Vortex trapping in suddenly connected Josephson junctions of Bose-Einstein condensates

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We investigate the problem of vortex trapping in cyclically coupled Bose-Josephson junctions. Starting with N independent Bose-Einstein condensates we couple the condensates through Josephson links and allow the system to reach a stable circulation by adding a dissipative term in our semiclassical equations of motion. The central question we address concerns the probability of trapping a vortex with winding number m. Our numerical simulations reveal that the final distribution of winding numbers is narrower than the initial distribution of total phases, indicating an increased probability for no-vortex configurations. Specifically, the final width of the distribution of winding numbers for N sites scales as λN^{α} , where $\alpha = 0.47 \pm 0.01$ and $\lambda < 0.67$ (the value predicted for the initial distribution). The actual value of λ is found to depend on the strength of dissipation. The nonlinearity of the problem also manifests itself in the result that it is possible to obtain a nonzero circulation starting with zero total phase around the loop.

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In the past few years, experiments on Josephson tunnel junctions in superconductors [1,2] and Bose-Einstein condensates (BECs) [3,4] have addressed the role of nonadiabaticity in the spontaneous production of topological defects, a question that has bearing on early-universe cosmology [5-8]. While a first type of experiments [2] used a temperature quench through a second-order phase transition from a normal to a superconducting phase, a second type [3,4] uses interference between initially independent condensates as a mechanism to trap vortices. In the case of superconductors, the Kibble-Zurek scaling law $\begin{bmatrix} 6 \end{bmatrix}$ relating the probability to trap vortices to the quench rate has been tested. Experiments connecting independent BECs have similarly tried to test the role of the merging rate in determining the probability for observing vortices in the final BEC. Motivated by these experiments, we have studied numerically the related problem of a ring-shaped Bose-Josephson junction array. We would like to stress that, while there are similarities between our initial conditions and those of the aforementioned experiments, there are also qualitative differences that will be discussed later. Nevertheless, it is quite conceivable that our findings here can be tested in future experiments with ultracold atomic gases [9].

The problem we study here is that of N independent Bose-Einstein condensates which upon sudden connection become arranged as a ring of weakly coupled condensates. We assume that the phase inside each condensate is uniform, the condition for which is outlined in Ref. [10]. This condition further ensures that no vortices form within the individual condensates, leaving us only with vortices caused by the phase variation along the ring. At t=0, simultaneous Josephson contacts are made between each adjacent pair of condensates. As shown in Ref. [11], for the case of two initially independent condensates, a relative phase is quickly established once a few condensate atoms have hopped from one side to another. Each pair of neighboring condensates behaves as if a random relative phase $\varphi \in (-\pi, \pi]$ is chosen locally. However, due to the single-valuedness of the macroscopic wave function, there are only N-1 independent variables. Therefore, in our simulations we choose N-1 relative phases independently, each following a flat distribution within the interval $(-\pi, \pi]$. The *N*th relative phase lies in the same interval and is determined by the constraint that the total phase variation around the ring should be $2\pi n$ ($n \in \mathbb{Z}$). From the central limit theorem, we know that for $N \rightarrow \infty$ the distribution of *n* approaches a normal distribution with full width at half maximum (FWHM) of $2.354\sigma N^{1/2}$, where $\sigma = 1/\sqrt{12}$ is the standard deviation for a flat distribution in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$. A key point is to realize that the classically stable fixed points correspond to all the relative phases being equal (modulo 2π) to a value $2\pi m/N$, where $m \in \mathbb{Z}$ is the winding number or charge of the final vortex conguration. To allow our system to converge to one of these fixed points, we let each link follow a semiclassical Josephson equation which includes a phenomenological dissipation term characterized by a single parameter γ . Such dynamics allows the system to go through phase slips at individual junctions. Thus, generally $m \neq n$. A number of interesting results are obtained.

(i) The distribution of the final winding number deviates from the initial distribution for all values of N and γ . That final distribution for m is narrower than the initial distribution for n, indicating an increased probability for low-charge vortex configurations (see Fig. 1).

(ii) The width of the final distribution scales with the size



FIG. 1. Initial distribution of total phases and final distribution of stable winding numbers for $N=10^3$ and $\gamma=5$ for 10^5 runs



FIG. 2. (Color online) Red plot (squares): scaling of the FWHM of the final distribution of winding numbers with *N* for γ =6. The scaling exponent is α =0.47±0.01 and the prefactor λ =0.55±0.05. Blue plot (triangles): scaling of the FWHM of the initial distribution of total relative phases; α =0.50±0.01, λ =0.67±0.05.

of the system as λN^{α} , where $\alpha = 0.47 \pm 0.01$, independent of γ and $\lambda < 0.67$ (the normal distribution value), indicating a shrinking of the basins of attraction for higher winding numbers (see Figs. 2 and 3). For $\gamma \leq 3$ the width of the final distribution shrinks upon decrease of γ (see inset of Fig. 3).

(iii) If one focuses on initial configurations with n=0, the final distribution of winding numbers in the limit of large N is still a Gaussian centered around m=0 with a nonzero spread (see Fig. 4). This reflects the fact that a finite fraction of the initial configurations with zero total phase have Josephson coupling energies higher than those that correspond to nonzero final winding numbers. The width of the final distribution generated from this initially restricted configuration is clearly smaller than the width of the final distribution for unrestricted initial conditions (see Figs. 1 and 4). This suggests that, while the reconnection process can result in the generation of a finite winding number with a sum total of zero for the initial phases, there is also significant evidence of memory of the initial conditions.

We start our analysis of the Josephson dynamics by stating a theorem: If N BECs with random relative phases are coupled by a nearest-neighbor Josephson coupling on a onedimensional lattice with periodic boundary conditions, a necessary condition to obtain a metastable nonzero circulation of winding number $2\pi m$ is N > 4m, the case of 4m links being marginal. The proof is as follows.

Let us assume that each Josephson junction is described by a two-mode Josephson Hamiltonian





FIG. 4. Restricted to configurations $\Sigma_i \phi_{i,i+1} = 0$, this histogram for final winding numbers shows that even in the high-friction limit one can obtain a nonzero circulation. The above simulation uses $N=10^3$ and $\gamma=50$.

$$H = -E_J \sum_{i} \cos \phi_{i,i+1} + (E_C/2) \sum_{i} n_i^2, \qquad (1)$$

where E_J is the Josephson coupling energy, E_C is the charging energy, $\phi_{i,i+1}$ is the relative phase between *i* and *i*+1 (with i=N+1 identified with i=1), and $n_i=N_i-N_i^{(0)}$ is the deviation of the number of particles N_i from the equilibrium value $N_i^{(0)}$ at condensate *i*. We assume all $N_i^{(0)}$'s to be the same and initially $n_i=0$, so that $\sum_i n_i=0$ throughout the entire evolution. In the classical limit, this Hamiltonian can be mapped into that of coupled rigid pendulums, with the first term denoting the "potential energy" and the second term the "kinetic energy" of the pendulum system. Now consider a system with N links and a total phase difference of $2\pi m$ around the loop. As stated earlier, the fixed point corresponding to a circulation of charge *m* is given by the configuration where all the phases are $\varphi_m = 2\pi m/N$ (modulo 2π). Hereafter, we simplify the notation $\phi_i \equiv \phi_{i,i+1}$. To determine whether this fixed point is stable we consider a configuration where $\phi_i = \varphi_m + \epsilon_i$ with $\Sigma_i \epsilon_i = 0$ and $\epsilon_i \rightarrow 0$. The potential energy of this new configuration with respect to the fixed point is, up to second order in ϵ_i , given by $\Delta E(\epsilon_i) = (\cos \varphi_m) \Sigma_i \epsilon_i^2$. For the fixed point to be stable we should have $\Delta E(\epsilon_i) > 0$, which requires N > 4m. This theorem can equally be applied to a system of XY spins coupled by Heisenberg interaction. A corollary is that final configurations satisfying $N/4 \le m \le N/2$ are unstable [12].

For a more generic analysis of the fixed points and their basins of attraction we derive from Hamiltonian (1) a set of semiclassical equations of motion for the relative phases and currents at each junction:

$$\ddot{\phi}_i(t) = E_C[2j_i(t) - j_{i+1}(t) - j_{i-1}(t)], \qquad (2)$$

$$j_i(t) = -\sin \phi_i(t) - \gamma \dot{\phi}_i(t).$$
(3)

Here time and energies are expressed in units of E_J^{-1} and E_J (\hbar =1), respectively. It is important to note that for cyclically coupled Josephson junctions the variable canonically conjugate to, say, ϕ_i is not $(n_i - n_{i+1})$ but rather the quantity $\int_0^t j_i(t) dt$. The detailed dynamics of the few-site case without dissipation has been studied by Dziarmaga *et al.* [13]. Here we have added a phenomenological dissipative term of the form $-\gamma \dot{\phi}_i$ in the equation of motion for j_i while neglecting finite-temperature noise [13]. It is important to add this term



FIG. 5. Potential energy landscape for N=10 and a certain class of configurations: m=2; $\phi_i=4\pi/10+\epsilon$, $\phi_j=4\pi/10-\epsilon/9$; $j\neq i$. Winding number zero is the global minimum of the energy landscape and here occurs at $\epsilon=3.6\pi$.

for the system to converge to one of the fixed points. From our knowledge of three or more coupled pendulums, we know that the system of Eqs. (2) and (3) is chaotic [14] and without any damping would typically explore the whole phase space without converging to a fixed point. To verify this point, we have investigated the dynamics of Lyapunov exponents for the case of N=3. To ensure that the system is in the Josephson regime we take $E_C/E_J=0.01$ in all our simulations. We find that three out of six Lyapunov exponents are positive, indicating chaotic behavior. We note that the Ohmic nature of the dissipative term is justified at temperatures higher than the chemical potential between sites [10] or at low temperatures if each condensate exists in a large box [15].

An interesting property of Eq. (2) is that $\sum_i \phi_i$ is a mathematical constant of motion. However, physically the system can still change its winding number by going through phase slips at any junction. It will be useful to incorporate the above constant of motion by imposing the restriction $\phi_i \in (-\pi, \pi]$ only at t=0 and removing it for later times. Of course, the physical quantity that is observed at the end of the evolution is the Josephson current at each junction, which depends on the relative phase modulo 2π . Thus, for accounting purposes we count states as different if they have had different histories, even if at the time in question they are physically indistinguishable.

In order to generate statistics, we consider a large number of different initial configurations, with the relative phases and numbers chosen as explained earlier. Equations (2) and (3) are then numerically integrated for each set of initial conditions. After the average current has reached its final equilibrium value, its magnitude equals $\sin(2\pi m/N)$ and the value of the final winding number m < N/4 is uniquely extracted. A histogram is then plotted for all values of *m* and its width is recorded. To obtain the scaling law, we have calculated the width as a function of *N* and fitted it to a function of the form λN^{α} . The process is repeated for different values of γ .

To get a qualitative idea of the dynamics and the role of dissipation, we consider a certain class of initial configurations where $\phi_1 = \varphi_m + \epsilon$ while $\phi_i = \varphi_m - \epsilon/(N-1)$ for $2 \le i \le N$. Given ϕ_1 , this configuration has the lowest potential energy. Figure 5 shows the potential energy for such a configuration as a function of ϵ for N=10 and m=2. The first minimum corresponds to the fixed point K_2 ($\phi_i = \varphi_2$ for all *i*) followed by the fixed point K_1 ($\phi_1 = \varphi_1 + 2\pi$; $\phi_i = \varphi_1$ for all i > 1) and so on. The global minimum of the energy landscape is the configuration K_0 with zero winding number. Starting with the initial configuration mentioned above, Fig. 5 shows the path of steepest descent from K_2 to K_0 . Starting from a local minimum, one can characterize the size of the basins of attraction by the value ϵ_c taken by ϵ at the next nearest local maximum. However, one should be warned that such an estimate applies only to the specific class of initial configurations described above.

The role played by dissipation can also be elucidated by studying that class of configurations. Suppose $\epsilon > \epsilon_{c1}$, where ϵ_{c1} is the first critical value of ϵ . The system starts at an unstable point, and, as it rolls down to the fixed point with one lower winding number, loses kinetic energy due to friction. If it arrives at the next stable point with kinetic energy less than what is needed to overcome the next barrier, then it settles down at the fixed point K_{m-1} . However, if it has enough kinetic energy to roll over the next barrier, then the final winding number will be less than (m-1). A similar role can be envisaged for dissipation in the general multidimensional landscape: For large γ , the system settles down in the nearest valley; for small γ , the particle may escape the initial basin and lower its winding number. Thus low friction enhances, by a moderate factor, the probability of ending in a low-charge configuration, as suggested by Fig. 5 and confirmed by Fig. 3.

For a semianalytical discussion of the basins of attraction, we focus on the case of N=5 (stable $m=0, \pm 1$) and high friction. Let P(m) be the probability of landing in a final vortex configuration of charge m, Q(n) the initial probability for $\sum_i \phi_i = 2\pi n$, and P(m|n) the probability to obtain a final charge *m* conditioned to $\Sigma_i \phi_i = 2\pi n$. Below we estimate P(1)and show that P(1) < Q(1). First we note that P(1)=P(1|1)Q(1)+P(1|0)Q(0)+P(1|-1)Q(-1). We therefore begin by estimating P(1|1). The limit of high friction ensures that the system follows the path of steepest descent toward the nearest stable fixed point. The system always resides on the hypersurface S_n defined by the constant of motion $\sum_i \phi_i = 2\pi n$. Note that, on the surface S_1 , most of the m=1 configurations correspond to the fixed point $\phi_i(t)$ $=2\pi/5$ (i=1,...,5), whereas m=0 can emerge from five different fixed points on S_1 , namely, those of the type $\phi_i(t)$ $=2\pi$ with $\phi_i(t)=0$ for all $j\neq i$ $(i=1,\ldots,5)$. Likewise, m=-1 is dominated by two sets of fixed points on S₁: five corresponding to one link having undergone a 4π total slip, and ten corresponding to two different links each having undergone a 2π slip. Note that, even for m=1 on S_1 , there are many other configurations different from the dominant ones mentioned above e.g., $\phi_i = 2\pi/5 + 2\pi$, $\phi_i = 2\pi/5 - 2\pi$, and $\phi_k = 2\pi/5$ for $k \neq i, j$ (i, j = 1, ..., 5). However, in the limit of large γ , those configurations involving many different, mutually canceling phase slips should have negligible probability.

To calculate the area of the basin of attraction for m=1, we define a set of five orthonormal vectors \hat{x}_i such that four of them lie on S_1 and the fifth vector is perpendicular to S_1 . We define our origin on S_1 by shifting that of S_0 along \hat{x}_5 by an amount $\varphi_1=2\pi/5$. The five vectors are then given by $\hat{x}_1 = (1/\sqrt{2})(1, -1, 0, 0, 0), \quad \hat{x}_2 = (1/\sqrt{2})(0, 0, 1, -1, 0), \\ \hat{x}_3 = (1/\sqrt{20})(1, 1, 1, 1, -4), \quad \hat{x}_4 = (1/2)(1, 1, -1, -1, 0), \text{ and } \\ \hat{x}_5 = (1/\sqrt{5})(1, 1, 1, 1, 1).$

To obtain the basin boundaries on the four-dimensional hypersurface we next write the four independent ϕ_i 's in terms of the in-plane basis vectors \hat{x}_i (i=1, ..., 4) and transform to spherical coordinates $(r, \theta_1, \theta_2, \theta_3)$. Now, the potential energy is given by $\mathcal{E}=-E_J\Sigma_i\cos\phi_i$ and the condition $\partial \mathcal{E}/\partial r=0$ defines the boundary of the basin of attraction. Shifting the origin back to S_0 , the basin boundary for m=1 on S_1 is then given by

$$f_1 \sin(rf_1 + \varphi_1) + f_2 \sin(rf_2 + \varphi_1) + f_3 \sin(rf_3 + \varphi_1) + f_4 \sin(rf_4 + \varphi_1) = 0,$$
(4)

where the various $f_k = f_k(\theta_1, \theta_2, \theta_3)$ are obtained from a coordinate transformation. The probability P(1|1) to end up with m=1 having started from any point on S_1 is given by the ratio A_1/B_1 , where A_1 is the area enclosed by the curve (4) on S_1 , and B_1 is the total area on S_1 subject to the initial constraints $\phi_i(0) \in (-\pi, \pi]$. Using a Monte Carlo integration method, we obtain P(1|1)=0.03. Similarly, we also calculate P(0|1) and P(0|0) by the Monte Carlo method, both yielding 0.94. Using this second result, the symmetry between m=1 and m=-1, and the fact that

P(1|0)+P(0|0)+P(-1|0)=1, we can also obtain P(1|0) = P(-1|0)=0.03. By contrast, the initial distributions are Q(0)=0.6 and Q(1)=Q(-1)=0.2. Hence, in the limit of large γ , P(1)/Q(1)=0.15, which indicates a shrinking of the initial distribution in favor of final zero winding number. Full scale simulations based on Eqs. (2) and (3) yield 0.14 for the same ratio. An exact agreement would require consideration of infinitely many phase-slip histories.

In passing, we would like to note that the above analysis holds true strictly in the Josephson regime. Experiments with fully merging independent BECs [3] or the scenario of quasicondensates in BEC formation as envisaged by Zurek [6] always go through an intermediate Josephson regime when adjacent condensates start to overlap. However, a complete study of the dynamics there would require going beyond the two-mode Josephson Hamiltonian (1) for each junction. This is clearly reflected in the outcome of experiments by Scherer *et al.* [3], where three independent BECs have been merged to form stable vortices in the final BEC.

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