

Population trapping in three-state quantum loops revealed by Householder reflections

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Population trapping occurs when a particular quantum-state superposition is immune to action by a specific interaction, such as the well-known dark state in a three-state Λ system. We here show that in a three-state loop linkage, a Hilbert-space Householder reflection breaks the loop and presents the linkage as a single chain. With certain conditions on the interaction parameters, this chain can break into a simple two-state system and an additional spectator state. Alternatively, a two-photon resonance condition in this Householder-basis chain can be enforced, which heralds the existence of another spectator state. These spectator states generalize the usual dark state to include contributions from all three bare basis states and disclose hidden population trapping effects and hence hidden constants of motion. Insofar as a spectator state simplifies the overall dynamics, its existence facilitates the derivation of analytic solutions and the design of recipes for quantum-state engineering in the loop system. Moreover, it is shown that a suitable sequence of Householder transformations can cast an arbitrary N -dimensional Hermitian Hamiltonian in a tridiagonal form. The implication is that a general N -state system, with arbitrary linkage patterns where each state connects to any other state, can be reduced to an equivalent chainwise-connected system, with nearest-neighbor interactions only, with ensuing possibilities for discovering hidden multidimensional spectator states and constants of motion.

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I. INTRODUCTION

Descriptions of the optical excitation of few-state quantum systems traditionally make use of the rotating-wave approximation (RWA), in which the Hilbert-space unit vectors (the bare quantum states) rotate with angular velocities that are fixed at various laser carrier frequencies and the Hamiltonian, with the neglect of rapidly varying terms, becomes a matrix of slowly varying Rabi frequencies and detunings [1,2]. For three states the usual electric-dipole selection rules of optical transitions produce a simple chain of interstate linkages, depicted as either a ladder, a Λ , or a V .

For some time it has been known that, either by means of a rotation of the arbitrary quantization axis for defining magnetic sublevels or by more general reorganization of the Hilbert-space basis states (a Morris-Shore transformation [3]), such patterns can be presented as a set of independent two-state excitations (bright states) together with spectator (dark) states that are unaffected by the specific radiation.

The presence of a third interaction, linking the two states that terminate the three-state chain, turns the linkage pattern into a loop; see Fig. 1. Such an interaction would violate the usual selection rules for electric-dipole radiation (which connects only states of opposite parity), but is possible for a variety of other interactions, such as occur with two-photon optical transitions or microwave transitions between hyperfine states. To avoid the presence of rapidly varying exponential phases in the RWA Hamiltonian, such a link should occur with carrier frequency suitably chosen.

Within the RWA there is no longer a distinction of the original bare-state energies; all that matters is the detunings—i.e., differences between a Bohr frequency and the associated laser-field carrier frequency. Nonetheless, it is traditional, when depicting the linkage pattern of laser-

induced interactions, to place representations of the states in a vertical direction ranked according to the original bare-state energies. Such a display convention is particularly useful in emphasizing the difference between low-energy stable or metastable states, unable to radiatively decay, and excited states, from which spontaneous emission is possible—visible as fluorescence.

With the loop pattern it is not immediately obvious that any simple restructuring of the Hilbert-space coordinates will produce a spectator state. For example, the loop system does not satisfy an essential condition for the Morris-Shore (MS) transformation [3]—namely, that the quantum states be classified into two sets, with transitions only between the sets, not within them.

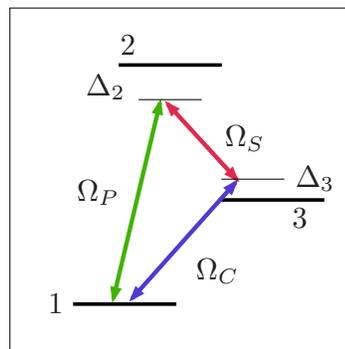


FIG. 1. (Color online) RWA linkage pattern for a loop, showing linkages: states ψ_1 and ψ_2 by Rabi frequency Ω_P , states ψ_2 and ψ_3 by Ω_S , and states ψ_1 and ψ_3 by Ω_C . The energy levels are shown with an ordering appropriate to state ψ_1 as ground state and state ψ_2 as excited state, but the symmetry of the loop linkage allows initial population in any state.

The three-state loop is the simplest example of discrete quantum states that can exhibit nontrivial probability-amplitude interferences, and hence it has attracted continuing interest [4–14]. A special case of the three-state loop was considered by Carroll and Hioe [6]. They presented analytical solutions for the probability amplitudes when three resonant laser pulses of different shapes were present and two of the couplings were real, while the third was purely imaginary. For this special case, the underlying SU(2) symmetry allows the three-state loop to be reduced to an effective two-state system.

Another resonant three-state loop was examined by Unanyan *et al.* [10]. In that work a pulsed quasistatic magnetic field supplemented the two optical pulses of a Λ linkage used for stimulated Raman adiabatic passage (STIRAP) [15]. This additional field provided a supplement to the usual adiabatic constraints and allowed a reduction of diabatic loss, thereby improving the population transfer efficiency.

The three-state loop was examined also by Fleischhauer *et al.* [11]. They showed that when each link was resonant, the dark state of STIRAP [15] could be modified to a higher-order trapping state, becoming an approximate constant of motion even for small pulse areas. This state adiabatically rotates, in Hilbert space, from the initial to the target state. This adiabatic motion leads to efficient population transfer, though at the expense of placing some population into the decaying atomic state.

Recently a three-state loop was shown to occur in physical processes where the free-space symmetry is broken, as it is in chiral systems [12,13]. Such quantum systems occur in left- and right-handed chiral molecules [12] or in “artificial atoms.” Loop linkages among discrete quantum states can also occur in superconducting quantum circuits [13] and in modeling entangled atoms coupled through an optical cavity [14].

We here consider loops that have less stringent constraints on the frequencies, although some do exist. We shall show in the following that it is possible, under appropriate conditions (including three-photon resonance), to break the loop into a chain. A further transformation of the basis states can then convert the linkage pattern into a pair of coupled states and a spectator state.

The required initial transformation, converting the loop into a simple chain, is taken to be a Householder reflection (HR) of the Hamiltonian matrix [16]. Such matrix manipulations, commonplace in works dealing with linear algebra [17], have recently been applied to quantum-state manipulations [18–21].

When acting upon an *arbitrary* square matrix a suitable sequence of HRs produces an upper-diagonal (or lower-diagonal) matrix. When acting upon a *unitary* matrix, such a sequence produces a *diagonal matrix*, with phase factors on the diagonal. This property has been used for decomposition, and therefore synthesis, of arbitrary preselected propagators in multistate systems [18–21]. We show here that, when utilized for a change of basis in Hilbert space, a suitable HR (or a sequence of HRs) can cast a (Hermitian) Hamiltonian into a *tridiagonal* form. This tridiagonalization implies a replacement of a general linkage pattern (for example, each state interacting with any other state) with an effective chainwise-

connected system where only nearest-neighbor interactions are present. We apply this tridiagonalization to the simplest nontrivial multistate system—a three-state loop system—and demonstrate its potential applications with examples ranging from effective chain breaking and spectator states to hidden two-photon resonances.

II. LOOP RWA HAMILTONIAN

We consider three fields, labeled pump (P), Stokes (S), and control (C),

$$\mathbf{E}_k(t) = \hat{\mathbf{e}}_k \mathcal{E}_k(t) \cos(\omega_k t + \phi_k) \quad (k = P, S, C). \quad (1)$$

The three carrier frequencies ω_k can be chosen arbitrarily, as long as they fulfill the three-photon resonance condition (Fig. 1)

$$\omega_C - \omega_P + \omega_S = 0. \quad (2)$$

This constraint is necessary for application of the RWA [1,2]. However, at the outset we impose no constraints on the single-photon detunings,

$$\hbar \Delta_P \equiv E_2 - E_1 - \hbar \omega_P, \quad (3a)$$

$$\hbar \Delta_S \equiv E_2 - E_3 - \hbar \omega_S. \quad (3b)$$

We introduce probability amplitudes $C_n(t)$ in the usual rotating Hilbert-space coordinates $\psi_n(t)$,

$$\Psi(t) = \exp(-i\zeta_0 t) [C_1(t)\psi_1 + C_2(t)\psi_2 + C_3(t)\psi_3], \quad (4)$$

where the rotations originate with field carrier frequencies, $\psi_2(t) \equiv \exp(-i\omega_P t)\psi_2$ and $\psi_3(t) \equiv \exp(-i\omega_C t)\psi_3$. From the time-dependent Schrödinger equation we obtain three coupled equations, conveniently written in the vector form

$$\frac{d}{dt} \mathbf{C}(t) = -i\mathbf{W}(t)\mathbf{C}(t), \quad (5)$$

where $\mathbf{C}(t) \equiv [C_1(t), C_2(t), C_3(t)]^T$ is a three-component column vector of probability amplitudes and $\hbar\mathbf{W}(t)$ is the slowly varying RWA Hamiltonian matrix. We take the overall phase factor ζ_0 to nullify the first diagonal element of $\mathbf{W}(t)$; it then has the structure

$$\mathbf{W}(t) = \frac{1}{2} \begin{bmatrix} 0 & \Omega_P(t)e^{i\phi_P} & \Omega_C(t) \\ \Omega_P(t)e^{-i\phi_P} & 2\Delta_2 & \Omega_S(t)e^{i\phi_S} \\ \Omega_C(t) & \Omega_S(t)e^{-i\phi_S} & 2\Delta_3 \end{bmatrix}, \quad (6)$$

where the interactions are parametrized by slowly varying real-valued Rabi frequencies $\Omega_k(t)$ ($k = P, S, C$). For simplicity and without loss of generality the C field is assumed real ($\phi_C = 0$); then, ϕ_P and ϕ_S represent the phase differences between the P and S fields, respectively, and the C field. The cumulative detunings are

$$\Delta_2 = \Delta_P, \quad \Delta_3 = \Delta_P - \Delta_S. \quad (7)$$

We note that Eq. (6) applies also to three-state loop systems with diagonal light-matter interaction terms, as in systems

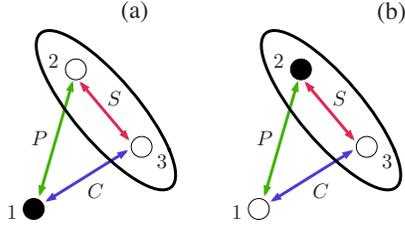


FIG. 2. (Color online) The Householder reflection leaves state ψ_1 unchanged. Initial population might be (a) in this state or (b) in one of the altered states. Relative energies of the original bare states are not relevant, only the couplings.

with broken symmetry, because the zero in the top left corner can always be produced by an appropriate phase transformation of the probability amplitudes.

III. HOUSEHOLDER REFLECTION

We seek a unitary transformation of the Hilbert-space basis states that will first replace the loop with a three-state chain. As we will show, the desired result can be produced by a Householder reflection acting upon the RWA Hamiltonian.

An N -dimensional Householder reflection is defined as the operator

$$\mathbf{M} = \mathbf{I} - 2|v\rangle\langle v|, \quad (8)$$

where \mathbf{I} is the identity operator and $|v\rangle$ is an N -dimensional normalized complex column vector. The Householder operator \mathbf{M} is Hermitian and unitary, $\mathbf{M} = \mathbf{M}^\dagger = \mathbf{M}^{-1}$; hence, \mathbf{M} is involutory, $\mathbf{M}^2 = \mathbf{I}$. The transformation is a reflection, so $\det \mathbf{M} = -1$. If the vector $|v\rangle$ is real, the Householder reflection has a simple geometric interpretation: it is a reflection with respect to an $(N-1)$ -dimensional plane normal to the vector $|v\rangle$.

The Householder reflection, acting upon an arbitrary N -dimensional matrix, uses $N-1$ operations to produce an upper or lower triangular matrix. This behavior makes the Householder reflection a powerful tool for many applications in classical data analysis [17]. For us, the reflection serves to transform the Hamiltonian from a full matrix to one that lacks one interaction—it breaks the loop into a chain.

The three-state system offers three basis vectors with which to define a Householder reflection. Because of the symmetry of the loop system, it is only necessary to consider one of these; the effect of others can be examined by a permutation of state labels. We shall take state ψ_1 as a fixed coordinate, within the plane of the reflection, and introduce an alteration of the Hilbert subspace spanned by the remaining unit vectors $\psi_2(t)$ and $\psi_3(t)$. Figure 2 illustrates the connection of the reflection with the basis states and the possible choices of the initial state.

With this choice the Householder vector reads

$$|v\rangle = [0, \sin(\theta/2)e^{-i\phi_P}, \cos(\theta/2)]^T \quad (9)$$

and the matrix representation of the Householder reflection is

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & e^{-i\phi_P} \sin \theta \\ 0 & e^{i\phi_P} \sin \theta & -\cos \theta \end{bmatrix}. \quad (10)$$

The angle θ , defined by the equation

$$\tan \theta \equiv \frac{\Omega_C}{\Omega_P}, \quad (11)$$

is twice the angle from the mirror normal to state ψ_2 (i.e., the twist of the mirror about the ψ_1 axis). Hereafter we omit explicit mention of time dependences; all Rabi frequencies are to be considered slowly varying in time, as are the Householder reflection \mathbf{M} and the angle θ .

The connection between the probability amplitudes $\tilde{\mathbf{C}}$ in the Householder basis and the amplitudes \mathbf{C} in the original (bare) basis is

$$\tilde{\mathbf{C}} = \mathbf{M}\mathbf{C}. \quad (12)$$

The transformed equation of motion reads

$$\frac{d}{dt}\tilde{\mathbf{C}} = -i\tilde{\mathbf{W}}\tilde{\mathbf{C}}, \quad (13)$$

where the Householder Hamiltonian is $\tilde{\mathbf{W}} = \mathbf{M}\mathbf{W}\mathbf{M} - i\mathbf{M}\dot{\mathbf{M}}$, with an overdot denoting a time derivative. Explicitly,

$$\tilde{\mathbf{W}} = \frac{1}{2} \begin{bmatrix} 0 & \tilde{\Omega}_P & 0 \\ \tilde{\Omega}_P^* & 2\tilde{\Delta}_2 & \tilde{\Omega}_S - 2ie^{-i\phi_P}\dot{\theta} \\ 0 & \tilde{\Omega}_S^* + 2ie^{i\phi_P}\dot{\theta} & 2\tilde{\Delta}_3 \end{bmatrix}, \quad (14)$$

with effective detunings

$$\tilde{\Delta}_2 = \frac{\Delta_3\Omega_C^2 + \Delta_2\Omega_P^2 + \Omega_P\Omega_C\Omega_S \cos(\phi_P + \phi_S)}{\Omega^2}, \quad (15a)$$

$$\tilde{\Delta}_3 = \frac{\Delta_2\Omega_C^2 + \Delta_3\Omega_P^2 - \Omega_P\Omega_C\Omega_S \cos(\phi_P + \phi_S)}{\Omega^2} \quad (15b)$$

and effective couplings

$$\tilde{\Omega}_P = e^{i\phi_P}\Omega, \quad (16a)$$

$$\begin{aligned} \tilde{\Omega}_S = \frac{1}{\Omega^2} [& 2e^{-i\phi_P}(\Delta_2 - \Delta_3)\Omega_P\Omega_C \\ & + (e^{-2i(\phi_P + \phi_S)}\Omega_C^2 - \Omega_P^2)e^{i\phi_S}\Omega_S], \end{aligned} \quad (16b)$$

with $\Omega \equiv \sqrt{\Omega_P^2 + \Omega_C^2}$. All of these elements acquire a time dependence from the pulses, though that is not shown explicitly.

The Hamiltonian in the Householder basis is that of a simple chain, $\psi_1 \leftrightarrow \tilde{\psi}_2 \leftrightarrow \tilde{\psi}_3$. By design the Householder reflection places the original two interactions of state ψ_1 into a single effective interaction with a superposition state $\tilde{\psi}_2$. This state, in turn, has an interaction with the other terminal state of the chain $\tilde{\psi}_3$, also a superposition state. The Householder states $\tilde{\psi}_n$ are superpositions of the original basis states ψ_n ,

$$\tilde{\psi}_1 = \psi_1, \quad (17a)$$

$$\tilde{\psi}_2 = \cos \theta \psi_2 + e^{-i\phi_P} \sin \theta \psi_3, \quad (17b)$$

$$\tilde{\psi}_3 = e^{i\phi_P} \sin \theta \psi_2 - \cos \theta \psi_3. \quad (17c)$$

When the initial population resides entirely in state ψ_1 , this chain is equivalent to a Λ or ladder system. When the initial population occurs in state $\tilde{\psi}_2$ it corresponds to the V linkage. The inherent interference in the loop system is now imprinted onto the Householder transformation and is absent in the Householder chain. The chain Hamiltonian (14) is conceptually simpler than the original loop Hamiltonian (6) for it allows only for nearest-neighbor interactions. The resulting chain linkage is easier to understand and treat analytically by a variety of exact or approximate approaches, and it allows one to use the considerable literature available on chainwise-connected three-state systems.

IV. SPECIAL CASES

In the remainder of this paper we consider special cases of the Householder Hamiltonian, obtained when we constrain the various pulse parameters, which lead to simplification of the resulting Hamiltonian matrix. Two simplifications are particularly interesting: (i) breaking the three-state Householder chain $\psi_1 \leftrightarrow \tilde{\psi}_2 \leftrightarrow \tilde{\psi}_3$ into a two-state system and a spectator state and (ii) two-photon resonance in the Householder basis. We shall identify conditions and deduce implications for these important special cases.

A. Effective two-state system and spectator state

Under appropriate conditions the three-state Householder chain $\psi_1 \leftrightarrow \tilde{\psi}_2 \leftrightarrow \tilde{\psi}_3$ breaks into two coupled states and a spectator state. This occurs whenever one of the Householder linkages vanishes. The vanishing of $\tilde{\Omega}_P$ requires that both Ω_P and Ω_C vanish, which is trivial and uninteresting. We hence assume the null linkage to be the coupling between states $\tilde{\psi}_2$ and $\tilde{\psi}_3$,

$$\tilde{\Omega}_S + 2ie^{-i\phi_P}\dot{\theta} = 0. \quad (18)$$

Under this condition state $\tilde{\psi}_3$, Eq. (17c), decouples from the other two states and becomes a *spectator state*: its population is trapped within a subspace of the full Hilbert space. The population distribution between states ψ_2 and ψ_3 may change, but in a manner that conserves the population of the spectator state (17c).

1. Conditions for chain breaking

One possible solution to Eq. (18) reads

$$\Delta_2 = \Delta_3, \quad (19a)$$

$$\phi_P = -\phi_S - \frac{\pi}{2}, \quad (19b)$$

$$\Omega_S = -2\dot{\theta}. \quad (19c)$$

The latter condition imposes a strict constraint on the pulse shapes. Given Ω_P and Ω_C , which determine $\dot{\theta}$ through Eq. (11), condition (19c) determines both the shape and the magnitude of Ω_S .

Another possible solution to Eq. (18) emerges when the P and C pulses have the same time dependence—say, $f(t)$, $\Omega_P(t) = \Omega_{P0}f(t)$, and $\Omega_C(t) = \Omega_{C0}f(t)$ —while the S pulse could differ, $\Omega_S(t) = \Omega_{S0}g(t)$. Then $\dot{\theta} = 0$ and condition (18) becomes $\tilde{\Omega}_S = 0$. This condition can be satisfied in several ways; cf. Eq. (16b). A simple realization for that condition occurs with the choice

$$\Delta_2 = \Delta_3, \quad (20a)$$

$$\phi_P = -\phi_S, \quad (20b)$$

$$\Omega_C = \Omega_P. \quad (20c)$$

Then $\theta = \pi/4$ and the spectator state reads

$$\tilde{\psi}_3 = \frac{1}{\sqrt{2}}(e^{-i\phi_S}\psi_2 - \psi_3). \quad (21)$$

2. Analytical three-state solutions

The dynamics of the two coupled Householder states ψ_1 and $\tilde{\psi}_2$, coupled by the interaction $\tilde{\Omega}_P$, offers other interesting possibilities. For the two-state system $\psi_1 \leftrightarrow \tilde{\psi}_2$, analytic solutions may be possible; these are known for many examples of pulse and detuning time dependences. Hence, by writing down the propagator in the Householder basis for a known two-state analytical solution and by using the transformation back to the original basis by the Householder reflection $\mathbf{M}(t)$, one can write down a number of analytic solutions for the three-state loop system. These would generalize the similar analytical solutions for a Λ system [22].

3. Population initially in state ψ_1

If only state ψ_1 is initially populated, then the dynamics remains confined within the effective two-state system $\psi_1 \leftrightarrow \tilde{\psi}_2$. In this two-state system we can enforce complete population return to state ψ_1 , complete population inversion to state $\tilde{\psi}_2$, or create a superposition of states ψ_1 and $\tilde{\psi}_2$.

Complete population transfer from state ψ_1 to the Householder state $\tilde{\psi}_2$ can be produced by a resonant π pulse [2], by adiabatic level-crossing adiabatic passage [23], or by a variety of novel, more sophisticated techniques [24–27]. Viewed in the original basis, the system ends up in a superposition of ψ_2 and ψ_3 ,

$$\cos \theta \psi_2 + e^{-i\phi_P} \sin \theta \psi_3, \quad (22)$$

with the angle θ given by (11); thus, the superposition is fully controlled by the ratio of Ω_C and Ω_P and has a relative phase ϕ_P .

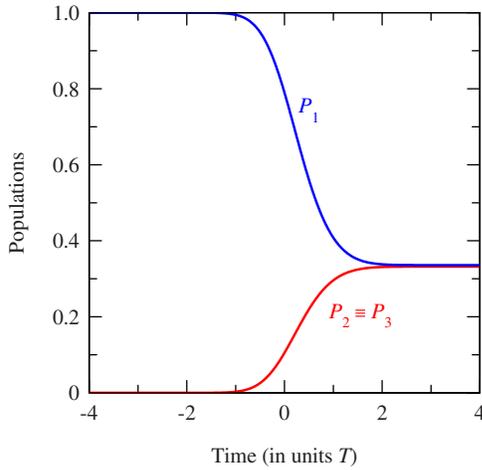


FIG. 3. (Color online) Creation of an equal superposition of states ψ_1 , ψ_2 , and ψ_3 for Gaussian pulses: $\Omega_P(t) = \Omega_{P0} e^{-t^2/T^2}$, $\Omega_C(t) = \Omega_{C0} e^{-t^2/T^2}$, $\Delta_2(t) = \Delta_3(t) = -\Omega_S(t)/2$, and $\Omega_S(t) = \Omega_{S0} e^{-(t-\tau)^2/T^2}$, with the parameters $\Omega_{P0} = \Omega_{C0} = 0.76/T$, $\Omega_{S0} = 1/T$, and $\tau = 0.5T$.

A predetermined *superposition* of states ψ_1 and $\tilde{\psi}_2$ can be created by resonant fractional- π pulses or by modifications of adiabatic-passage techniques—for example, half Stark-chirped rapid adiabatic passage (half-SCRAP) [28] and two-state STIRAP [29]. Such techniques allow, for instance, the creation of an arbitrary predetermined, maximally coherent superposition of the three states ψ_1 , ψ_2 , and ψ_3 . For example, one can create a maximally coherent superposition using fractional- π pulses that obey conditions (20a)–(20c) and which are resonant in the Householder basis: $\Delta_2(t) = \Delta_3(t) = -\Omega_S(t)/2$. Such an example is demonstrated in Fig. 3.

4. Population initially in state ψ_2

Let us assume now that it is state ψ_2 that is populated initially. (The symmetric case of state ψ_3 initially populated is just a matter of relabeling the states.) If the C pulse precedes the P pulse, then we are in the dark state (17c) and this is a situation similar to STIRAP; then, there will occur complete population transfer to state ψ_3 . The resonant case of this process was discussed and explained earlier [10,11]. If we are in state ψ_2 and we apply the pulses in the intuitive order (the P pulse precedes the C pulse), then we are in the bright state and depending on the pulses we can have complete population transfer to state ψ_1 or end up in a superposition of states ψ_1 , ψ_2 , and ψ_3 .

B. Effective two-photon resonance

We assume now that the P and C pulses have the same time dependence and consider a resonance condition between states ψ_1 and $\tilde{\psi}_3$,

$$\tilde{\Delta}_3 = 0. \quad (23)$$

The resulting Householder Hamiltonian is exactly that of the Λ linkage on the two-photon resonance used for STIRAP

[15]. The traditional dark state of the STIRAP process appears here as

$$\begin{aligned} \Phi_D &= \cos \varphi \psi_1 - \sin \varphi \tilde{\psi}_3 \\ &= \cos \varphi \psi_1 - e^{i\phi_P} \sin \varphi \sin \theta \psi_2 + \sin \varphi \cos \theta \psi_3, \end{aligned} \quad (24)$$

where $\tan \varphi = \tilde{\Omega}_P / \tilde{\Omega}_S$. The state Φ_D is a spectator state because it is not affected by the specified radiation, but it has components of all three original basis states. One can use this kind of spectator state, with the traditional STIRAP pulse sequence of $\tilde{\Omega}_S$ preceding $\tilde{\Omega}_P$, to move the initial population from state ψ_1 to a superposition of state $\tilde{\psi}_2$ and state $\tilde{\psi}_3$. The superposition is controlled by the ratio of Ω_C and Ω_P and has the phase ϕ_P .

Condition (23) can always be satisfied for an appropriate choice of the (time-dependent) detuning $\Delta_2(t)$ [or $\Delta_3(t)$]. However, the specific time dependence, although possible in principle, might be complicated and difficult to produce experimentally.

Condition (23) can be satisfied with constant detunings when the P and C pulses share the same time dependence: $\Omega_P(t) = \Omega_{P0}f(t)$ and $\Omega_C(t) = \Omega_{C0}f(t)$. Then the mixing angle θ is constant [see Eq. (11)] and $\dot{\theta} = 0$. Two options provide the needed pulses.

(i) The conditions

$$\phi_P + \phi_S = \pi/2 \quad (25a)$$

and

$$\Delta_3 = -\Delta_2 \frac{\Omega_C^2}{\Omega_P^2} \quad (25b)$$

hold. Then the S field can be arbitrary and both detunings Δ_2 and Δ_3 can be constant.

(ii) The S field is constant. Then condition (23) can be fulfilled for constant detunings that obey the relation

$$\Delta_3 = -\Delta_2 \frac{\Omega_C^2}{\Omega_P^2} + \frac{\Omega_C}{\Omega_P} \Omega_S \cos(\phi_P + \phi_S). \quad (26)$$

Therefore, the usual two-photon resonance condition, necessary for the emergence of a spectator (dark) state in the original basis, is replaced by a condition for the detuning Δ_3 : either (i) Eq. (25b), for arbitrary S field, but with the phase relation (25a), or (ii) Eq. (26), for constant S field.

If we now start initially in state ψ_1 and apply the S pulse before the P pulse, then the following superposition is formed:

$$-e^{i\phi_P} \sin \theta \psi_2 + \cos \theta \psi_3. \quad (27)$$

The superposition characteristics are fixed by the S -field phase and the angle θ defined by Eq. (11). Figure 4 illustrates how, starting in state ψ_1 and applying the S pulse before the P pulse, we obtain an equal superposition of ψ_2 and ψ_3 .

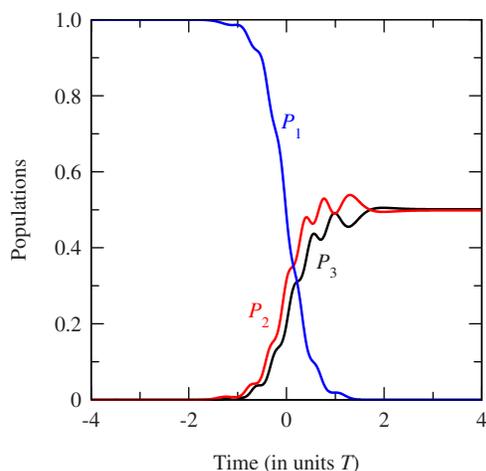


FIG. 4. (Color online) Creation of an equal superposition of states ψ_2 and ψ_3 , with the couplings and detunings $\Omega_p(t) = \Omega_0 \cos \theta e^{-(t-\tau)^2/T^2}$, $\Omega_c(t) = \Omega_0 \sin \theta e^{-(t-\tau)^2/T^2}$, $\Omega_s(t) = \Omega_0 e^{-(t+\tau)^2/T^2}$, and $\Delta_2(t) = \Delta_3(t) = 0$, where the parameters are $\theta = \pi/4$, $\Omega_0 = 30/T$, and $\tau = 0.5T$.

V. REDUCTION OF AN ARBITRARY N -DIMENSIONAL QUANTUM SYSTEM TO A CHAIN

The Householder transformation introduced here for a three-state loop system is readily extended to a general N -state quantum system with arbitrary linkages, even in the most general case when each state connects to any other state. A suitable sequence of at most $N-2$ Householder transformations can cast the Hamiltonian, which is a Hermitian matrix, into a *tridiagonal* form. In this sequence the Householder vector for the n th reflection is chosen as

$$|v_n\rangle = \frac{|x_n\rangle - |x_n\rangle|e_{n+1}\rangle}{\| |x_n\rangle - |x_n\rangle|e_{n+1}\rangle \|}. \quad (28)$$

Here $|e_{n+1}\rangle$ is a unit vector that defines the $(n+1)$ st axis—i.e., its components are zero except for being unity at the $(n+1)$ st place—and $|x_n\rangle$ is the n th column vector of the transformed Hamiltonian after the n th step.

The tridiagonalization of the Hamiltonian implies that in the Householder basis, each of the basis states is connected only to its nearest-neighbor states, thus forming a *chainwise* linkage pattern. The chain is conceptually simpler, and analytically easier, to treat, with a variety of exact and approximate approaches available in the literature.

VI. CONCLUSIONS AND OUTLOOK

A two-parameter Householder reflection can break the loop-linkage pattern of a three-state system, providing instead a simple chain. For the three-state system the result can appear either as a Λ system (with initial population at one end of the chain) or as a V system (with initial population in the middle state of the chain). In either case the system can be transformed further into a pair of coupled states and a spectator state, within which population remains trapped. This is a different kind of spectator state involving all three basis states; it contrasts with the conventional dark states that have no excited-state component.

These results hold intrinsic interest because the three-state loop is the simplest discrete-state quantum system in which nontrivial interference occurs. The present solutions may therefore offer opportunities for manipulating the quantum states of such systems.

Our objective in this paper has been to introduce this important transformation and with it to show that a loop system is equivalent to a chain system. The examples presented of the uses of the Householder transformation in a three-state loop system, being by no means exhaustive, have indicated a number of potential applications based on analytical approaches, ranging from hidden chain breaking and spectator states to hidden two-photon resonances and analytic solutions. These allow one to establish generic features of the interaction dynamics and engineer interactions that, for instance, can produce various superposition states at will.

The results in this work for three-state systems are readily extended to N -state systems. More general sequences of Householder reflections can replace their arbitrary complicated linkages with simple chain linkages. Hence an N -state system wherein each state is coupled to any other state can be reduced to an equivalent chain system with nearest-neighbor interactions only. In so doing, the Householder reflection produces effective Rabi frequencies and detunings, i.e., it alters both the off-diagonal and diagonal elements of the Hamiltonian. Then one can apply various available analytical approaches to the Householder chain to reveal interesting features of the multistate dynamics, including Hilbert-space factorization, hidden spectator states, and ensuing dynamical invariants.

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