Density correlations in cold atomic gases: Atomic speckles in the presence of disorder

Peter Henseler
Physikalisches Institut, Universität Bonn, Nussallee 12, 53115 Bonn, Germany

Boris Shapiro
Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel

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The phenomenon of random intensity patterns, for waves propagating in the presence of disorder, is well known in optics and in mesoscopic physics. We study this phenomenon for cold atomic gases expanding, by a diffusion process, in a weak random potential. We show that the density-density correlation function of the expanding gas is strongly affected by disorder and we estimate the typical size of a speckle spot, i.e., a region of enhanced or depleted density. Both a Fermi gas and a Bose-Einstein condensate (in a mean-field approach) are considered.

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I. INTRODUCTION

A wave propagating in a random medium undergoes multiple scattering from inhomogeneities. As a result, a complicated, highly irregular, wave intensity pattern is formed. It is characterized in statistical terms, with the help of various correlation functions and probability distributions. Such intensity patterns have been thoroughly studied for electromagnetic waves in disordered dielectric media (optical speckles) [1]. They are also well known for the Schrödinger waves in disordered electronic conductors, where, due to the electron scattering on impurities, a random distribution of electron density and currents is established within the sample. These "Schrödinger speckles" manifest themselves in various mesoscopic effects, for instance, in the sample-to-sample conductance fluctuations [1].

Cold atomic gases offer a new playground for studying matter waves in the presence of disorder. Unlike the case of electrons in disordered conductors, for atomic gases one can directly measure the atomic spatial density or, more precisely, density integrated along the direction of imaging. There have been a number of recent experiments [2–7], as well as some theoretical work [8,9] on the effect of disorder on atomic Bose-Einstein condensates (BEC), in quasi-one-dimensional geometry. Some of these experiments reported observation of apparently random albeit reproducible interference patterns—the atomic analog of the Schrödinger wave speckles in disordered conductors.

In the present paper, we develop a theory for such "atomic speckles" in two and three spatial dimensions when, unlike the one-dimensional (1D) case, atoms can propagate by diffusion. The diffusion of the BEC in a random potential has been recently studied [10–12] and it was pointed out that only some fraction of the condensate diffuses away, while the rest stays localized near its initial location (Anderson transition for BEC) [11,12]. Only the average density of the evolving atomic cloud was studied in Refs. [11,12]. The atomic speckles, however, are characterized by more complicated quantities, such as the density-density correlation function studied in the present paper. Before plunging into calculations, let us emphasize that we assume no (or negligibly small) disorder within the trap, i.e., the atoms are released from a "clean" trap and diffuse in the random potential. (A different setup is when the disorder, created within the trap, has a strong effect already at the equilibrium state of the system, causing a transition to an insulating phase [13].) We focus mainly on the case of Fermi gases (Sec. II). A brief discussion of the BEC case is given in Sec. III.

II. ATOMIC SPECKLES IN A DIFFUSING FERMI GAS

We consider N fermions, initially occupying the N lowest energy eigenstates in a trap. Assuming zero temperature and neglecting interactions we can write the quantum expectation value of the particle density as

$$\langle \hat{n}(\mathbf{r}) \rangle_0 = g_s \sum_n f_n |\phi_n(\mathbf{r})|^2, \tag{1}$$

where $g_s = 2S+1$ is the spin-degeneracy factor and $\phi_n(\mathbf{r})$ is the orbital part of an eigenstate in the trap. The factor f_n is unity for an occupied site and zero otherwise.

At time $t=0$ the Fermi gas is released from the trap and the single-particle wave functions start to evolve according to the Schrödinger equation

$$i\hbar \partial_t \psi_n(\mathbf{r},t) = -\frac{\hbar^2}{2m} \Delta \psi_n(\mathbf{r},t) + V(\mathbf{r}) \psi_n(\mathbf{r},t), \tag{2}$$

where $V(\mathbf{r})$ is the external static random potential. The quantum expectation value of the particle density, at time t, for a given realization of randomness is equal to

$$\begin{aligned} \langle \hat{n}(\mathbf{r},t) \rangle &= g_s \sum_n f_n |\psi_n(\mathbf{r},t)|^2 \\ &= g_s \int d^d R \int d^d R' G(\mathbf{r},\mathbf{R},t) G^*(\mathbf{r},\mathbf{R}',t) \\ &\quad \times \sum_n f_n \phi_n(\mathbf{R}) \phi_n^*(\mathbf{R}'), \end{aligned} \tag{3}$$

where $G(\mathbf{r},\mathbf{R},t)$ is the retarded Green's function of the Hamiltonian in Eq. (2). The sum in Eq. (3) can conveniently be written as

$$\sum_n f_n \phi_n(\mathbf{R}) \phi_n^*(\mathbf{R}') = \int_{-\infty}^{\varepsilon_F} d\varepsilon \sum_n \phi_n(\mathbf{R}) \phi_n^*(\mathbf{R}') \delta(\varepsilon - \varepsilon_n)$$
$$= -\frac{1}{\pi} \int_{-\infty}^{\varepsilon_F} d\varepsilon \ \mathrm{Im}\ \mathcal{G}(\mathbf{R}', \mathbf{R}, \varepsilon), \qquad (4)$$

where ε_n are the particle eigenenergies in the trap and ε_F is the Fermi energy. The Green's function $\mathcal{G}(\mathbf{R}', \mathbf{R}, \varepsilon)$ refers to a particle in the trap and it should not be confused with the function $G(\mathbf{r}, \mathbf{R}, t)$ which describes propagation from point $\mathbf{R}$ to an observation point $\mathbf{r}$, upon the release of the gas from the trap.

The density $\langle \hat{n}(\mathbf{r}, t) \rangle$ is still a random variable, in the sense that its value depends on the specific realization of the random potential $V(\mathbf{r})$. Denoting averaging over realizations by an overbar, we write

$$\overline{\langle \hat{n}(\mathbf{r}, t) \rangle} = -\frac{g_s}{\pi} \int d^d R \int d^d R' \overline{G(\mathbf{r}, \mathbf{R}, t) G^*(\mathbf{r}, \mathbf{R}', t)}$$
$$\times \int_{-\infty}^{\varepsilon_F} d\varepsilon \ \mathrm{Im}\ \mathcal{G}(\mathbf{R}', \mathbf{R}, \varepsilon). \qquad (5)$$

The last integral in this expression contains information about the initial state of the gas in the trap. The product of the two propagators, $\overline{GG^*}$, propagates this information in space and time.

In experiment, an ensemble of many random samples can be prepared either by creating different realizations of the random potential or by making slight changes in the Fermi energy ε_F. It is well known in mesoscopic physics that, for a fixed realization of randomness, various properties of a system are extremely sensitive to the precise position of ε_F [1]. Indeed, an "ensemble of disordered conductors" is often produced from a single sample, whose randomness is fixed by technology, by changing the Fermi energy. In this respect, cold atomic gases constitute a more convenient system for studying disorder related effects. A different realization of disorder can be easily created by changing the optical speckle pattern, on which the atoms are being scattered. Furthermore, since the precise value of ε_F cannot be controlled, from one experimental run to another, to an accuracy higher than a few percent it appears that in experimenting with cold atoms an "ensemble of random samples" is created automatically, in a natural way.

The average product of the two Green's functions in Eq. (5) is a standard object in the theory of disordered systems. In the diffusion regime, considered in the present paper, it can be transformed as [12]

$$\overline{G(\mathbf{r}, \mathbf{R}, t) G^*(\mathbf{r}, \mathbf{R}', t)} = -\frac{1}{\pi} \int d\varepsilon P_\varepsilon(\mathbf{r}, \mathbf{R}, t) \mathrm{Im}\ \bar{G}(\mathbf{R} - \mathbf{R}', \varepsilon), \qquad (6)$$

where

$$P_\varepsilon(\mathbf{r}, \mathbf{R}, t) = \frac{e^{-|\mathbf{r} - \mathbf{R}|^2 / 4 D_\varepsilon t}}{(4\pi D_\varepsilon t)^{d/2}} \qquad (7)$$

is the diffusion propagator, in d dimensions ($d=2,3$) and D_ε is the diffusion coefficient for a particle at energy ε. The average Green's function is given by [1]

$$\bar{G}(\Delta R, \varepsilon) = G_0(\Delta R, \varepsilon) e^{-\Delta R / 2 l_\varepsilon}, \quad \Delta R = |\mathbf{R} - \mathbf{R}'|, \qquad (8)$$

where G_0 is the free Green's function and l_ε is the particle mean free path. The Eqs. (7) and (8) for P and $\bar{G}$ are valid only for sufficiently large values of the energy parameter ε, namely if $k_\varepsilon l_\varepsilon \gg 1$, where $k_\varepsilon = \sqrt{2m\varepsilon/\hbar^2}$. Therefore, $k_F l_F$ is the essential parameter which determines the overall behavior of the gas, after switching off the trap, where k_F and l_F denote the value of the wave number and of the mean free path at the Fermi energy ε_F. Only for $k_F l_F \gg 1$, which is the case assumed in this paper, most of the atomic cloud will diffuse away from the trap. (A similar condition, with the chemical potential μ replacing the Fermi energy, is required for the diffusive behavior of a BEC cloud [11].)

To facilitate analytic treatment, we assume that the size of the trap is much smaller than its distance from the observation point $\mathbf{r}$. Then, choosing the coordinate origin somewhere inside the trap, one can set $\mathbf{R}=0$ in the second argument of the diffusion propagator, writing it simply as $P_\varepsilon(r, t)$ ($r = |\mathbf{r}|$). Moreover, since upon release from the trap, each particle goes on its own (long) diffusive trajectory, the actual shape of the trap is of no importance. It is convenient to replace the actual harmonic trap by a cubic trap of size L, with periodic boundary conditions. Finally, Fourier transforming $\bar{G}$ in (6) with respect to $\mathbf{R} - \mathbf{R}'$, and performing integration over $\mathbf{R}$ and $\mathbf{R}'$ in (5), we obtain

$$\overline{\langle \hat{n}(r, t) \rangle} = \frac{g_s L^d}{\pi^2} \int \frac{d^d k}{(2\pi)^d} \int d\varepsilon P_\varepsilon(r, t) \mathrm{Im}\ \overline{G}(\mathbf{k}, \varepsilon)$$
$$\times \int_{-\infty}^{\varepsilon_F} d\varepsilon' \ \mathrm{Im}\ \mathcal{G}(\mathbf{k}, \varepsilon'). \qquad (9)$$

Since we assume no disorder in the trap, it follows that $\mathrm{Im}\ \mathcal{G}(\mathbf{k}, \varepsilon') = -\pi \delta(\varepsilon' - \varepsilon_k)$ and integration over ε' results in a step function $\Theta(k_F - k)$. Furthermore, since $k_F l_F \gg 1$, it follows that the weak disorder condition, $k_\varepsilon l_\varepsilon \gg 1$, is satisfied for the great majority of k's in Eq. (9). Therefore, $\mathrm{Im}\ \bar{G}(\mathbf{k}, \varepsilon)$ can be approximated by $-\pi \delta(\varepsilon - \varepsilon_k)$ and integration over ε can be carried out, resulting in

$$\overline{\langle \hat{n}(r, t) \rangle} = g_s L^d \int_{k < k_F} \frac{d^d k}{(2\pi)^d} P_k(r, t), \qquad (10)$$

where the diffusion kernel $P_k(r, t)$ is given by Eq. (7), with $R=0$ and ε set equal to $\hbar^2 k^2 / 2m$. Equation (10) has a very simple interpretation: Particles, prior to their release from the trap, occupy all states up to k_F, with $L^d d^d k / (2\pi)^d$ being the number of particles in the element $d^d k$. When the trap is switched off, particles start diffusing and $P_k(r, t)$ is the probability density that a particle with wave number k will reach point $\mathbf{r}$ in time t. Integration over $\mathbf{k}$ gives the average particle density $\overline{\langle \hat{n}(r, t) \rangle}$. An equation similar to (10) exists also for a BEC [11]. It was emphasized there, and the same is true for the present case of a Fermi gas as well, that some fraction of the released particles will not propagate by diffusion but will

remain localized near their original location. For $k_F l_F \gg 1$ the fraction of such particles is small and we ignore them in the present work.

We turn now to the calculation of the density-density correlation function. Let us first note that, already in the absence of disorder, a Fermi gas possesses some subtle density correlations of purely quantum nature [14]. The expectation value, $\langle \hat{n}(\mathbf{r},t)\rangle$, is ignorant about these correlations. However, as has been particularly emphasized in [15,16], a single imaging experiment (with sufficient resolution) does not measure $\langle \hat{n}(\mathbf{r},t)\rangle$ but rather one particular event, i.e., some particular density pattern, $n(\mathbf{r},t)$, whose probability is dictated by the many-body wave function of the system. The density correlation function can be extracted from such noisy density patterns by averaging the product $n(\mathbf{r})n(\mathbf{r}')$ over many experimental realizations, taken under (as far as possible) identical experimental conditions, for fixed $\mathbf{r},\mathbf{r}'$ and t. Such averaging would yield the theoretically calculated quantum expectation value $\langle \hat{n}(\mathbf{r})\hat{n}(\mathbf{r}')\rangle$. It should be noted, however, that already a single experimental image, although noisy and "grainy," contains information about the density correlation function. This information will be revealed by taking the product $n(\mathbf{r})n(\mathbf{r}+\Delta\mathbf{r})$ and averaging it over many points $\mathbf{r}$, for fixed $\Delta\mathbf{r}$ (the equivalence of the two averaging procedures constitutes the "ergodic assumption" in mesoscopic physics [1]).

For fermions confined to a trap of size L, the density-density correlation function exhibits rapidly decaying oscillations with a characteristic period $\Delta x_0 \approx k_F^{-1} \simeq L N^{-1/d}$ [14,17]. When the gas is released from the trap it starts expanding, and, for times $t > (L/v_F) \equiv t_0$, the size of the cloud grows linearly with time and so does the correlation length $\Delta x(t) \approx \Delta x_0 t/t_0$ [18]. Thus, roughly speaking, the free, ballistic expansion amplifies the scale of correlations by the factor t/t_0.

Below we show that in the presence of a random potential, i.e., when the expansion is diffusive instead of ballistic, the picture is different: The size of the atomic cloud grows as $\sqrt{t}$ whereas the short-range correlations do not get amplified at all. The density-density correlation function in the presence of disorder is defined as

$$C(\mathbf{r},\mathbf{r}',t) = \overline{\langle \hat{n}(\mathbf{r},t)\hat{n}(\mathbf{r}',t)\rangle} - \overline{\langle \hat{n}(\mathbf{r},t)\rangle\langle \hat{n}(\mathbf{r}',t)\rangle} \times - \delta(\mathbf{r}-\mathbf{r}')\overline{\langle \hat{n}(\mathbf{r},t)\rangle}. \tag{11}$$

The last term describes trivial correlations, which exist already in a classical ideal gas and which are commonly subtracted, in order to isolate the nontrivial correlations [14]. There are two kinds of averaging in (11): The quantum mechanical averaging, for a given realization of disorder, and averaging over the ensemble of different realizations. The first averaging is straightforward and leads to [14]

$$\langle \hat{n}(\mathbf{r},t)\hat{n}(\mathbf{r}',t)\rangle = g_s^2 \sum_n f_n |\psi_n(\mathbf{r},t)|^2 \sum_m f_m |\psi_m(\mathbf{r}',t)|^2 - g_s \left| \sum_n f_n \psi_n^*(\mathbf{r},t)\psi_n(\mathbf{r}',t) \right|^2 + g_s \delta(\mathbf{r}-\mathbf{r}') \sum_n f_n |\psi_n(\mathbf{r},t)|^2, \tag{12}$$

where the absence of overbars indicates that this expression refers to a specific realization of the random potential. Next, we must average (12) over the disorder. This involves averaging products of four single-particle wave functions. For short-range correlations, i.e., on a scale smaller than the mean free path l_ε, such averages decouple into products of pairwise averages [1]. For instance,

$$\overline{\psi_n^*(\mathbf{r},t)\psi_n(\mathbf{r}',t)\psi_m^*(\mathbf{r}',t)\psi_m(\mathbf{r},t)} \approx \overline{\psi_n^*(\mathbf{r},t)\psi_n(\mathbf{r}',t)}\,\overline{\psi_m^*(\mathbf{r}',t)\psi_m(\mathbf{r},t)} + \overline{\psi_n^*(\mathbf{r},t)\psi_m(\mathbf{r},t)}\,\overline{\psi_m^*(\mathbf{r}',t)\psi_n(\mathbf{r}',t)}, \quad |\mathbf{r}-\mathbf{r}'| < l. \tag{13}$$

Performing such decoupling in Eq. (12) and subtracting the product of averaged densities, $\overline{\langle n(\mathbf{r},t)\rangle}\,\overline{\langle n(\mathbf{r}',t)\rangle}$, we obtain

$$C(\mathbf{r},\mathbf{r}',t) = \sum_{n,m} f_n f_m [- g_s A_{nn}(\mathbf{r},\mathbf{r}',t) A_{mm}^*(\mathbf{r},\mathbf{r}',t) - g_s A_{nm}(\mathbf{r},\mathbf{r},t) A_{nm}^*(\mathbf{r}',\mathbf{r}',t) + g_s^2 |A_{nm}(\mathbf{r},\mathbf{r}',t)|^2], \tag{14}$$

where

$$A_{nm}(\mathbf{r},\mathbf{r}',t) \equiv \overline{\psi_n^*(\mathbf{r},t)\psi_m(\mathbf{r}',t)}. \tag{15}$$

The first term in (14) describes quantum correlations, due to the Pauli exclusion principle. In particular, for $\mathbf{r}' \to \mathbf{r}$, it approaches the value $-g_s \overline{\langle \hat{n}(\mathbf{r},t)\rangle}^2$ and, thus, it is proportional to N^2. The third term is of "classical" origin, in the sense that it originates from the interference between multiply scattered waves. It contributes positive correlations, similarly to speckle pattern in optics. However, in contrast to a single frequency laser speckle, here there are many waves with different frequencies. Since contributions from different frequencies should be added incoherently, i.e., intensities (rather than amplitudes) are summed up, the third term is proportional to N and it will be neglected. The second term in (14) is a combination of quantum and classical correlations. Its sign and the factor g_s originate from the exclusion principle. Since, however, wave functions at different energies are essentially uncorrelated, this term is also proportional to N and can be neglected in the large-N limit. Thus, keeping only the first term in (14), we obtain

$$C(\mathbf{r},\mathbf{r}',t) = - g_s \left| \sum_n f_n A_{nn}(\mathbf{r},\mathbf{r}',t) \right|^2 \equiv - g_s |F(\mathbf{r},\mathbf{r}',t)|^2. \tag{16}$$

The function $F(\mathbf{r},\mathbf{r}',t) \equiv \Sigma_n f_n \overline{\psi_n^*(\mathbf{r},t)\psi_n(\mathbf{r}',t)}$ can be expressed in terms of the Green's functions, in complete analogy with the earlier derivation for the average density. The only difference is that now there are two "observation points," $\mathbf{r}$ and $\mathbf{r}'$. The resulting expression for $F(\mathbf{r},\mathbf{r}',t)$ [compare to Eq. (5)] is

$$F(\mathbf{r},\mathbf{r}',t) = -\frac{1}{\pi}\int d^dR \int d^dR' \overline{G(\mathbf{r},\mathbf{R},t)G^*(\mathbf{r}',\mathbf{R}',t)} \times \int_{-\infty}^{\varepsilon_F} d\varepsilon \ \mathrm{Im}\ \mathcal{G}(\mathbf{R}',\mathbf{R},\varepsilon). \quad (17)$$

In order for the average product of Green's functions not to be exponentially small, the pair of points $\mathbf{r},\mathbf{r}'$ should not be separated by more than the mean free path. On the other hand, $|\mathbf{r}-\mathbf{R}|$ is much larger than l_F. The computation of the average product in (17) is straightforward and the result is

$$\overline{G(\mathbf{r},\mathbf{R},t)G^*(\mathbf{r}',\mathbf{R}',t)} = -\frac{1}{\pi}\int d\varepsilon f(\Delta r,\varepsilon) P_\varepsilon(\mathbf{r},\mathbf{R},t)\mathrm{Im}\ \bar{G}(\mathbf{r}-\mathbf{R}',\varepsilon), \quad (18)$$

where the extra factor $f(\Delta r,\varepsilon)$, as compared to Eq. (6), is given by

$$f(\Delta r,\varepsilon) = -\frac{1}{\pi\nu_\varepsilon}\mathrm{Im}\ \bar{G}(\Delta r,\varepsilon), \quad \Delta r = |\mathbf{r}-\mathbf{r}'|. \quad (19)$$

Following the same line of derivation as for the average density, Eq. (10), we arrive at

$$F(\mathbf{r},\mathbf{r}',t) = L^d \int_{k<k_F} \frac{d^dk}{(2\pi)^d} P_k(r,t) f(\Delta r,\varepsilon_k). \quad (20)$$

For $d=3$ [1],

$$f(\Delta r,\varepsilon_k) = \frac{\sin(k\Delta r)}{k\Delta r} e^{-\Delta r/2l_k}. \quad (21)$$

While the kernel P_k and the mean free path l_k are slowly changing with k, the function $f(\Delta r,\varepsilon_k)$, for $\Delta r \gg k_F^{-1}$, contains a rapidly oscillating factor within the integration region in (20). Taking the "slow" functions out of the integral and computing the remaining integral we obtain

$$C(\mathbf{r},\mathbf{r}',t) = -g_s L^{2d} P_{k_F}(r,t)^2 e^{-\Delta r/l_F} \frac{k_F^6}{4\pi^4} \times \frac{[\sin(k_F\Delta r) - (k_F\Delta r)\cos(k_F\Delta r)]^2}{(k_F\Delta r)^6}, \quad (22)$$

where P_k and l_k are taken at $k=k_F$. It is convenient to normalize the correlation function by the average density which yields

$$I(\mathbf{r},\mathbf{r}',t) \equiv \frac{C(\mathbf{r},\mathbf{r}',t)}{\langle\hat{n}(r,t)\rangle\langle\hat{n}(r',t)\rangle} = -\frac{9}{g_s} e^{-\Delta r/l_F} \frac{[\sin(k_F\Delta r) - (k_F\Delta r)\cos(k_F\Delta r)]^2}{(k_F\Delta r)^6}. \quad (23)$$

The analogous calculation for $d=2$ yields

$$I(\mathbf{r},\mathbf{r}',t) = -\frac{4}{g_s} e^{-\Delta r/l_F} \frac{J_1(k_F\Delta r)^2}{(k_F\Delta r)^2}, \quad (24)$$

where $J_1(k_F\Delta r)$ is the Bessel function of the first kind.

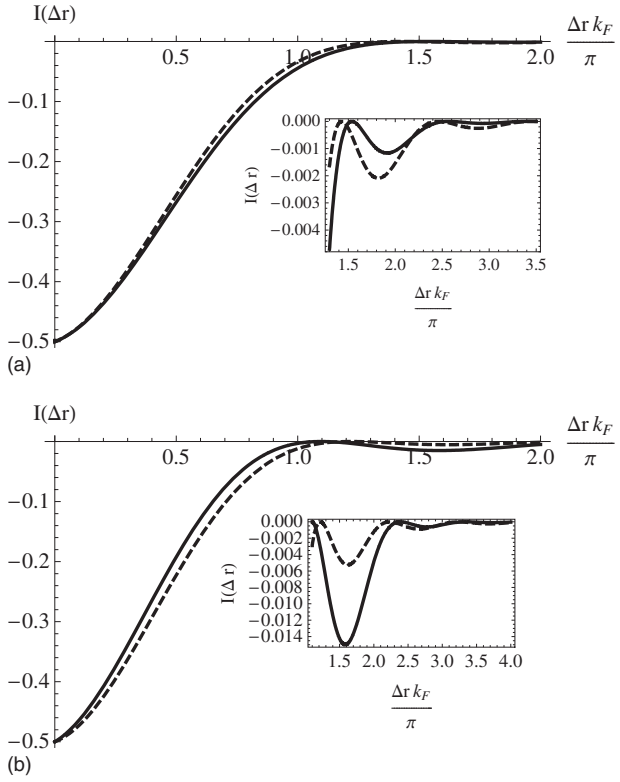

FIG. 1. The normalized density-density correlation function $I(\Delta r,t)$ of a Fermi gas with $g_s=2$, for $k_F l_F=10$, $r=50l_F$, and $t=3r^2/(2dD_{k_F})$. (a) corresponds to $d=3$ and (b) to $d=2$. The dashed curves in (a) and (b) correspond to Eqs. (23) and (24), respectively. The solid curves are the result of an exact numerical integration, as described in the text. The insets show the oscillating decay of $I(\Delta r,t)$ with increasing Δr.

The normalized density-density correlation functions, given in Eqs. (23) and (24) are plotted in Fig. 1 (dashed lines). The solid lines in this figure represent the corresponding functions obtained by computing the integral in Eq. (20) numerically, without taking the "slow" functions out of the integral. In this computation a white noise random potential has been used which yields $l_k=$const in three dimensions and $l_k \sim k$ in two dimensions (this implies that the diffusion coefficient is proportional to k and k^2 in three or two dimensions, respectively). Figure 1 demonstrates that for the decaying envelope of $I(\Delta r)$ the agreement between the numerically exact results and the approximate expressions, Eqs. (23) and (24), is quite good. For the oscillations the agreement is only qualitative. Due to the rapid decay of the envelope function, the oscillations are rather small and clearly visible only on an amplified scale, as shown in the insets to the figure.

It is quite remarkable that, while the atomic cloud keeps expanding, the local normalized density correlations, Eqs. (23) and (24), do not depend on time. In particular, the characteristic length of oscillations, $\Delta r \sim k_F^{-1}$, remains the same as for the gas in the trap, prior to its release. This behavior is in sharp contrast with that for a free ballistic expansion, when the spatial oscillation period is growing linearly in time, just as the average interparticle distance. The essential

difference between ballistic and diffusive cases can be traced to the evolution of the phase of a wave function. In the ballistic expansion the characteristic spatial period of a wave packet keeps increasing. [This statement can be easily verified by taking an initial wave packet of a Gaussian shape, $\phi(r) \sim \exp(-\frac{r^2}{a^2}+i\frac{r^2}{b^2})$, with $b \ll a$, whose time evolution is exactly soluble.] On the contrary, in the diffusive expansion the phase of a wave function gets randomized after few scattering events and no long-range order in the phase can be established. An oscillating wave function with some period b, subjected to diffusive evolution, will locally look as a plane wave, with the same period, where "locally" means on a scale smaller than the mean free path. Thus, at any time t, the diffusing wave function can be viewed as made up of "patches" of plane waves, of size l_F each, but with no phase relation among different patches. This observation explains the somewhat counterintuitive behavior of the correlations, namely, that the correlation functions in Eqs. (23) and (24) remain stationary, while the interparticle distance increases under the expansion. This leads to an increase of the relative fluctuation of the particle number, in a given volume. For a homogeneous Fermi gas in equilibrium (in three dimensions) the particle number variance $\overline{\Delta N^2}$, in a certain volume, is not equal to the average number of particles $\bar{N}$ (in the same volume) but is proportional to $\bar{N}^{2/3} \ln \bar{N}$ [19,20]. This means that, due to correlations, the Fermi gas possesses some kind of "rigidity." (The effect is particularly spectacular in 1D, where $\overline{\Delta N^2}$ grows only as $\ln \bar{N}$.) Free expansion of the gas, in the absence of disorder, does not affect this rigidity, because, as was already mentioned, the scale of correlations is amplified in exact proportion to the interparticle distance. The disorder disrupts this proportionality and leads to destruction of rigidity and to the $\overline{\Delta N^2}=\bar{N}$ behavior (in the long time limit). Therefore, the image of a Fermi gas, expanding in the presence of disorder, should look more "grainy" than the image of a freely expanding gas. This effect might be observable experimentally.

III. CORRELATIONS IN A DIFFUSING BEC: A MEAN-FIELD APPROACH

So far we considered the dynamics of a degenerate Fermi gas. We now briefly discuss the case of a BEC, expanding in the presence of a random potential. Within the mean-field approach the BEC is described by a macroscopic wave function, $\Psi(\mathbf{r},t)$, whose dynamics satisfies the Gross-Pitaevskii equation

$$i\hbar\partial_t\Psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\Delta\Psi(\mathbf{r},t) + V(\mathbf{r})\Psi(\mathbf{r},t) + g|\Psi(\mathbf{r},t)|^2\Psi(\mathbf{r},t), \tag{25}$$

where g is the interaction parameter related to the scattering length (we assume positive g, i.e., repulsive interactions). Equation (25) describes the evolution of the condensate, in the random potential $V(\mathbf{r})$, upon its release from the trap. We assume an isotropic harmonic trap, characterized by frequency ω. For weak randomness the expansion can be separated into two distinct stages [8,11,12]: A rapid ballistic "explosion," during the time of order $(1/\omega)$, followed by an essentially linear evolution. The first stage is dominated by the nonlinearity. At the second stage, however, most of the interaction energy had been already converted into the kinetic (flow) energy, so that the nonlinearity becomes weak and is neglected [21]. Thus, below we consider the linear equation

$$i\hbar\partial_t\Psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\Delta\Psi(\mathbf{r},t) + V(\mathbf{r})\Psi(\mathbf{r},t). \tag{26}$$

The initial condition for this equation is supplied by the wave function $\Phi(r)$ at the end of the first stage of the expansion, i.e., at time of order $1/\omega$. Qualitatively, this wave function is of the form [22,23]

$$\Phi(r) = \mathcal{F}(r)\exp(ir^2/a_0^2), \tag{27}$$

where $a_0=(\hbar/m\omega)^{1/2}$ is the oscillator size of the trap and $\mathcal{F}(r)$ is an envelope function which decays on the characteristic distance $R_0 \gg a_0$, where R_0 is the initial size of the BEC in the trap. The envelope is often approximated by a Gaussian, $\mathcal{F}(r)=A\exp(-r^2/2R_0^2)$, or by the inverted parabola $\mathcal{F}(r)=A\sqrt{1-(r/R_0)^2}$, with A being the normalization constant.

The emerging linear problem, Eqs. (26) and (27), is considerably simpler than the fermionic problem treated above. This is because the BEC is described by a single coherent wave function which can be treated as a classical field. Within such mean-field description, there are no density fluctuations in the absence of disorder. The notion of density correlations, with their inherent statistical features, becomes meaningful only in the presence of an external random potential. Thus, the problem becomes similar to that considered in the theory of optical speckles, where a classical electromagnetic wave or a scalar wave propagates and gets scattered on a random potential [1]. The essential difference is that in the theory of optical speckles one usually assumes a monochromatic field, whereas the BEC wave function, Eq. (26), contains a broad spectrum of wave numbers. For a monochromatic field $\psi_\omega(\mathbf{r})$ with a wave number k_ω, the disorder-induced intensity-intensity correlation function, $C_\omega(\mathbf{r},\mathbf{r}')=\overline{|\psi_\omega^*(\mathbf{r})\psi_\omega(\mathbf{r}')|^2}$, was calculated in [24],

$$C_\omega(\mathbf{r},\mathbf{r}') = \overline{n_\omega(\mathbf{r})}\,\overline{n_\omega(\mathbf{r}')}\left(\frac{\sin(k_\omega\Delta r)}{k_\omega\Delta r}\right)^2 e^{-\Delta r/l_\omega} \quad (d=3), \tag{28}$$

$$C_\omega(\mathbf{r},\mathbf{r}') = \overline{n_\omega(\mathbf{r})}\,\overline{n_\omega(\mathbf{r}')}J_0^2(k_\omega\Delta r)e^{-\Delta r/l_\omega} \quad (d=2), \tag{29}$$

where $\overline{n_\omega(\mathbf{r})}=\overline{|\psi_\omega(\mathbf{r})|^2}$ is the average intensity of the wave.

The extension to the nonmonochromatic case of a BEC is quite straightforward. Let us start with the average density of the BEC, $n(\mathbf{r})$. The BEC wave function at time t is given by

$$\Psi(\mathbf{r},t) = \int d^dR G(\mathbf{r},\mathbf{R},t)\Phi(R). \tag{30}$$

Using (6), one obtains in the large t (and large r) limit [12]

$$\overline{n(\mathbf{r})} = \overline{|\Psi(\mathbf{r},t)|^2}$$
$$= -\frac{1}{\pi}\int d\varepsilon P_\varepsilon(r,t)\int \frac{d^dk}{(2\pi)^d}\mathrm{Im}\,\bar{G}(\mathbf{k},\varepsilon)|\tilde{\Phi}(k)|^2, \tag{31}$$

where $\tilde{\Phi}(k)$ is the Fourier transform of the initial condition (27). The (equal time) field-field correlation function, $C_{\mathrm{field}}(\Delta r,t)\equiv\overline{\Psi^*(\mathbf{r},t)\Psi(\mathbf{r}',t)}$, differs from (31) only by an extra factor, $f(\Delta r,\varepsilon)$, defined in (19),

$$C_{\mathrm{field}}(\Delta r,t)$$
$$= -\frac{1}{\pi}\int d\varepsilon P_\varepsilon(r,t)\int \frac{d^dk}{(2\pi)^d}f(\Delta r,\varepsilon)\mathrm{Im}\,\bar{G}(\mathbf{k},\varepsilon)|\tilde{\Phi}(k)|^2. \tag{32}$$

Approximating, as above, $\mathrm{Im}\,\bar{G}(\mathbf{k},\varepsilon)=-\pi\delta(\varepsilon-\varepsilon_k)$, we obtain the final expression for the field-field correlation function,

$$C_{\mathrm{field}}(\Delta r,t) = \int \frac{d^dk}{(2\pi)^d}P_k(r,t)f(\Delta r,\varepsilon_k)|\tilde{\Phi}(k)|^2. \tag{33}$$

The short-range density-density correlation function is given by

$$C(\Delta r,t) \equiv \overline{n(\mathbf{r})n(\mathbf{r}')} - \overline{n(\mathbf{r})n(\mathbf{r}')} - \delta(\mathbf{r}-\mathbf{r}')\overline{n(\mathbf{r})}$$
$$= |C_{\mathrm{field}}(\Delta r,t)|^2. \tag{34}$$

Equations (33) and (34), supplemented by the expression (19) for the f function, provide the general solution for the density correlations in a BEC diffusing in a weak random potential.

To get an estimate for the three-dimensional normalized correlation function,

$$I(\Delta r,t) = \frac{C(\Delta r,t)}{\overline{n(\mathbf{r}',t)n(\mathbf{r}',t)}}, \tag{35}$$

we assume a Gaussian envelope, $\mathcal{F}(r)=A\exp(-r^2/2R_0^2)$. Then the Fourier transform of the BEC wave function, Eq. (27), takes the form

$$|\tilde{\Phi}(k)|^2 = \tilde{A}\exp(-k^2/k_0^2), \tag{36}$$

with the normalization constant $\tilde{A}$ and $k_0\approx 2R_0/a_0^2\simeq 1/\xi$, where ξ is the healing length. The numerical evaluation of Eq. (35) (see Fig. 2), shows that, quite similar to the case of the Fermi gas, $I(\Delta r,t)$ decays on a scale $\Delta r\sim k_0^{-1}$. This scale remains nearly constant in time. Since the "cutoff function," $|\tilde{\Phi}(k)|^2$, under the integral in Eq. (33) is not as sharp as the step function in Eq. (20), the oscillations of the normalized correlation function are even weaker than for the fermionic case. The short-range correlations, shown in Fig. 2, imply that the image of a condensate, diffusing in a weak random potential, should exhibit a random pattern of particle density (speckle). The typical size of each speckle spot is of the order of a few healing lengths.

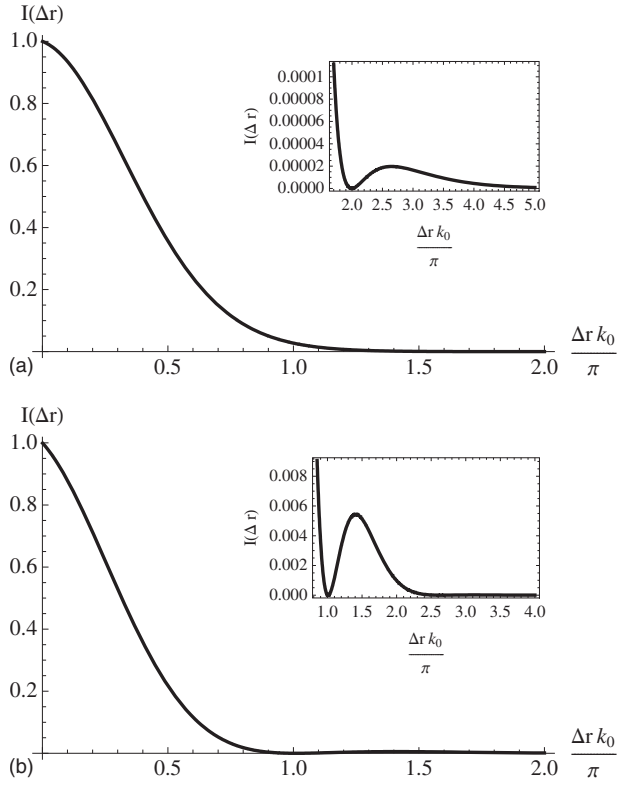

FIG. 2. The normalized density-density correlation function $I(\Delta r,t)$ of a BEC for (a) $d=3$ and (b) $d=2$. As in Fig. 1 the parameters were set to $k_0l_0=10$, $r=50l_0$, $t=3r^2/(2dD_{k_0})$, $D_k=D_{k_0}(k/k_0)^{4-d}$, $l_k=l_0(k/k_0)^{3-d}$.

IV. CONCLUSIONS

We have studied density fluctuations in a cloud of cold atoms, expanding in the presence of a weak random potential. Only the simplest, two-point correlation function, $C(\Delta r,t)$, was considered. We find that the disorder has a strong effect on $C(\Delta r,t)$, for either a Fermi gas or for a BEC. In both cases we obtain a random density pattern consisting of speckles of high and low density. The typical speckle size is determined by the decay length of the correlation function. For fermions this size is of the order of the Fermi wavelength $\lambda_F=2\pi/k_F$, where k_F is the Fermi wave number in the trap. It is interesting, and somewhat counterintuitive, that while the gas keeps expanding, the typical speckle size does not change. This is in contrast to the "clean" case of the free expansion, when a fixed, time-independent relation exists between the correlation length and the interparticle distance. For a BEC we confined ourselves to a mean-field treatment, based on the time-dependent Gross-Pitaevskii equation. In this case the speckle structure is caused solely by the random potential. The typical speckle size is given by the healing length of the condensate in the trap and, once again, this size does not change in the process of the expansion. If one were to go beyond the mean-field description of a BEC, then density correlations (bunching) would appear even in the absence of disorder. The combined effect of quantum fluctuations and disorder on density correlations for bosons (similarly to what has been done for fermions in Sec. II) is an

interesting problem which is, however, beyond the scope of this paper [25]. The only important parameter in our theory is $k_F l_F$, for fermions, and $k_0 l_0$ for the BEC. We have assumed zero temperature in all of the calculations. The extension to finite temperatures, at least for the Fermi case, is quite straightforward but has not been done in the present paper.

Only short-range correlations were considered. It is well known in optics, as well as in the mesoscopic physics of disordered conductors, that in addition to the strong short-range correlations there are also weak long-range correlations. Such correlations manifest themselves as fluctuations in the transmission coefficient through a disordered slab or as the universal conductance fluctuations in disordered conductors [1]. In order to observe similar effects for cold atoms one should create a random potential in some region (say, of a shape of a slab) and then let an atomic cloud impinge on that region. One could then take images of the transmitted, as well as of the reflected clouds.

Our numerical estimates have been made for a white noise random potential, which is the case when the correlation radius of the potential, R_c, is smaller than the relevant wavelength, λ, of the matter waves (the Fermi wavelength for fermions or the healing length for the BEC). Since the random potential for atoms is commonly produced by creating a random pattern of light intensity (an optical speckle), one can assume that R_c is of the order of a few light wavelengths. If R_c exceeds λ, one enters the regime of the correlated potential (colored noise). In that regime the diffusion coefficient is determined by the transport mean free path, l_{tr}, which is much longer than the scattering mean free path l_k appearing in the average Green's function. However, since both only weakly depend on k our results should remain qualitatively correct, at least in the well-developed diffusion regime, i.e., when the cloud has spread to a distance much larger than l_{tr}.

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