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Time delay in thin slabs with self-focusing Kerr-type nonlinearity

G. Isić,[1,*] V. Milanović,[2] J. Radovanović,[2] Z. Ikonić,[1] D. Indjin,[1] and P. Harrison[1]
[1]*School of Electronic and Electrical Engineering, University of Leeds, Leeds LS2 9JT, United Kingdom*
[2]*School of Electrical Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, 11120 Belgrade, Serbia*
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Time delays for an intense transverse electric (TE) wave propagating through a Kerr-type nonlinear slab are investigated. The relation between the bidirectional group delay and the dwell time is derived and it is shown that the difference between them can be separated into three terms. The first one is the familiar self-interference time, due to the dispersion of the medium surrounding the slab. The other two terms are caused by the nonlinearity and oblique incidence of the TE wave. It is shown that the electric field distribution along the slab may be expressed in terms of Jacobi elliptic functions while the phase difference introduced by the slab is given in terms of incomplete elliptic integrals. The expressions for the field-intensity-dependent complex reflection and transmission coefficients are derived and the multivalued oscillatory behavior of the delay times for the case of a thin slab is demonstrated.

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I. INTRODUCTION

It is well known that tunneling represents a typically quantum-mechanical phenomenon. Soon after the discovery of tunneling, Condon raised the question of the speed of the tunneling process (in 1931) [1]. The papers published in the 1950's [2–4], have provided analytical expressions for the time delays, suggesting those times to be very short but finite. Since then, the matter of defining various delay times and the interpretation of obtained expressions, has been the focus of research of both theoretical and applied quantum mechanics, which is illustrated by the large number of review papers on this subject [5–7].

On the other hand, given the deep analogy between the Schrödinger equation and the Helmholtz equation, and the fact that the tunneling is present in the propagation of electromagnetic waves through optically heterogeneous media, a certain amount of attention has been devoted to the problem of finding delay times in these conditions, as well. In that respect, the following papers have been influential: a paper by Winful [8], and the experimental work of Enders and Nimtz [9], Steinberg [10], and Spielmann [11].

In this paper, we apply the formalism of delay times to investigate the temporal aspects of TE wave propagation through a nonlinear slab [12]. At perpendicular or only slightly oblique incidence, such as assumed in this paper, the TE waves are always propagating through the nonlinear slab (i.e., there is no evanescent decay) so, strictly speaking, there is no tunneling phenomena. However, we believe that the delay times are a useful concept even in this case since they cast more light on the very complicated dynamics of nonlinear wave propagation.

II. THEORETICAL MODELING AND NUMERICAL EXAMPLES

When illuminated by light of a very high intensity, such as a laser beam, some media exhibit a highly nonlinear response. If the material may be considered isotropic, its relative permittivity ϵ may be written as

$$\epsilon = \epsilon_L + \alpha_{NL}|\mathbf{E}|^2 \quad (1)$$

with only the lowest order of nonlinearity taken into account. Consider a slab of thickness L made of such a material, placed in a material with relative permittivity ϵ_1 and irradiated with a transverse electric (TE) wave as in Fig. 1. We shall label the axis perpendicular to the slab with x, let the electric field be pointed along the y axis and assume that the propagation constant along the z axis is $\beta = \sqrt{\epsilon_1} k_0 \sin\theta$, where θ is the angle of incidence with respect to the x axis. Further, assume that the angular frequency spectrum of the wave is sharply centered around ω and, therefore, that the vacuum propagation constant of the TE plane wave incident on the slab is $k_0 = \omega / c$. The Helmholtz equation within the slab reads

$$\frac{d^2E_y}{dx^2} + (\kappa^2 + \alpha_{\mathrm{NL}} k_0^2 |E_y|^2) E_y = 0, \quad \kappa^2 = \epsilon_L k_0^2 - \beta^2,$$

$$0 < x < L, \quad (2)$$

with E_y being the complex amplitude of the y component of the electric field. Introducing $E_y = \eta \exp[i\phi(x)]$, with real η

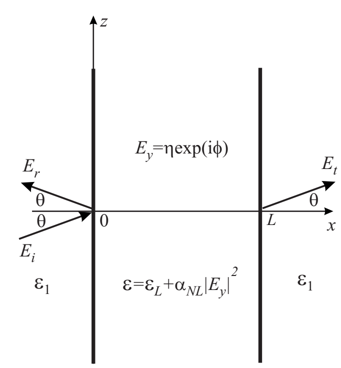

FIG. 1. Diagram shows a TE wave being obliquely incident on a Kerr-type nonlinear slab.

*elgi@leeds.ac.uk

>0 and ϕ, Eq. (2) can be separated into two equations involving real functions. From the imaginary part, we obtain

$$\eta^2 \frac{d\phi}{dx} = C_1 = 2\omega\mu_0 P_x, \tag{3}$$

where P_x is the x component of the time-averaged Poynting vector $\mathbf{P}$. The real part of Eq. (2) leads to

$$\left(\frac{d\eta}{dx}\right)^2 = C_2 - C_1\eta^{-2} - \kappa^2\eta^2 - \frac{\alpha_{\rm NL}}{2}k_0^2\eta^4, \tag{4}$$

with C_2 given by

$$C_2 = \left(\gamma^2 + \kappa^2 + \frac{\alpha_{\rm NL}\omega\mu_0}{\gamma}P_x k_0^2\right)\frac{2\omega\mu_0}{\gamma}P_x, \quad \gamma = \sqrt{\epsilon_1}k_0 \cos\theta. \tag{5}$$

We can rewrite Eq. (4) as

$$\left(\frac{d\eta}{dx}\right)^2 = -\frac{\alpha_{\rm NL}k_0^2}{2\eta^2}(\eta^2 - I_1)(\eta^2 - I_2)(\eta^2 - I_3), \quad I_3 = \frac{2\omega\mu_0}{\gamma}P_x, \tag{6}$$

with

$$I_{1/2} = -\left(\frac{\kappa^2}{\alpha_{\rm NL}k_0^2} + \frac{I_3}{2}\right) \mp \sqrt{\left(\frac{\kappa^2}{\alpha_{\rm NL}k_0^2} + \frac{I_3}{2}\right)^2 + \frac{2\gamma^2}{\alpha_{\rm NL}k_0^2}I_3}. \tag{7}$$

Assuming that the Kerr-type slab is of self-focusing type ($\alpha_{\rm NL}>0$) and that it is optically denser than the surrounding medium ($\epsilon_L > \epsilon_1$), it is easy to verify that

$$I_3 > I_2 > 0 > I_1 \quad \text{and} \quad I_2 \le \eta^2 \le I_3, \tag{8}$$

because η is real so the right-hand side of Eq. (6) must be positive. To integrate Eq. (6) we note that for $x<0$ we have $E_y = E_i + E_r$ and for $x>L$ there is only the transmitted wave, $E_y = E_t$ with

$$E_i = E_0 \exp(i\gamma x), \quad E_r = RE_0 \exp(-i\gamma x),$$

and

$$E_t = TE_0 \exp(i\gamma x), \tag{9}$$

where we introduced the field intensity dependent reflection and transmission coefficients $R=R(|E_0|)$ and $T=T(|E_0|)$, respectively. To find η in the above equations, we need to specify P_x which is uniquely determined by the transmitted wave amplitude $|E_t| = |TE_0|$. The inconvenience of using boundary conditions in $x<0$ stems from the fact that the response of the slab depends on $|E_0|$ so a self-consistent problem needs to be solved. However, for $x>L$ there is only one plane wave component so the field magnitude is constant and we can easily relate the field boundary conditions with the power flow in the x direction. Therefore, using $I_3 = |E_t|^2$ and integrating Eq. (6) from $x=L$ to any given point x in the slab, we can obtain the solution for η^2 in a closed form as a function of parameter $|E_t|$:

$$\pm k_0 A\sqrt{\frac{\alpha_{\rm NL}}{2}}(L-x) = A\int_0^{\sqrt{I_3-\eta^2(x)}} \frac{du}{\sqrt{(A^2-u^2)(B^2-u^2)}},$$

$$u = I_3 - \eta^2 \le B^2 < A^2, \tag{10}$$

with $B^2 = I_3 - I_2$ and $A^2 = I_3 - I_1$. Finally, the solution for η^2 is given by

$$\eta^2 = |E_t|^2 - B^2 \mathrm{sn}^2\left(Ak_0\sqrt{\frac{\alpha_{\rm NL}}{2}}(L-x), \frac{B}{A}\right), \tag{11}$$

where $\mathrm{sn}(u,k)$ is the Jacobi elliptic function with argument u and modulus k [13]. Using this result, we can integrate Eq. (3) to obtain the phase difference across the slab $\Delta\phi = \phi(L) - \phi(0)$:

$$\Delta\phi = \gamma\left(Ak_0\sqrt{\frac{\alpha_{\rm NL}}{2}}\right)^{-1}\Pi\left[\frac{B^2}{|E_t|^2}, F^{-1}\left(Ak_0\sqrt{\frac{\alpha_{\rm NL}}{2}}L, \frac{B}{A}\right), \frac{B}{A}\right], \tag{12}$$

where $\Pi(n,\varphi,k)$ is the incomplete elliptic integral of the third kind and $F^{-1}(u,k)$ is the inverse of the incomplete elliptic integral of the first kind[13].

To obtain R and T, we use the fact that E_y and $\frac{\partial E_y}{\partial x}$ are continuous at $x=0$ and $x=L$. Denoting $|E_y(x=0)|$ by $\eta(0)$ and $\frac{\partial|E_y(x=0)|}{\partial x}$ by $\eta'(0)$, we arrive at

$$R = \frac{\gamma\eta^2(0) - \gamma|E_t|^2 + i\eta(0)\eta'(0)}{\gamma\eta^2(0) + \gamma|E_t|^2 - i\eta(0)\eta'(0)}$$

and

$$T = \frac{2\gamma\eta(0)|E_t|\exp(i\Delta\phi)}{\gamma\eta^2(0) + \gamma|E_t|^2 - i\eta(0)\eta'(0)}. \tag{13}$$

Since we only specify the amplitude of the transmitted wave $|E_t|$, the phase of E_0 is arbitrary, i.e., our system is not sensitive to the phase of E_0 because it is stationary. If we choose the arbitrary phase so that $\phi(x=0)=0$, E_0 is given by

$$E_0 = \frac{1}{2\gamma}\left(\gamma\eta(0) + \gamma\frac{|E_t|^2}{\eta(0)} - i\eta'(0)\right). \tag{14}$$

Note that both $\eta(0)$ and $\eta'(0)$ are found in closed analytic form using Eq. (11) and some elementary properties of Jacobi elliptic functions. Since the values of $R(|E_0|)$, $T(|E_0|)$ and $|E_0|$ itself are given in terms of parameter $|E_t|$, in general, there will be more than one value of $R(|E_0|)$ and $T(|E_0|)$ corresponding to a given value of $|E_0|$. However, each of these solutions will have a different power flow in the x direction.

Using Eq. (11) we can easily analyze the behavior of $\eta = |E_y|$ inside the slab: η is a periodic function with period of $\frac{2K}{Ak_0}\sqrt{\frac{2}{\alpha_{NL}}}$ where K is the complete elliptic integral of the first kind, $K=F(\frac{\pi}{2},k)$ with modulus $k=\frac{B}{A}$. The peaks of η, $\eta_{\max} = \eta(x_{\max}^m) = |E_t|$, are located in points $x_{\max}^m$ satisfying $x_{\max}^m = L - 2m\frac{K}{Ak_0}\sqrt{\frac{2}{\alpha_{NL}}}$, $m=0,1,2,\ldots$, and starting from $x_{\max}^0 = L$. The minima of η, $\eta_{\min} = \eta(x_{\min}^m) = \sqrt{I_2}$, are located in points $x_{\min}^m = L-(2m+1)\frac{K}{Ak_0}\sqrt{\frac{2}{\alpha_{NL}}}$, $m=0,1,2,\ldots$. The condition of resonant transmission $|T|=1$ is that $|E_0| = |E_t|$ with zero reflected wave, i.e., $\eta(0) = \eta(L) = |E_t|$, hence the condition is that there is a positive integer m such that

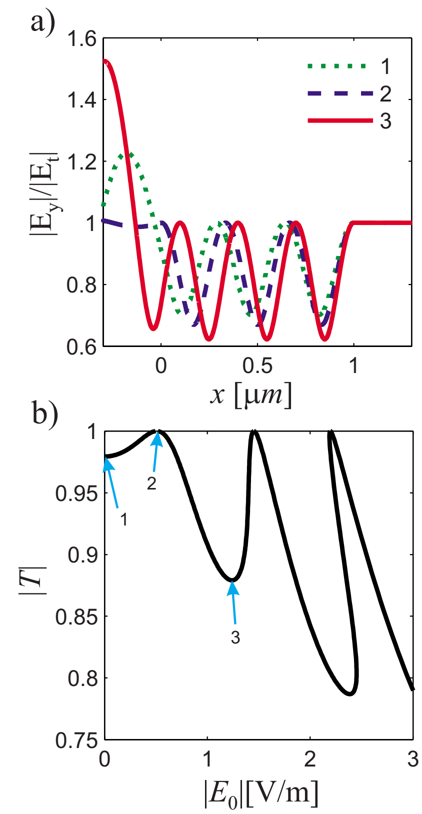

FIG. 2. (Color online) (a) Distribution of the normalized electric field magnitude $\frac{|E_y|}{|E_t|}$ for three different values of $|E_0|$. (b) Dependence of transmission magnitude $|T|$ on $|E_0|$. Points corresponding to different curves in (a) are labeled with arrows. The parameters are $\theta=10°$, $L=\lambda_0=1\ \mu$m, $\epsilon_1=1$, $\epsilon_L=2$, and $\alpha_{\mathrm{NL}}=1\frac{m^2}{V^2}$.

$$L = m\frac{2K}{Ak_0}\sqrt{\frac{2}{\alpha_{NL}}}. \qquad (15)$$

To illustrate the dependence of wave reflection and transmission on the intensity $|E_0|$ of the incident wave, we consider a slab with arbitrarily chosen $\epsilon_L=2$, $\alpha_{NL}=1\frac{m^2}{V^2}$, and $L=\lambda_0=1\ \mu$m (vacuum wavelength) placed in vacuum ($\epsilon_1=1$). Figure 2(a) shows the field distribution for three different values of $|E_0|$ and Fig. 2(b) shows the dependence of $|T|$ on $|E_0|$ with markers showing the points corresponding to curves in Fig. 2(a).

In the remainder of this paper, we derive the connection between two well-established delay times, the bidirectional group delay and the dwell time. Finally, we use the above given results to calculate the dependence of various delay times on the incident field intensity for the slab from Fig. 2.

The overall electromagnetic energy W, within the slab is obtained from the Poynting theorem assuming that the dispersion may be neglected in a narrow frequency band around ω:

$$W = \frac{S\epsilon_0}{2}\left(\int_0^L |E_y|^2\epsilon dx - \frac{\epsilon_1\cos\theta}{k_0}|E_0|^2\,\mathrm{Im}(R)\right), \qquad (16)$$

where S is the cross-sectional surface of the structure, perpendicular to the x axis. To determine the delay times through the thin slab and the way they are interrelated, we use the same procedure as in Ref. [14]. Starting from Eq. (2) and differentiating it with respect to ω and, subsequently multiplying it by E_y^*, we obtain the first expression. Then, we conjugate Eq. (2) and multiply it by $\frac{\partial E_y}{\partial\omega}$ to obtain the second expression which, when subtracted from the first one, yields

$$\frac{\partial}{\partial x}\left(E_y^*\frac{\partial^2 E_y}{\partial\omega\,\partial x} - \frac{\partial E_y}{\partial\omega}\frac{\partial E_y^*}{\partial x}\right) = -k_0^2|E_y|^2\left(\frac{2\tilde{\epsilon}}{\omega} + \frac{\partial\tilde{\epsilon}}{\partial\omega}\right),$$

$$\tilde{\epsilon} = \epsilon - \epsilon_1\sin^2\theta. \qquad (17)$$

Integrating Eq. (17) from $x=0^-$ to $x=L^+$, we arrive to

$$\tilde{\tau}_g + \mathrm{Im}(R)\frac{1}{\gamma}\frac{\partial\gamma}{\partial\omega} = \frac{k_0}{2\epsilon_1\cos\theta|E_0|^2}\int_0^L\left(\frac{2}{\omega}\tilde{\epsilon} + \frac{\partial\tilde{\epsilon}}{\partial\omega}\right)|E_y|^2 dx. \qquad (18)$$

The bidirectional group delay $\tilde{\tau}_g$ is defined by $\tilde{\tau}_g=|T|^2\frac{\partial\phi_0}{\partial\omega}+|R|^2\frac{\partial\phi_r}{\partial\omega}$, ($\phi_0=\gamma L+\phi_t$) while ϕ_r and ϕ_t are the arguments of the complex reflection and transmission coefficients, respectively. By defining the dwell time as $\tau_d=W/P_{\mathrm{in}}$, where $P_{\mathrm{in}}=\frac{S\sqrt{\epsilon_1}k_0\cos\theta}{2\omega\mu_0}|E_0|^2$ is the x component of the incoming power flux and using Eq. (16) we have

$$\tau_d = \frac{1}{c\epsilon_1\cos\theta|E_0|^2}\left(\int_0^L \epsilon|E_y|^2 dx - \frac{\epsilon_1\cos\theta|E_0|^2}{k_0}\mathrm{Im}(R)\right), \qquad (19)$$

so Eq. (18) can be rewritten as

$$\tilde{\tau}_g = \tau_d + \mathrm{Im}(R)\left(\frac{1}{\omega} - \frac{1}{\gamma}\frac{\partial\gamma}{\partial\omega}\right) + \tau_{\mathrm{NL}} - \tau_t, \qquad (20)$$

$$\tau_{\mathrm{NL}} = \frac{k_0}{2\epsilon_1\cos\theta|E_0|^2}\int_0^L \alpha_{\mathrm{NL}}|E_y|^2\frac{\partial(|E_y|^2)}{\partial\omega}dx, \qquad (21)$$

$$\tau_t = \frac{\sin^2\theta}{c\cos\theta|E_0|^2}\left(\frac{\omega}{2\epsilon_1}\frac{\partial\epsilon_1}{\partial\omega} + 1\right)\int_0^L |E_y|^2 dx. \qquad (22)$$

The second term on the right-hand side of Eq. (20) is called the self-interference time, i.e., $\tau_i=\mathrm{Im}(R)\left(\frac{1}{\omega}-\frac{1}{\gamma}\frac{\partial\gamma}{\partial\omega}\right)$. It describes the effect of dispersion in the surrounding medium in analogy with the quantum tunneling case [14]. However, in the case of a dispersionless surrounding medium, τ_i is equal to zero. This follows from the fact that within our model the waveguide width in the z direction is not limited, yielding the propagation constant along this direction $\beta=k_0\sin\theta\sqrt{\epsilon_1}$. Consequently, in the Helmholtz equation analogous to Eq. (2), written for semi-infinite layers surrounding the slab, the term in parentheses becomes $\gamma^2=\frac{\epsilon_1\omega^2}{c^2}-\beta^2=\frac{\epsilon_1\omega^2}{c^2}\cos^2\theta$. Thus, the self-interference term vanishes. However, if the waveguide width in the z direction is limited (as described in Ref. [8]), then values of β become quantized in terms of $\frac{l\pi}{a}$ (where l is an integer and a is the waveguide width) and the self-interference time τ_i remains finite.

The third term in Eq. (20), τ_{NL}, is the explicit contribution of the nonlinearity. The presence of the fourth term, τ_t, can

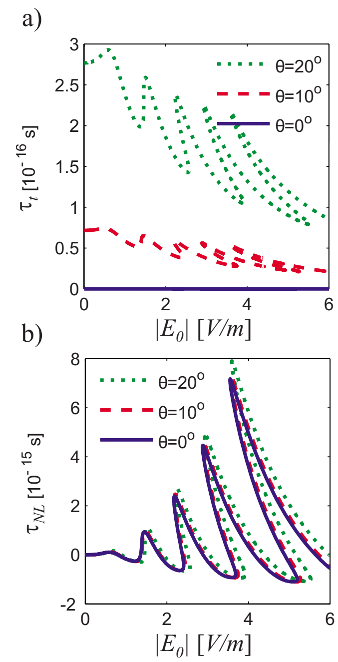

FIG. 3. (Color online) Dependence of (a) the transversal time τ_t and (b) the nonlinear term τ_{NL} on the intensity $|E_0|$ of the incident plane wave. The structure parameters are the same as in Fig. 2.

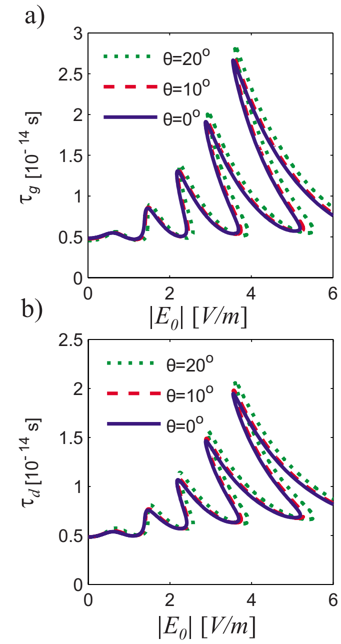

FIG. 4. (Color online) Dependence of (a) the bidirectional group delay $\tilde{\tau}_g$ and (b) the dwell time τ_d on $|E_0|$. Parameters are given in Fig. 2. Both $\tilde{\tau}_g$ and τ_d are at least an order of magnitude greater than τ_t and τ_{NL}. Since the group delay in absence of the slab is approximately 0.3×10^{-14} s, the velocities corresponding to these times are subluminal.

be explained by the following reasoning: when the wave front is tilted, any pulse to arrive to a point (x_O, y_O), will have been started off at some point (x_S, y_S) lying on the same wave front as $(x_P, y_P{=}y_O)$, whereas the expression for $\tilde{\tau}_d$ assumes that the pulse propagates from (x_P, y_P) to (x_O, y_O) which is why it has to be reduced by τ_t, a quantity accounting for the transversal propagation. Finally, the bidirectional group delay, $\tilde{\tau}_g$, may be written in the familiar form

$$\tilde{\tau}_g = \tau_d + \tau_i + \tau_{\mathrm{NL}} - \tau_t, \qquad (23)$$

with the last two terms going to zero for perpendicular incidence on a linear slab, $\alpha_{\mathrm{NL}}{=}0$.

In the case of previously considered slab in vacuum, the self-interference time goes to zero, $\tau_i{=}0$. Figures 3 and 4 show the dependence of τ_t, τ_{NL} and $\tilde{\tau}_g$, τ_d for several different values of the angle of incidence. The oscillatory field behavior is reflected in the delay times, as well. From Figs. 3 and 4 we see that the increased field intensity $|E_0|$ is followed by an increased oscillation amplitude and multivalued behavior with several stable states. The order of magnitude of $|E_0|$ leading to pronounced nonlinear behavior can be estimated by finding the first occurrence of the resonant transmission given by Eq. (15) and $m{=}1$. In the case of perpendicular incidence from a dispersionless surrounding medium on a linear slab, the familiar result [14] $\tilde{\tau}_g{=}\tau_d$ is recovered.

Reference [15] provides a general relation for traversal time of electromagnetic waves in terms of transmission and reflection amplitudes, ascribing a real and an imaginary component to this time. If we annul the nonlinearity in our expression for the dwell time and limit the analysis to normal incidence $(\theta{=}0)$, a suitable correlation can be established between that result and the real part of the traversal time from Ref. [15]. This stems from the fact that the results for traversal time presented therein rely on a more complex model for the linear regime, comprising the contribution of the Faraday effect.

III. CONCLUSION

This paper provides a comprehensive analysis of the problem of calculating the delay times (dwell time, bidirectional group delay, interference time) which characterize the transmission of electromagnetic waves through a thin slab with Kerr-type nonlinearity present. Particular consideration is given to the complex task of determining the field distribution within the slab. For this purpose, the Helmholtz equation is decomposed into two equations, one describing the amplitude of the field, and the other describing the phase of the field. While the second equation can easily be reduced to a simple integral equation, the solutions of the first one are given via elliptic functions. A simple analysis shows that all the required constants can be obtained if the integration is carried out backward. By expressing the phase shift along the slab via incomplete elliptic integrals, we arrived to a closed analytic expression for the complex reflection and transmission coefficients. Upon resolving the field distribution, in the second part of the paper, we derive the appropriate expressions for all three types of delay times and identify two additional terms τ_{NL} and τ_t. Finally, by calculating the delay times for an arbitrarily chosen thin slab, we show that these become very sensitive to changes in the incoming wave amplitude when it goes above the first resonant transmission condition. In this regime, an oscillatory behavior of the delay times with the increased field intensity is observed. Our results indicate that bistability and multivalued behavior are present in the delay times, as well. As pointed out in Ref. [15], the transversal electric field present in a slab of material exhibiting Kerr-type nonlinearity can be utilized to measure the interaction time of the electromagnetic waves in given region. Hence, by drawing on the theory presented there, it is possible to analyze traversal and reflection times of electromagnetic waves through the slab, exploiting the Kerr effect as an electric clock.

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