Solitons in dispersion-managed mode-locked lasers

Mark J. Ablowitz and Theodoros P. Horikis*
Department of Applied Mathematics, University of Colorado, 526 UCB, Boulder, Colorado 80309-0526, USA

Boaz Ilan
School of Natural Sciences, University of California, Merced, P.O. Box 2039, Merced, California 95344, USA
(Received 31 October 2007; published 6 March 2008)

The dynamics and propagation of ultrashort optical pulses generated in mode-locked lasers are investigated. The mode locking process depends crucially on the gain and loss mechanisms inside the cavity. Analytical models are introduced that include dispersion management and gain-loss terms with energy and power saturation. Comparisons of the pulse dynamics with previously well-known models are discussed. Stable soliton solutions are found for wide ranges of the parameters; the only significant requirement being sufficient gain in the system. The evolution of the system "locks" into the pure soliton solutions obtained independently with mode finding algorithms. Such solutions can be viewed as soliton wave attractors. Finally, it is shown that these solutions satisfy an asymptotic averaged system and can be approximated by elementary, Gaussian-type, functions.

DOI: 10.1103/PhysRevA.77.033814 PACS number(s): 42.65.Tg, 42.55.Wd, 42.65.Re, 42.65.Sf

Solitons and solitary waves in general have been predicted and observed in an abundance of physical systems. A prototypical example of a system that supports solitary waves, which is also of great technological importance, is mode-locked (ML) lasers. ML lasers have a long history dating back for more than 4 decades [1,2]. It is only been in recent years that researchers have begun to utilize their potential for science and technology. Many modern applications employ ultrashort pulses of the order of a few femtoseconds to a picosecond. Applications utilizing ML lasers include communications [2], high quality optical oscillators (optical clock technology) [3,4], high harmonic generation [5], and measurements of the fundamental constants of nature [6]. Various types of ML lasers have been used to produce ultrashort pulses, they include Ti:sapphire, Sr-fosterite, and fiber lasers among others. To produce ultrashort soliton pulses these lasers are often dispersion-managed systems in which the underlying normal dispersion is suitably compensated so that the average over one cavity round-trip is in the anomalous regime.

Mode locking can be achieved in a laser by the use of an active element (active mode locking) or a passive element (passive or Kerr-lens mode locking). The latter produces the shortest pulses and hence we shall be concerned with the dynamics of passive ML lasers. Many different models have been used to describe pulse propagation in mode-locked lasers. The best known mode-locking system is the so-called master equation [2,7,8]. The master equation models the effects of nonlinearity, dispersion, bandwidth limited gain, energy saturation, and intensity discrimination in the laser cavity. Gain is saturated with energy while loss is represented by a cubic nonlinearity. For a narrow range of these parameters [9] this equation has stable soliton solutions with mode-locking evolution. Otherwise the solitons are found to be unstable; either dispersing to radiation or evolving into nonlocalized quasiperiodic states. The amplitude with more general initial data can also grow rapidly under evolution. Thus the basic master equation captures some qualitative aspects of pulse propagation in a laser cavity, however, there is only a small range of the parameter space for which stable mode-locked soliton pulses exist.

To overcome this sensitivity various modifications have been proposed; e.g., higher-order nonlinear (quintic) terms or more complex absorber terms have been introduced. Ginzburg-Landau type equations with higher order nonlinearities have also been extensively studied. Such equations support many types of solutions such as pulsating, chaotic, and periodically growing or decaying ("exploding") localized states [10–12], the latter occurring when the laser operates at a critical point. On the other hand, there are wide operating regimes where soliton mode locking is observed [13].

Modeling ultrashort pulse ML lasers must take into account dispersion management and amplification and feedback. Recently a modified system was introduced [14,15] which takes into account these effects. There the gain medium is modeled by a distributive saturated energy term, whereas the Kerr-lens effect is modeled as a lumped fast absorber saturated with power. This equation was employed to model the propagation of large amplitude self-similar pulses for Ti:sapphire lasers in the normal dispersion regime.

Similar to the above system, we propose a normalized *distributed* dispersion-managed, power saturation model. We find this system to naturally describe the locking and evolution of pulses in ML lasers operating in the soliton regime. Further, the essential features of the lumped model are also included in this distributive equation. For a pulse with amplitude $u(z,t)$, power $P(z,t)=|u|^2$, and energy $E(z)=\int_{-\infty}^{+\infty}|u|^2dt$, which is propagating along the z direction, our model equation takes the form

*theodoros.horikis@colorado.edu

$$i\frac{\partial u}{\partial z}+\frac{d(z)}{2}\frac{\partial^2 u}{\partial t^2}+n(z)|u|^2u = \frac{ig}{1+E/E_{\text{sat}}}u+\frac{i\tau}{1+E/E_{\text{sat}}}u_{tt}-\frac{il}{1+P/P_{\text{sat}}}u, \qquad (1)$$

where the parameters g, τ, l, E_{sat}, and P_{sat} are positive. It is the averaged variation along the propagation direction of the dispersion and nonlinearity maps, given by $d=d(z)$ and $n=n(z)$, respectively, that enables the temporal localization (i.e., along t) of the pulses. Hereafter, the right-hand side of Eq. (1) will be denoted by $Q[u]$. The first term on the right-hand side represents saturable gain, the second is nonlinear filtering, and the third saturable loss. For our analysis, $\tau\neq 0$, so that spectral filtering is present. An important observation is that when the loss term is approximated in the weakly nonlinear regime by the first-order Taylor polynomial we obtain the master equation. Hence the master equation is included in the power saturated model as a first-order approximation. The master equation and the lumped model have parameters which are obtained from experiments [2,14]. The same holds for the distributive model, Eq. (1). We refer to this equation as the perturbed nonlinear Schrödinger (PNLS) equation. When there is insufficient gain, pulses dissipate to zero. On the other hand, remarkably, a distinguishing feature of this model is that even under large gain, pulses do not blow up nor do they exhibit instabilities. On the contrary: when the gain is greater than some threshold value $g=g^*$, during the evolution, the pulse readjusts itself as it mode locks into a stable soliton solution which we call a soliton wave attractor (SWA). Furthermore, all pulses that evolve into SWAs can be obtained independently using a mode-finding algorithm.

Power saturation models also arise in other problems in nonlinear optics and are important in the underlying theory; for example, in the study of the dynamics of localized lattice modes (solitons, vortices, etc.) propagating in photorefractive nonlinear crystals [16,17]. If the nonlinear term in these equations was simply a cubic nonlinearity, without saturation, two-dimensional fundamental lattice solitons would be vulnerable to blow up singularity formation, which is not observed. Thus saturable terms are crucial in these problems.

The introduction of dispersion and nonlinear management induces rapidly varying dynamics which often obscure the main features. To overcome this difficulty we employ an averaged, or mean-field theory, and work with the underlying averaged equation. Solitons as solutions of equations with constant dispersion are localized modes whose amplitude is constant in time. On the other hand dispersion managed (DM) solitons exhibit rapid breathing behavior, namely their amplitude changes according to the dispersion map. A convenient feature of the averaged model is that the modes represent pulses averaged over one cavity round-trip, as though the pulses were propagating in a system with the (constant) average cavity dispersion. In the averaged equation the DM solitons have constant amplitude.

We model the effect of dispersion management by splitting the dispersion $d(z)$ into two components [18] $d(z)=d_0+1/z_a\Delta(z/z_a)$. The variable z_a is the dispersion-map period, which measures the ratio of the characteristic nonlinear distance to the characteristic dispersion length. Typically z_a is small, i.e., $z_a\ll 1$. The function $d(\zeta)=d(z/z_a)=d_0+\Delta(z/z_a)/z_a$ is large and periodic and $n(z/z_a)=n(\zeta)$ is $O(1)$ and periodic. The path-averaged dispersion is d_0 and $\Delta(z)$ is rapidly varying and has zero average. We focus on the case of positive average dispersion, $d_0>0$, which corresponds to the soliton regime. Within each map period, the dispersion flips its sign as follows: $\Delta(\zeta)=\{-\Delta_1,\ 0<\zeta<1/2,\ \Delta_1,\ 1/2<\zeta<1\}$, whereas the propagation is periodically linear-nonlinear-linear, i.e., $n(\zeta)=\{0,\ 0<\zeta<1/2,\ n_0,\ 1/2<\zeta<1\}$. A key parameter is the map strength $s=\Delta_1/4$, which models the variability of the dispersion around the average. To illustrate the features of this system, we set $d_0=1$, $n_0=1$, and allow s to vary.

To obtain the averaged equation we introduce multiple scales and apply perturbation theory [18]. Define the new variables for distance $\zeta=z/z_a$ and $Z=z$ representing the short- and long-scale dynamics, respectively. Next expand u in powers of $z_a\ll 1$ as $u(\zeta,Z,t)=u^{(0)}(\zeta,Z,t)+z_a u^{(1)}(\zeta,Z,t)+O(z_a^2)$. In this way Eq. (1) breaks into a series of equations corresponding to the different powers of z_a. At $O(z_a^{-1})$

$$i\frac{\partial u^{(0)}}{\partial\zeta}+\frac{\Delta(\zeta)}{2}\frac{\partial^2 u^{(0)}}{\partial t^2}=0. \qquad (2)$$

To leading order the evolution of the pulse is determined by the large variations of $d(z)$ about the mean and nonlinearity and residual dispersion represent only a small perturbation to the linear equation. The linear equation, Eq. (2), can be solved using Fourier transforms, namely

$$\hat{u}^{(0)}(\zeta,Z,\omega)=\exp\left[-i\frac{\omega^2}{2}C(\zeta)\right]\hat{U}_0(Z,\omega), \qquad (3)$$

where $C(\zeta)=\int_0^{\zeta}\Delta(\zeta')d\zeta'$, and $\hat{U}_0(Z,\omega)=\hat{u}^{(0)}(\zeta=0,Z,\omega)$. The Fourier transform of any function, $f(t)$, is denoted

$$\hat{f}(\omega)=\mathcal{F}\{f(t)\}\equiv\int_{-\infty}^{+\infty}f(t)\exp(i\omega t)dt.$$

The function $\hat{U}_0$ represents the slowly evolving amplitude of $\hat{u}$, whereas the fast oscillations induced by the local values of the dispersion are included in the exponential term. The function $\hat{U}_0$ is arbitrary at this stage and is determined by removing secular terms at the next order of perturbation.

This procedure determines an equation for $\hat{U}_0(Z,\omega)$ which is given by

$$i\frac{\partial\hat{U}_0}{\partial Z}-\frac{d_0}{2}\omega^2\hat{U}_0+\int_0^1\exp\left[-i\frac{\omega^2}{2}C(\zeta)\right](n(\zeta)\mathcal{F}\{|u^{(0)}|^2u^{(0)}\} - \mathcal{F}\{Q[u^{(0)}]\})d\zeta=0. \qquad (4)$$

This is a nonlocal equation for $\hat{U}_0(Z,\omega)$ and describes the averaged dynamics of the pulse envelope. We refer to Eq. (4) as the dispersion-managed gain-loss nonlinear Schrödinger (DMNLS-GL) equation.

A central issue is how to compute localized, i.e., soliton, solutions for this equation. Various techniques have been used, e.g., shooting, relaxation, self-localization, etc. Here we use the method introduced in Ref. [19] which generalizes the method of Ref. [20] to more general nonlinearities. This technique takes advantage of the fact that in a physical system dispersion tends to widen and break the soliton apart whereas nonlinearity has the opposite effect. When these effects balance a stable pulse forms. This spectrally accurate, iterative method adjusts in each iteration the ratio between the dispersive and the nonlinear parts of the equation in order to "balance" the two effects.

To find soliton solutions of Eq. (4) first assume a localized solution of the form $U_0(Z,t)=\exp(i\mu Z)q(t)$, where μ is the propagation constant or the soliton eigenvalue. Substitute this into Eq. (4) to obtain

$$-\mu\hat{q}-\frac{d_0}{2}\omega^2\hat{q}+\int_0^1 \exp\left[-i\frac{\omega^2}{2}C(\zeta)\right](n(\zeta)\mathcal{F}\{|u^{(0)}|^2u^{(0)}\} - \mathcal{F}\{Q[u^{(0)}]\})d\zeta=0. \quad (5)$$

This is a nonlinear eigenvalue problem for q and μ with boundary conditions $q\to 0$ as $t\to\pm\infty$. In order to construct a solution whose amplitude does not grow indefinitely nor tends to zero we introduce $v(t)$ such that $q(t)=\lambda v(t)\Leftrightarrow \hat{q}(\omega)=\lambda\hat{v}(\omega)$ where $\lambda\neq 0$ is called the renormalization constant and is to be determined. Multiplying by $\hat{v}^*$ and integrating over the entire space ω we find an algebraic relation, at say the jth iteration, relating v and λ

$$\int_{-\infty}^{+\infty}(\mu+d_0\omega^2/2)|v_j|^2d\omega-\int_{-\infty}^{+\infty}\int_0^1\exp\left[-i\frac{\omega^2}{2}C(\zeta)\right] \times(n(\zeta)\mathcal{F}\{|\lambda_j w_j|^2w_j\}-\mathcal{F}\{Q[\lambda_j w_j]/\lambda_j\})\hat{v}_j^*d\zeta d\omega=0,$$

where $u^{(0)}=\lambda w$. Given v_j, this can be solved for λ_j using suitable numerical root finding methods, e.g., Newton's method, thus determining the values of the renormalization constant, λ_j. The solution to Eq. (5) is obtained by iterating as follows:

$$\hat{v}_{j+1}=\frac{\int_0^1\exp\left[-i\frac{\omega^2}{2}C(\zeta)\right]}{\mu+d_0\omega^2/2}(n(\zeta)\mathcal{F}\{|\lambda_j w_j|^2w_j\} - \mathcal{F}\{Q[\lambda_j w_j]/\lambda_j\})d\zeta,$$

subject to the additional constraint that $\mathrm{Im}\{\lambda_j\}=0$. To begin the iteration, at $j=0$, an initial guess is given, e.g., a Gaussian.

For realistic lasers one cannot hope to have a soliton as the initial input. Hence we first study the mode-locking mechanism of more general pulses inserted in the cavity at $z=0$. To study such an evolution, we integrate the DMNLS-GL Eq. (4) with Eq. (3) (using fourth order Runge–Kutta) with a given initial profile $U_0(0,t)=\exp(-t^2)$. In order to lock onto stable soliton solutions the gain parameter g needs to be sufficiently large to counter the two lossy terms; noting that the filtering in the equation acts as an additional loss term for the system.

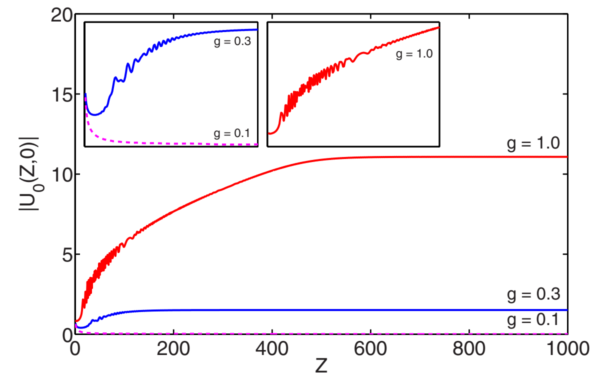

FIG. 1. (Color online) Evolution of the soliton's peak of a unit Gaussian using the DMNLS-GL equation with $s=1$ and a wide range of gain parameters $g=0.1$ (damped evolution), 0.3, and 1.0. Inset: the early evolution of the peaks is shown.

In what follows we fix $\tau=l=0.1$ and $E_{\mathrm{sat}}=P_{\mathrm{sat}}=1$. Let us first consider $g=0.1$, $s=1$. In this case the pulse vanishes quickly due to the effect of excessive loss in the system. No oscillatory behavior nor complex dynamics are noticed, the pulse simply decays to zero, in this damped evolution. Next, we take $g=0.3,1$ and $s=1$. Two distinct evolutions are observed in Fig. 1. When $g=0.3$ and due to the loss in the system the pulse amplitude initially undergoes a sharp decrease, relative to its amplitude. It then rapidly recovers and with further propagation locks to a stable solution. Note the difference with the master equation. When too much loss was present in the master equation either the pulse amplitude decayed into radiation or the pulse evolved to a nonlocalized quasiperiodic or chaotic state. The evolution reflected the fact that the solitons were unstable for a wide range of parameter values. In this power saturation model, the solitons, when they exist, are stable. For a given map strength s, when g is above a critical value g^*, the resulting evolution always mode locks into solitons (which can be obtained by the mode finding algorithm presented above as will be shown below). Even under extreme gain, when $g=1$, $s=1$, a stable solution is reached, see Fig. 1. Additional insights into this behavior will be discussed after we discuss the soliton solutions of the DMNLS-GL equation. Even when one introduces detailed gain dynamics where the gain coefficient g varies in a more complicated fashion [21], mode locking is still expected to occur, in which case one would replace g by an effective value.

We now proceed to solve for solitons from Eq. (5) for the parameter values used above. We vary the gain parameter g and the map strength s. When $g<g^*(s)$ Eq. (5) has no solution, i.e., we do not find values of μ for which the numerical iteration converges to a localized solution. This implies that when the effect of loss is stronger than the gain the only acceptable solution is the trivial solution. The solutions throughout have been compared and verified to solutions of Eq. (1) obtained by direct simulations. We study Eq. (5) because it exhibits solutions with uniform evolution.

On the other hand, when $g>g^*(s)$, there exists a solution to Eq. (5) for an appropriate value of μ; the value of the propagation constant is unique for the specific values of the other parameters. The solutions of the DMNLS-GL equation

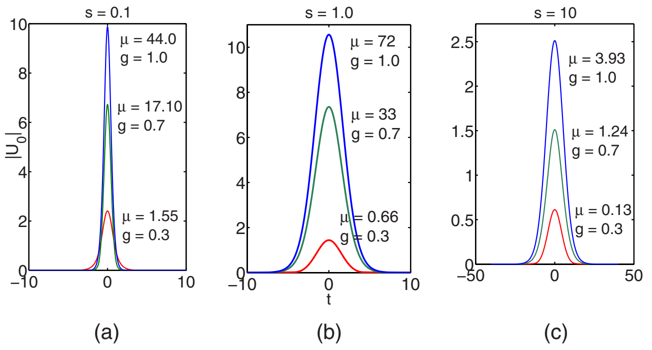

FIG. 2. (Color online) Solitons of the DMNLS-GL equation corresponding to different map strengths and gain parameters. Notice that these solitons occur only for a specific value of μ in contrast to the unperturbed case where solutions exist for $\mu > 0$, thus providing the mode-locking mechanism.

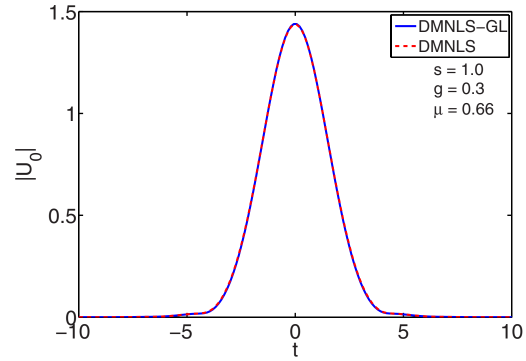

FIG. 3. (Color online) Comparison between the solutions of DMNLS-GL [solid (blue) curve] and pure DMNLS [dashed (red) curve] for $s=1$ and $g=0.3$.

in the different regimes are presented in Fig. 2. From this figure we see that for given map strengths the amplitude of the pulses increases with g. In addition, the pulses also become broader as the map strength s increases. As the gain parameter increases both the energy and the amplitude of the pulse increase and tend to the "pure" (unperturbed) DMNLS solution–without additional gain-loss terms. Blowup does not occur, even under extreme gain. Indeed, this is what one would expect from Eq. (5). If blowup were to occur that would mean that both the amplitude and the energy of the pulse are infinite, or very large, and the equation reduces to the pure DMNLS equation, $Q[u]\to 0$, for which stable soliton solutions exist for all values of the propagating constant; this is a Hamiltonian system. Hence for sufficiently large values of the propagation constant where $Q[u]\to 0$ the solutions of the pure DMNLS and the perturbed equation are comparable. Similarly if one considers relatively small values for the parameters such that $Q[u]$ can be regarded as a small perturbation of the unperturbed equation the modes of DMNLS-GL and DMNLS must be again close. This suggests that the influence of $Q[u]$ is only the mode-locking mechanism and that when that occurs the modes correspond to the modes of the unperturbed equation with the same propagation constant μ. The difference between the two cases is that when μ and the amplitude is large enough the system becomes Hamiltonian and any choice of μ beyond that will effectively result in a soliton solution. We call these solutions quasisolitons or near-soliton solutions of the DMNLS-GL equation.

In Fig. 3 we show the solutions of both the pure DMNLS and the DMNLS-GL equations and for a moderate dispersion map when $g=0.3$. As mentioned above, the solutions of the two equations are compared for the same value of μ. The very good agreement between these modes suggest that we can use the unperturbed DMNLS equation for the same value of the propagation constant to approximate modes of the perturbed equation. This is also particularly useful because modes of the DMNLS equation can be approximated analytically by Gaussians [22], thus suggesting that the same will also hold for the perturbed system.

To conclude, we have presented a distributive model equation for the study of pulse propagation in mode-locked lasers. This model captures the qualitative features of different types of mode-locked lasers operating in the soliton regime. Advantages of the DMNLS-GL model over other models are discussed. Depending on the size of the gain parameter, pulses are either damped, i.e., decay to zero, or are asymptotically attracted to a stable solitary wave. Instabilities and blowup are not observed for a wide range of the parameters, even when the perturbing effects cannot be considered small. The energy-saturated gain and filtering and power-saturated loss are crucial for the mode-locking mechanism. The ensuing modes are essentially those of the unperturbed system for the same propagation constant.

This research was partially supported by the U.S. Air Force Office of Scientific Research under Grant No. FA4955-06-1-0237 and by the National Science Foundation under Grant No. DMS-0505352.

[1] W. E. Lamb, Jr., Phys. Rev. **134**, A1429 (1964).
[2] H. Haus, IEEE J. Sel. Top. Quantum Electron. **6**, 1173 (2000).
[3] S. Diddams, J. Bergquist, S. Jefferts, and C. Oates, Science **306**, 1318 (2004).
[4] J. Ye and S. Cundiff, *Femtosecond Optical Frequency Comb: Technology, Principles, Operation & Application* (Springer, New York, 2005).
[5] H. Kapteyn, O. Cohen, I. Christov, and M. Murnane, Science **371**, 775 (2007).
[6] M. Fischer *et al.*, Phys. Rev. Lett. **92**, 230802 (2004).
[7] H. Haus, J. Appl. Phys. **46**, 3049 (1975).
[8] H. Haus, J. Fujimoto, and E. Ippen, IEEE J. Quantum Electron. **28**, 2086 (1992).
[9] J. Kutz, SIAM Rev. **48**, 629 (2006).

[10] J. M. Soto-Crespo and N. Akhmediev, Phys. Rev. E **66**, 066610 (2002).
[11] N. Akhmediev and A. Ankiewicz, *Solitons, Nonlinear Pulses and Beams* (Chapman and Hall, London, 1997).
[12] S. T. Cundiff, J. M. Soto-Crespo, and N. Akhmediev, Phys. Rev. Lett. **88**, 073903 (2002).
[13] Q. Quraishi, S. T. Cundiff, B. Ilan, and M. J. Ablowitz, Phys. Rev. Lett. **94**, 243904 (2005).
[14] S. Wieczorek and W. W. Chow, Phys. Rev. Lett. **92**, 213901 (2004).
[15] F. Ilday, F. Wise, and F. Kaertner, Opt. Express **12**, 2731 (2004).
[16] O. Cohen, T. Schwartz, J. W. Fleischer, M. Segev, and D. N. Christodoulides, Phys. Rev. Lett. **91**, 113901 (2003).
[17] C. Lou, X. Wang, J. Xu, Z. Chen, and J. Yang, Phys. Rev. Lett. **98**, 213903 (2007).
[18] M. Ablowitz and G. Biondini, Opt. Lett. **23**, 1668 (1998).
[19] M. Ablowitz and Z. Musslimani, Opt. Lett. **30**, 2140 (2005).
[20] V. Petviashvili, Sov. J. Plasma Phys. **2**, 257 (1976).
[21] C. Menyuk, J. Wahlstrand, J. Willits, R. Smith, T. Schibli, and S. Cundiff, Opt. Express **15**, 6677 (2007).
[22] M. Ablowitz, G. Biondini, and E. Olson, in *Massive WDM and TDM Soliton Transmission Systems*, edited by A. Hasegawa (Kluwer Academic, Kyoto, Japan, 2000), pp. 362–367.