}_s = -\boldsymbol{\nabla}\Psi_{\mathcal{R}}\partial\Psi_{\mathcal{R}}/\partial t$ (scalar case) or from the Poynting vector $\boldsymbol{S}_v = (\boldsymbol{E}\times\boldsymbol{H})$ (vectorial case, the condition being that $\Psi_{\mathcal{R}}$ be a single component, A_z, of the vector potential $\boldsymbol{A}$) [6]. We have here indicated by $\Psi_{\mathcal{R}}$ the real part of Ψ. For $v=0$, Eq. (2) becomes

$$\omega = b \equiv \omega_0,$$

so that the particular subluminal waves endowed with null velocity are actually monochromatic beams. [Incidentally, let us seize the present opportunity for recalling that only for superluminal LWs one can meet a rigid motion not only of the field amplitude, but also of its real and imaginary parts: In the most general case, the field magnitude does keep its shape while propagating (according to the definition of LW), but its real and imaginary parts suffer local variations; in the sense that the latter are still LWs, but their envelope (which determines the field shape) appears multiplied by a plane wave, which entails the already mentioned local variations. For instance, the real part of the simple Mackinnon solution, Eq. (10), appears as the product of a nondiffracting envelope and a plane wave. This can be verified, more in general, by inspection of Eq. (8). Of course, both the single real, or imaginary, parts carry energy and momentum, too. We had just recourse to the real part only, since the particular fields considered by us, for the scalar optical or acoustic cases, are obviously real (even if one always adopts the customary complex formalism, mainly for elegance reasons), and the power flux associated with Ψ must be obtained from the real part.]

It may be stressed that the present (second) method does yield exact solutions, without any need of the paraxial approximation, which, on the contrary, is so often used when looking for expressions representing beams, such as the Gaussian ones. Let us recall that, when having recourse to the paraxial approximation, the obtained beam expressions are valid only when the envelope sizes (e.g., the beam spot) vary in space much more slowly than the beam wavelength. For instance, the usual expression for a Gaussian beam [56] holds only when the beam spot $\Delta\rho$ is much larger than $\lambda_0 \equiv \omega_0/(2\pi c) = b/(2\pi c)$, so that those beams cannot be very localized. By contrast, our method overcomes such problems, since it yields, as we have seen above, exact expressions for (well localized) beams with sizes of the order of their wavelength. Notice, moreover, that the already known exact solutions—for instance, the Bessel beams—are nothing but particular cases of our solution (31).

An example. On choosing (with $0 \le q_- < q_+ \le 1$) the spectral double-step function

$$S(k_z) = \begin{cases} \dfrac{c}{\omega_0(q_+ - q_-)} & \text{for } q_-\omega_0/c \le k_z \le q_+\omega_0/c, \\ 0 & \text{elsewhere,} \end{cases} \quad (32)$$

the coefficients of Eq. (31) become

$$A_n = \frac{ic}{2\pi n\omega_0(q_+ - q_-)}(e^{-iq_+\pi n} - e^{-iq_-\pi n}). \quad (33)$$

The double-step spectrum (32), with regard to the longitudinal wave number, corresponds to the mean value $\bar{k}_z = \omega_0(q_+ + q_-)/2c$ and to the width $\Delta k_z = \omega_0(q_+ - q_-)/c$. From these relations, it follows that $\Delta k_z/\bar{k}_z = 2(q_+ - q_-)/(q_+ + q_-)$.

For values of q_- and q_+ that do not satisfy the inequality $\Delta k_z/\bar{k}_z \ll 1$, the resulting solution will be a nonparaxial beam.

Figure 3 shows the exact solution corresponding to $\omega_0 = 1.88\times 10^{15}$ Hz (i.e., $\lambda_0 = 1$ μm) and to $q_- = 0.3$, $q_+ = 0.9$, which results to be a beam with a spot-diameter of 0.6 μm, and, moreover, with a rather good longitudinal localization. In the case of Eqs. (32) and (33), about 21 terms ($-10 \le n \le 10$) in the sum entering Eq. (31) are quite enough for a good evaluation of the series. The beam considered in this example is highly nonparaxial (with $\Delta k_z/\bar{k}_z = 1$), and therefore could not have been obtained by ordinary Gaussian

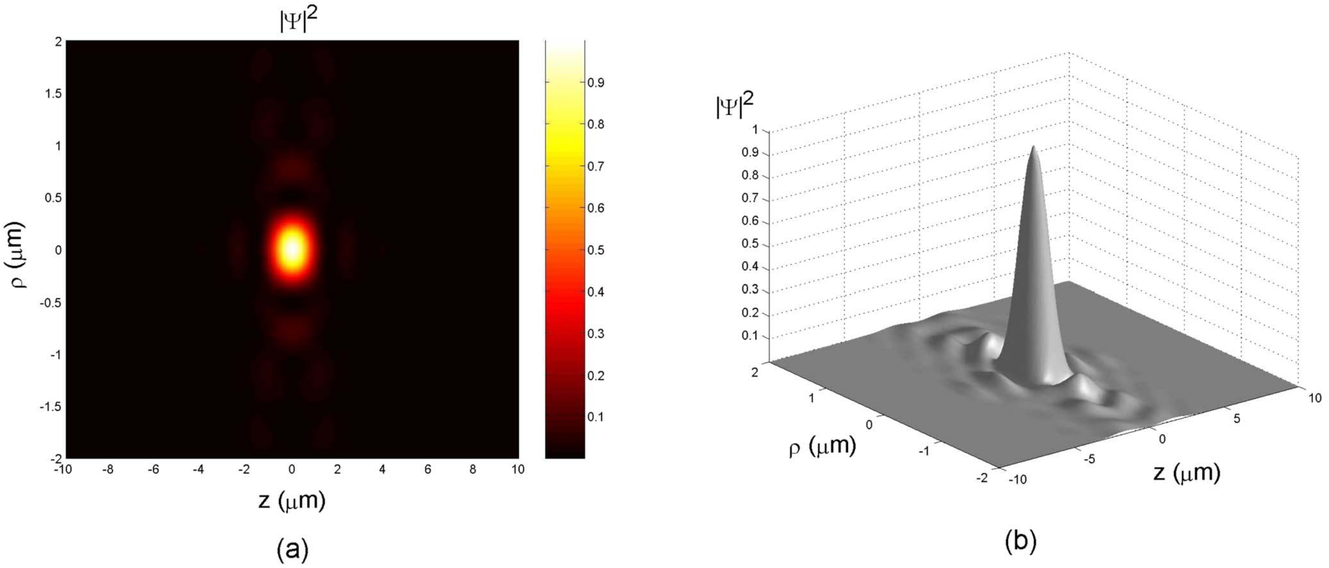

FIG. 3. (Color online) (a) Orthogonal projection of the three-dimensional intensity pattern of the beam (a null-speed subluminal wave) corresponding to spectrum (32); (b) 3D plot of the field intensity. The beam considered in this example is highly nonparaxial.

beam solutions (which are valid in the paraxial regime only).[3]

Let us now emphasize that a noticeable property of our present method is that it allows a spatial modeling even of monochromatic fields (that correspond to envelopes at rest; so that, in the electromagnetic cases, one can speak, e.g., of the modeling of "light-fields at rest"). Such a property—rather interesting, especially for applications [55]—was already investigated, under different assumptions, in Refs. [49,63,16], where the stationary fields with static envelopes were called "frozen waves" (FW). Namely, in the quoted references, discrete superpositions of Bessel beams were adopted in order to get a predetermined longitudinal (on-axis) intensity pattern, inside the desired space interval $0<z<L$. In other words, in Refs. [49,63,16], the frozen waves have been written in the form

$$\Psi(\rho,z,t) = e^{-i\omega_0 t} e^{iQz} \sum_{n=-N}^{N} B_n J_0(\rho k_{\rho n}) e^{i2n\pi z/L} \tag{34}$$

with

$$B_n = \frac{1}{L}\int_0^L dz F(z) e^{-i2n\pi z/L}, \tag{35}$$

quantity $|F(z)|^2$ being the desired longitudinal intensity shape, chosen *a priori*. In Eq. (34), it is $k_{\rho n}^2 = \omega_0^2/c^2 - k_{zn}^2$, and $0 \leq Q + 2N\pi/L \leq \omega_0/c$, where we set $k_{zn} \equiv Q + 2n\pi/L$. As we see from Eq. (34), the FWs have been represented in the past in terms of discrete superpositions of Bessel beams. But, now, the method exploited in this paper allows us to go on to dealing with continuous superpositions. In fact, the continuous superposition analogous to Eq. (34) are written as

$$\Psi(\rho,z,t) = e^{-i\omega_0 t}\int_{-\omega_0/c}^{\omega_0/c} dk_z S(k_z) J_0(\rho k_\rho) e^{izk_z}, \tag{36}$$

which, actually, is nothing but our previous equation (22) with $v=0$ (and therefore $\zeta=z$): That is, Eq. (36) does just represent a null-speed subluminal wave. To be clearer, let us recall that the FWs were expressed in the past as discrete superposition, mainly because it was not known at that time how to analytically treat a continuous superposition as Eq. (36). Only by following the method presented in this work one can eventually extend the FW approach [49,63,16] to the case of integrals: without numerical simulations, but in terms once more of analytic solutions.

Indeed, the exact solution of Eq. (36) is given by Eq. (31), with coefficients (29′). One can choose the spectral function $S(k_z)$ in such a way that Ψ assumes the on-axis prechosen static intensity pattern $|F(z)|^2$. Namely, the equation to be satisfied by $S(k_z)$, to such an aim, comes by associating Eq. (36) with the requirement $|\Psi(\rho=0,z,t)|^2 = |F(z)|^2$, which entails the integral relation

$$\int_{-\omega_0/c}^{\omega_0/c} dk_z S(k_z) e^{izk_z} = F(z). \tag{37}$$

Equation (37) would be trivially soluble in the case of an integration between $-\infty$ and $+\infty$, since it would merely be a Fourier transformation; but obviously this is not the case, because its integration limits are finite. Actually, there are functions $F(z)$ for which Eq. (37) is not soluble, in the sense that no spectra $S(k_z)$ exist obeying the last equation. Namely, if we consider the Fourier expansion

[3]We are considering here only scalar wave fields. In the case of nonparaxial optical beams, the vector character of the field must be considered.

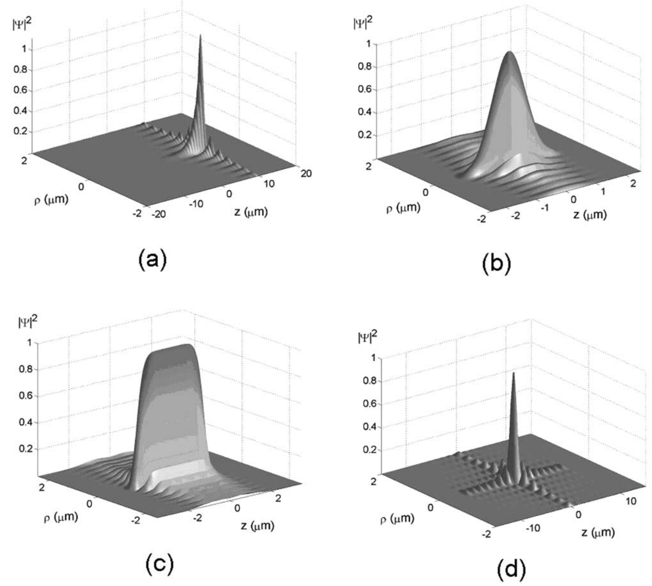

FIG. 4. Frozen waves with the on-axis longitudinal field pattern chosen as (a) exponential; (b) Gaussian; (c) super-Gaussian; (d) zero-order Bessel function.

$$F(z)=\int_{-\infty}^{\infty}dk_z\tilde{S}(k_z)e^{izk_z},$$

when $\tilde{S}(k_z)$ does assume non-negligible values outside the interval $-\omega_0/c<k_z<\omega_0/c$, then in Eq. (37) no $S(k_z)$ can forward that particular $F(z)$ as a result.

However, a way out can be devised, such that one can nevertheless find out a function $S(k_z)$ that approximately (but satisfactorily) complies with Eq. (37).

The first way out consists in writing $S(k_z)$ in the form

$$S(k_z)=\frac{1}{K}\sum_{n=-\infty}^{\infty}F\left(\frac{2n\pi}{K}\right)e^{-i2n\pi k_z/K}. \tag{38}$$

where, as before, $K=2\omega_0/c$. Then, one can easily verify Eq. (38) to guarantee that the integral in Eq. (37) yields the values of the desired $F(z)$ at the discrete points $z=2n\pi/K$. Indeed, the Fourier expansion (38) is already of the same type as Eq. (28), so that in this case the coefficients A_n of our solution (31), appearing in Eq. (29), do simply become

$$A_n=\frac{1}{K}F\left(-\frac{2n\pi}{K}\right). \tag{39}$$

This is a powerful way for obtaining a desired longitudinal (on-axis) intensity pattern, especially for very small spatial regions, because it is not necessary to solve any integral to find out the coefficients A_n, which by contrast are given directly by Eq. (39).

Figure 4 depicts some interesting applications of this method. A few desired longitudinal intensity patterns $|F(z)|^2$ have been chosen, and the corresponding frozen waves calculated by using Eq. (31) with the coefficients A_n given in Eq. (39). The desired patterns are enforced to exist within very small spatial intervals only, in order to show the capability of our method to model the field intensity shape also under such strict requirements.

In the four examples below, we considered a wavelength $\lambda=0.6$ μm, which corresponds to $\omega_0=b=3.14\times10^{15}$ Hz. The first longitudinal (on-axis) pattern considered by us is that given by

$$F(z)=\begin{cases}e^{a(z-Z)} & \text{for } 0\le z\le Z,\\ 0 & \text{elsewhere,}\end{cases}$$

i.e., a pattern with an exponential increase, starting from $z=0$ until $z=Z$. The chosen values of a and Z are $Z=10$ μm and $a=3/Z$. The intensity of the corresponding frozen wave is shown in Fig. 4(a).

The second longitudinal pattern (on axis) taken into consideration is the Gaussian one, given by

$$F(z)=\begin{cases}e^{-q(z/Z)^2} & \text{for } -Z\le z\le Z,\\ 0 & \text{elsewhere,}\end{cases}$$

with $q=2$ and $Z=1.6$ μm. The intensity of the corresponding frozen wave is shown in Fig. 4(b).

In the third example, the desired longitudinal pattern is supposed to be a super-Gaussian,

$$F(z)=\begin{cases}e^{-q(z/Z)^{2m}} & \text{for } -Z\le z\le Z,\\ 0 & \text{elsewhere,}\end{cases}$$

where m controls the edge sharpness. Here we have chosen $q=2$, $m=4$, and $Z=2$ μm. The intensity of the frozen wave obtained in this case is shown in Fig. 4(c).

Finally, in the fourth example, let us choose the longitudinal pattern as being the zero-order Bessel function

$$F(z)=\begin{cases}J_0(qz) & \text{for } -Z\le z\le Z,\\ 0 & \text{elsewhere,}\end{cases}$$

with $q=1.6\times10^6$ m^{-1} and $Z=15$ μm. The intensity of the corresponding frozen wave is shown in Fig. 4(d).

Let us observe that, of course, any static envelopes of this type can be easily transformed into propagating pulses by the mere application of Lorentz transformations.

Another way out exists for evaluating $S(k_z)$, based on the assumption that

$$S(k_z)\simeq\tilde{S}(k_z), \tag{40}$$

which constitutes a good approximation whenever $\tilde{S}(k_z)$ assumes negligible values outside the interval $[-\omega_0/c,\omega_0/c]$. In such a case, one can have recourse to the method associated with Eq. (28) and expand $\tilde{S}(k_z)$ itself in a Fourier series, obtaining eventually the relevant coefficients A_n by Eq. (29′). Let us recall that it is still $K\equiv k_z\max-k_z\min=2\omega_0/c$.

It may be interesting to call attention to the circumstance that, when constructing FWs in terms of a sum of discrete superpositions of Bessel beams (as it was done by us in Refs. [49,63,16,55]), it was easy to obtain extended envelopes such as, e.g., "cigars," where easy means having recourse to a few terms of the sum. By contrast, when we construct FWs—following this section—as continuous superpositions, then it is easy to get highly localized (concentrated) envelopes. Let us explicitly mention, moreover, that the method presented in this section furnishes FWs that are no longer periodic along the z axis (a situation that, with our old method [49,63], was obtainable only when the periodicity interval tended to infinity).

V. MENTIONING THE ROLE OF SPECIAL RELATIVITY AND OF LORENTZ TRANSFORMATIONS

Strict connections exist between, on one hand, the principles and structure of special relativity and, on the other hand, the whole subject of subluminal, luminal, superluminal localized waves, in the sense that it is expected for a long time that *a priori* they are transformable one into the other via suitable Lorentz transformations (cf. Refs. [61,62,65], besides work of our own in progress).

Let us first confine ourselves to the cases faced in this paper. Our subluminal localized pulses, that may be called "wave bullets," behave as particles: Indeed, our subluminal pulses [as well as the luminal and superluminal (X-shaped) ones, that have been amply investigated in the past literature] do exist as solutions of any wave equations, ranging from electromagnetism and acoustics or geophysics, to elementary particle physics (and even, as we discovered recently, to gravitation physics). From the kinematical point of view, the velocity composition relativistic law holds also for them. The same is true, more in general, for any localized waves (pulses or beams).

Let us start for simplicity by considering, in an initial reference frame O, just a (ν-order) Bessel beam

$$\Psi(\rho,\phi,z,t) = J_\nu(\rho k_\rho) e^{i\nu\phi} e^{izk_z} e^{-i\omega t}; \tag{41}$$

in Ref. [66]—whose philosophy, which in part goes back to Refs. [61,62], has been constantly shared by us—it was first shown, by applying the appropriate Lorentz boost, that a second reference frame O', moving with respect to O with speed u—along the positive z axis and in the positive direction, for simplicity's sake—will observe the Bessel beam

$$\Psi(\rho',\phi',z',t') = J_\nu(\rho' k'_{\rho'}) e^{i\nu\phi'} e^{iz' k'_{z'}} e^{-i\omega' t'}. \tag{42}$$

Let us now pass to subluminal pulses. One can investigate the action of a Lorentz transformation (LT), by expressing them either via the first method (Sec. II) or via the second one (Sec. III). Let us consider for instance, in the frame O, a v-speed (subluminal) pulse, given by Eq. (3). When one goes on to a second observer O' moving with the same speed v w.r.t. frame O, and, still for the sake of simplicity, passing through the origin O of the initial frame at time $t=0$, the observer O' will see, as explicitly noticed [66] by applying again the suitable LT, the pulse

$$\Psi(\rho',z',t') = e^{-it'\omega_0'} \int_{\omega_-}^{\omega_+} d\omega S(\omega) J_0(\rho' k'_{\rho'}) e^{iz' k'_{z'}}, \tag{43}$$

with

$$k'_{z'} = \gamma^{-1}\omega/v - \gamma b/v, \quad \omega' = \gamma b \equiv \omega_0{}', \quad k'_{\rho'} = \omega_0'/c^2 - k'^2_{z'}. \tag{44}$$

Notice that $k'_{z'}$ is a function of ω, as expressed by the first one of the three relations in the Eqs. (44); and that here ω' is a constant. It is interesting that Eq. (43) can be written as

$$\Psi(\rho',z',t') = \gamma v e^{-it'}\omega_0{}' \int_{-\omega_0{}'/c}^{\omega_0{}'/c} dk'_{z'} \bar{S}(k'_{z'}) J_0(\rho' k'_{\rho'}) e^{iz' k'_{z'}}, \tag{45}$$

with $\bar{S}(k'_{z'}) = S(\gamma v k'_{z'} + \gamma^2 b)$. Equation (45) describes monochromatic beams with axial symmetry (and does coincide also with what was derived within our second method, in Sec. III, when posing $v=0$).

One can therefore conclude, in agreement with Ref. [66], that a subluminal pulse, given by Eq. (3), which appears as a v-speed pulse in a frame O, will appear in another frame O' (traveling w.r.t. observer O with the same speed v in the same direction z) just as the monochromatic beam in Eq. (45) endowed with angular frequency $\omega_0' = \gamma b$, whatever be the pulse spectral function in the initial frame O: Even if the kind of monochromatic beam one arrives to does of course depend on the chosen $S(\omega)$. [One gets in particular a Bessel-type beam when S is a Dirac's δ function, $S(\omega)=\delta(\omega-\omega_0)$; let us moreover notice that, on applying a LT to a Bessel beam, one obtains another Bessel beam, with a different axicon angle.] The vice versa is also true, in general.

Let us set forth explicitly an observation that has not been noticed in the existing literature yet. Namely, let us mention that, when starting not from Eq. (3) but from the most general solutions which—as we have already seen—are sums of solutions (3) over the various values b_m of b, then a Lorentz transformation will lead us to a sum of monochromatic beams; actually, of harmonics (rather than to a single monochromatic beam). In particular, if one wants to obtain a sum of harmonic beams, one must apply a LT to more general subluminal pulses.

Let us add that also the various superluminal localized pulses get transformed one into the other by the mere application [66] of ordinary LTs; while it may be expected that the subluminal and the superluminal LWs are to be linked (apart from some known technical difficulties, that require a particular caution) by the superluminal Lorentz "transformations" expounded long ago, e.g., in Refs. [62,67,65,61] and references therein.[4] Let us recall once more that, in the years 1980–1982, special relativity, in its nonrestricted version, predicted that, while the simplest subluminal object is obviously a sphere (or, in the limit, a space point), the simplest

[4]One should pay due attention to the circumstance that, as we mention, the topic of superluminal LTs is a delicate [62,67,65,61] one, at the extent that the majority of the recent attempts to readdress this question and its applications seem to be defective (sometimes they do not even keep the necessary covariance of the wave equation itself).

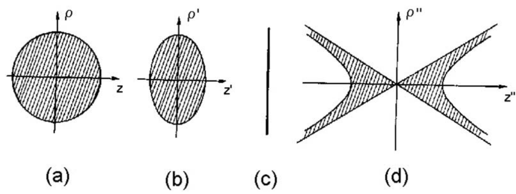

FIG. 5. Let us consider an object that is intrinsically spherical, i.e., that is a sphere in its rest frame [panel (a)]. After a generic subluminal LT along z, i.e., under a subluminal z boost, it is predicted by special relativity (SR) to appear as ellipsoidal due to Lorentz contraction [panel (b)]. After a superluminal z boost [62,67,65] (namely, when this object moves [60] with superluminal speed V), it is predicted by SR, in its nonrestricted version (ER), to appear [61] as in panel (d), i.e., as occupying the cylindrically symmetric region bounded by a two-sheeted rotation hyperboloid and an indefinite double cone. The whole structure, according to ER, is expected to move rigidly and, of course, with the speed V, the cotangent square of the cone semiangle being $(V/c)^2-1$. Panel (c) refers to the limiting case when the boost speed tends to c, either from the left or from the right (for simplicity, a space axis is skipped). It is remarkable that the shape of the localized (subluminal and superluminal) pulses, solutions to the wave equations, appears to follow the same behavior; this can have a role for a better comprehension even of de Broglie and Schrödinger wave mechanics. The present figure is taken from Refs. [61,62]. See also Fig. 6.

superluminal object is on the contrary an X-shaped pulse (or, in the limit, a double cone): cf. Fig. 5, taken from Refs. [61,62].

The circumstance that also the pattern of the localized solutions to the wave equations does meet this prediction is rather interesting, and is expected to be useful—in the case, e.g., of elementary particles and quantum physics—for a deeper comprehension of de Broglie's and Schrödinger's wave mechanics. With regard to the fact that the simplest subluminal LWs, solutions to the wave equation, are "ball-like," let us depict by Fig. 6, in the ordinary three-dimensional (3D) space, the general shape of the Mackinnon's solutions as expressed by Eq. (10) for $v \ll c$: In such figures we graphically represent the field isointensity surfaces, which in the considered case result to be (as expected) just spherical.

We have also seen that, even if our first method (Sec. II) cannot yield directly zero-speed envelopes, such envelopes "at rest," in Eq. (31), can be however obtained by applying a v-speed LT to Eq. (16). In this way, one starts from many frequencies [Eq. (16)] and ends up with one frequency only, since γb gets transformed into the frequency of the monochromatic beam.

VI. NONAXIALLY SYMMETRIC SOLUTIONS: THE CASE OF HIGHER-ORDER BESSEL BEAMS

Let us stress that until now we have paid attention to exact solutions representing axially symmetric (subluminal) pulses only: That is to say, to pulses obtained by suitable superpositions of zero-order Bessel beams.

It is however interesting to look also for analytic solutions representing nonaxially symmetric subluminal pulses, which can be constructed in terms of superpositions of ν-order Bessel beams, with ν a positive integer. This can be attempted both in the case of Sec. II (first method), and in the case of Sec. III (second method).

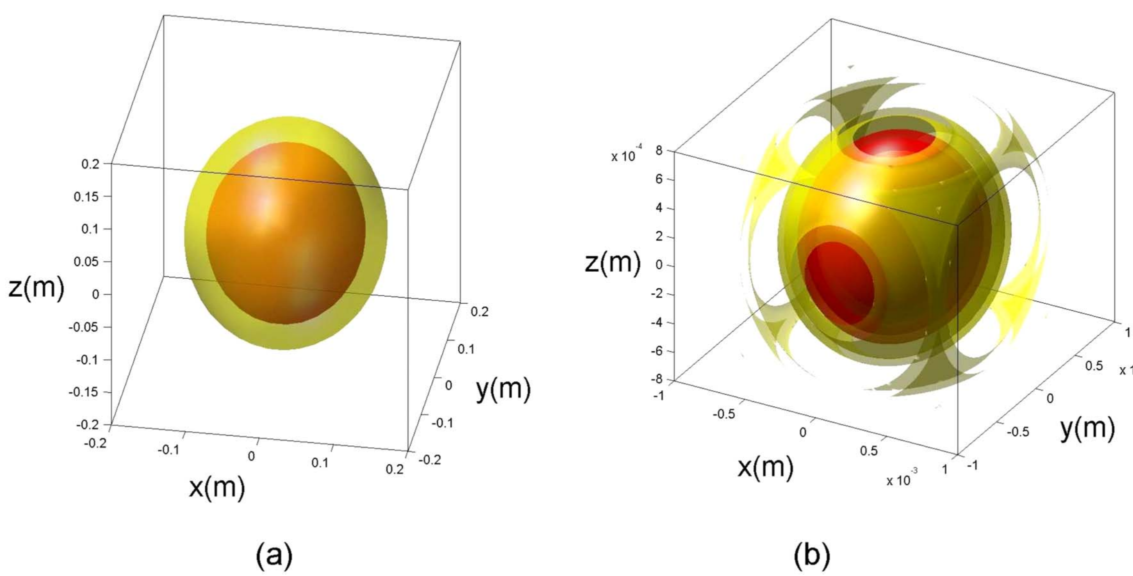

FIG. 6. (Color online) In Fig. 5 we have seen how SR, in its nonrestricted version (ER), predicted [61,62] that, while the simplest subluminal object is obviously a sphere (or, in the limit, a space point), the simplest superluminal object is on the contrary an X-shaped pulse (or, in the limit, a double cone). The circumstance that the localized solutions to the wave equations do follow the same pattern is rather interesting, and is expected to be useful—in the case, e.g., of elementary particles and quantum physics—for a deeper comprehension of de Broglie's and Schrödinger's wave mechanics. With regard to the fact that the simplest subluminal LWs, solutions to the wave equations, are "ball-like," let us depict by these figures, in the ordinary 3D space, the general shape of the Mackinnon's solutions as expressed by Eq. (10), numerically evaluated for $v \ll c$. In (a) and (b) we graphically represent the field isointensity surfaces, which in the considered case result to be (as expected) just spherical.

For brevity's sake, let us take only the first method (Sec. II) into consideration. One is immediately confronted with the difficulty that no exact solution is known for the integral in Eq. (8) when $J_0(\cdots)$ is replaced with $J_\nu(\cdots)$.

One can overcome this difficulty by following a simple method, which will allow us to obtain "higher-order" subluminal waves in terms of the axially symmetric ones. Indeed, it is well known that, if $\Psi(x,y,z,t)$ is an exact solution to the ordinary wave equation, then $\partial^n\Psi/\partial x^n$ and $\partial^n\Psi/\partial y^n$ are also exact solutions.[5] One should notice that, on the contrary, when working in cylindrical coordinates, if $\Psi(\rho,\phi,z,t)$ is a solution to the wave equation, $\partial\Psi/\partial\rho$ and $\partial\Psi/\partial\phi$ are not solutions, in general. Nevertheless, it is not difficult at all to reach the noticeable conclusion that, once $\Psi(\rho,\phi,z,t)$ is a solution, then also

$$\bar{\Psi}(\rho,\phi,z,t) = e^{i\phi}\left(\frac{\partial\Psi}{\partial\rho} + \frac{i}{\rho}\frac{\partial\Psi}{\partial\phi}\right) \tag{46}$$

is an exact solution. For instance, for an axially symmetric solution of the type $\Psi = J_0(k_\rho\rho)\exp(ik_z)\exp(-i\omega t)$, Eq. (46) yields $\bar{\Psi} = -k_\rho J_1(k_\rho\rho)\exp(i\phi)\exp(ik_z)\exp(-i\omega t)$, which is actually another analytic solution.

In other words, it is enough to start for simplicity from a zero-order Bessel beam, and to apply Eq. (46), successively, ν times, in order to obtain as a solution $\bar{\Psi} = (-k_\rho)^\nu J_\nu(k_\rho\rho)\exp(i\nu\phi)\exp(ik_z)\exp(-i\omega t)$, which is a ν-order Bessel beam.

In such a way, when applying ν times Eq. (46) to the (axially symmetric) subluminal solution $\Psi(\rho,z,t)$ in Eqs. (16), (15), and (14) [obtained from Eq. (3) with spectral function $S(\omega)$], we are able to obtain the subluminal nonaxially symmetric pulses $\Psi_\nu(\rho,\phi,z,t)$ as analytic solutions, consisting as expected in superpositions of ν-order Bessel beams,

$$\Psi_\nu(\rho,\phi,z,t) = \int_{\omega_-}^{\omega_+} d\omega S'(\omega) J_\nu(k_\rho\rho) e^{i\nu\phi} e^{ik_z z} e^{-i\omega t}, \tag{47}$$

with $k_\rho(\omega)$ given by Eq. (4), and quantities $S'(\omega) = [-k_\rho(\omega)]^\nu S(\omega)$ being the spectra of the pulses. If $S(\omega)$ is centered at a certain carrier frequency (it is a Gaussian spectrum, for instance), then $S'(\omega)$ too will approximately be of the same type.

Now, if we wish the solution $\Psi_\nu(\rho,\phi,z,t)$ to possess a predefined spectrum $S'(\omega)=F(\omega)$, we can first take Eq. (3) and set $S(\omega)=F(\omega)/[-k_\rho(\omega)]^\nu$ in its solution (16), and afterwards apply to it, ν times, the operator $U \equiv \exp(i\phi)[\partial/\partial\rho + (i/\rho)\partial/\partial\phi]$: As a result, we will obtain the desired pulse, $\Psi_\nu(\rho,\phi,z,t)$, endowed with $S'(\omega)=F(\omega)$.

An example. On starting from the subluminal axially symmetric pulse $\Psi(\rho,z,t)$, given by Eq. (16) with the Gaussian spectrum (17), we can obtain the subluminal, nonaxially symmetric, exact solution $\Psi_1(\rho,\phi,z,t)$ by simply calculating

[5]Let us mention that even $\partial^n\Psi/\partial z^n$ and $\partial^n\Psi/\partial t^n$ will be exact solutions.

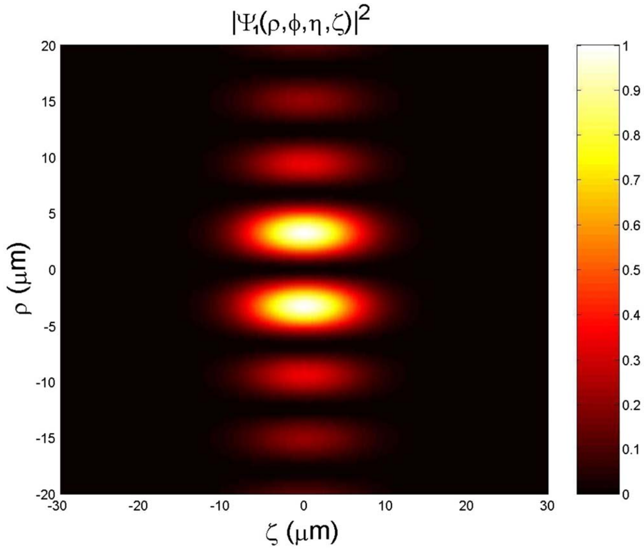

FIG. 7. (Color online) Orthogonal projection of the field intensity corresponding to the higher-order subluminal pulse, $\Psi_1(\rho,\phi,\eta,\zeta)$, represented by the exact solution equation (49), quantity Ψ being given by Eq. (16) with the Gaussian spectrum (17). The pulse intensity happens to have a "donut"-like shape.

$$\Psi_1(\rho,\phi,z,t) = \frac{\partial\Psi}{\partial\rho}e^{i\phi}, \tag{48}$$

which actually yields the "first-order" pulse $\Psi_1(\rho,\phi,z,t)$, which can be more compactly written in the form

$$\Psi_1(\rho,\phi,\eta,\zeta) = 2\frac{b}{c}v\gamma^2 \exp\left(i\frac{b}{c}\beta\gamma^2\eta\right)\sum_{n=-\infty}^{\infty} A_n \exp\left(in\frac{\pi}{\beta}\right)\psi_{1n} \tag{49}$$

with

$$\psi_{1n}(\rho,\phi,\eta,\zeta) \equiv \frac{b^2}{c^2}\gamma^2\rho Z^{-3}(Z\cos Z - \sin Z)e^{i\phi}, \tag{50}$$

where

$$Z \equiv \sqrt{\frac{b^2}{c^2}\gamma^2\rho^2 + \left(\frac{b}{c}\gamma^2\zeta + n\pi\right)^2}. \tag{51}$$

This exact solution, let us repeat, corresponds to superposition (47), with $S'(\omega)=k_\rho(\omega)S(\omega)$, quantity $S(\omega)$ being given by Eq. (17). It is represented in Fig. 7. The pulse intensity has a "donutlike" shape.

VII. CONCLUSIONS

As in the well-known superluminal case [1], the subluminal localized waves can be obtained by superposing Bessel beams. However, they have been scarcely considered in the past, for the reason that the superposition integral must run in this case over a finite interval (which makes it mathematically difficult to work out analytic expressions for them). In this paper, however, we have obtained nondiffracting subluminal pulses as exact analytic solutions to the wave equations: For arbitrarily chosen frequencies and bandwidths, avoiding any recourse to the backward-traveling components, and in a simple way.

Indeed, only one closed-form subluminal LW solution, ψ_{cf}, to the wave equations was known [9]: It was obtained by choosing, in the relevant integration, a constant weight function $S(\omega)$; while all other solutions had been previously obtained only by numerical simulations. By contrast, we have shown that, for instance, a subluminal LW can be obtained in closed form by adopting any spectra $S(\omega)$ that are expansions in terms of ψ_{cf}. In fact, the initial disadvantage, of having to deal with a limited bandwidth, may be turned into an advantage, since in the case of "truncated" integrals the spectrum $S(\omega)$ can be expanded in a Fourier series.

More in general, it has been shown in this paper how one can arrive at exact solutions both by integration over the Bessel beams' angular frequency ω, and by integration over their longitudinal wave number k_z. Both methods are expounded above. The first one appears to be comprehensive enough; we have studied the second method as well, however, since it allows tackling also the limiting case of zero-speed solutions (thus furnishing a second way, in terms of continuous spectra, for obtaining the so-called "frozen waves," quite promising also from the applicative point of view). We have briefly treated the case, moreover, of nonaxially symmetric solutions, that is, of higher-order Bessel beams.

At last, some attention has been paid to the role of special relativity, and to the fact that the localized waves are to be transformed one into the other by suitable Lorentz transformations. Moreover, our results seem to show that in the subluminal case the simplest LW solutions are (for $v \ll c$) "ball"-like, as expected since long ago [61] on the mere basis of special relativity [62]: More precisely, already in the years 1980–1982 it had been predicted that, if the simplest subluminal object is a sphere (or, in the limit, a space point), then the simplest superluminal object is an X-shaped pulse (or, in the limit, a double cone); and vice versa: cf. Fig. 5. It is rather interesting that the same pattern appears to be followed by the localized solutions of the wave equations. For the subluminal case, see, e.g., Fig. 6. The localized pulses, endowed with a finite energy, or merely truncated, will be constructed in another presentation.

In the present work we have fixed our attention on acoustics and optics. However, analogous results are valid whenever an essential role is played by a wave equation (such as electromagnetism, seismology, geophysics, gravitation, elementary particle physics, etc.).

ACKNOWLEDGMENTS

This work was partially supported by FAPESP and CNPq (Brazil), and by INFN, MIUR (Italy). The authors are indebted to H. E. Hernández-Figueroa for his continuous collaboration, and thank L. R. Baldini, I. M. Besieris, A. Castoldi, C. Conti, J-y. Lu, G. C. Maccarini, R. Riva, P. Saari, A. M. Shaarawi, and M. Tygel for useful discussions or kind interest. The authors are moreover deeply grateful to an anonymous referee for many, very useful remarks.

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