Quantum-state decorrelation

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We address the general problem of removing correlations from quantum states while preserving local quantum information as much as possible. We provide a complete solution in the case of two qubits by evaluating the minimum amount of noise that is necessary to decorrelate covariant sets of bipartite states. We show that two harmonic oscillators in an arbitrary Gaussian state can be decorrelated by a Gaussian covariant map. Finally, for finite-dimensional Hilbert spaces, we prove that states obtained from most cloning channels (e.g., universal and phase-covariant cloning) can be decorrelated only at the expense of a complete erasure of information about the copied state. More generally, in finite dimension, cloning without correlations is impossible for continuous sets of states. On the contrary, for continuous variables cloning, a slight modification of the customary setup for cloning coherent states allows one to obtain clones without correlations.

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I. INTRODUCTION

The processing of quantum information is subjected to a number of restrictions imposed by the laws of quantum mechanics, which forbid basic tasks as state cloning [1], or the universal-NOT gate [2]. Such limitations, however, are sometimes proved useful for applications, e.g., the no-cloning theorem, which is at the core of quantum cryptography, since it prevents an eavesdropper from creating perfect copies of a transmitted quantum state. Moreover, the study of these no-go theorems allows us to broaden our understanding of quantum mechanics itself.

In a recent paper [3] we have posed the following question: "Is there any intrinsic limitation in removing correlations between quantum systems?" We are interested in the possibility of decorrelating quantum states—namely, removing any quantum and classical correlation—nontrivially, while keeping some local information encoded on each system. Notice that, although extensive studies have been carried out on the separability problem, in order to distinguish classical correlation from entanglement, very little was known before on the problem of decorrelatability of quantum states. Linearity of quantum mechanics forbids exact decorrelation of an unknown density matrix [4], i.e., there exists no quantum channel that can map an unknown multipartite quantum state to the tensor product of its local reduced density matrices. What about then unfaithful decorrelation that allows some additional noise on the output decorrelated local states, and what about if the input state is not completely unknown, i.e., it is drawn from a smaller set of states, such as a set with some symmetry? Other questions that are naturally raised are: how decorrelatable are the states from optimal universal cloning? Is it possible to approximately clone without correlating the copies? Is the infinite-dimensional case (continuous variables) analogous to the finite-dimensional one (qudits)? In Ref. [3] we answered these questions.

The following facts about cloning and state estimation motivate further the interest in the problem of quantum state decorrelation. We know that quantum information cannot be copied or broadcast exactly, due to the no-cloning theorem. Nevertheless, one can find approximate optimal cloning channels which increase the number of copies of a state at the expense of the quality. In the presence of noise, however, (i.e., when transmitting "mixed" states), it can happen that we are able to increase the number of copies without loosing the quality, if we start with sufficiently many identical originals. Indeed, it is even possible to purify in such a broadcasting process—the so-called superbroadcasting [5,6]. Clearly, a larger number of copies cannot increase the available information about the original input state, and this is due to the fact that the final copies are not statistically independent, and the correlations between them limit the extractable information [7]. It is now natural to ask if we can remove such correlations and make the output systems independent again. Clearly, such quantum decorrelation cannot be achieved exactly, otherwise we would increase the information on the state. *A priori* it is not excluded, however, that it is possible to decorrelate clones at the expense of introducing some additional noise—such that state estimation fidelity after decorrelation is not greater than before. One of the results of this paper is that clones obtained by most cloning machines (e.g., universal, covariant) cannot be decorrelated even within this relaxed condition (see Sec. V). This also implies that the nonincreasing of distinguishability of states is not in general a sufficient condition for decorrelatability. Apart from this negative result, we will provide examples of sets of states for which decorrelation is possible, and calculate the optimal local noise that needs to be added to achieve the task.

After review and further discussing the results of Ref. [3] with a thorough derivation, we present new general results on the state-decorrelation problem. We will prove that for qudits uncorrelated cloning is impossible, even probabilistically, for any set of states containing a finite arch of states of the form $|\phi\rangle=\sqrt{p}|0\rangle+\sqrt{1-p}e^{i\phi}|1\rangle$, with $\langle 1|0\rangle=0$. On the other hand, we will show that, quite surprisingly, this no-go theorem does not hold for continuous variables. In fact, we will show that we can make uncorrelated cloning with a slight modification of the customary setup for cloning coherent states.

The paper is organized as follows. In Sec. II we review the general problem of optimal state decorrelation. In Sec. III we show the general structure of the quantum channels that erase correlations for covariant sets of states, when both different and identical signals are encoded on the local states of a multipartite density matrix. In Sec. IV the theory is specialized to the case of two qubits with detailed derivation of the results, and the special form of the set of decorrelatable states is obtained. In Sec. V we give the proof that approximate cloning without correlations for continuous sets of qudit states is impossible. The case of continuous variables is reviewed in Sec. VI, where we show that an arbitrary set of bipartite Gaussian states can be decorrelated in a covariant way with respect to the group of displacement operators, i.e., independently of the coherent signal. Moreover, we show that it is possible to realize continuous variable cloning without correlation between the copies. Section VII is devoted to the conclusions and discussion of open problems.

II. THE PROBLEM OF OPTIMAL DECORRELATION

We say that a quantum channel $\mathcal{D}$ decorrelates exactly an N-partite state ρ if the following equation holds:

$$\mathcal{D}(\rho) = [\rho]_1 \otimes \cdots \otimes [\rho]_N, \tag{1}$$

where $[\rho]_i$ is the local state of the ith party, which is given by the reduced density matrix of ρ

$$[\rho]_i := \mathrm{Tr}_{1,\ldots,i-1,i+1,\ldots,N}[\rho]. \tag{2}$$

The problem of state-decorrelatability is the following: given a set of states S, we ask whether there exists a quantum channel $\mathcal{D}$ that satisfies Eq. (1) for every state $\rho \in \mathsf{S}$. As for the no-cloning theorem, the answer will strongly depend on the set of states S. In particular, if the set S consists of only one element ρ, then the problem of decorrelatability is trivial (one considers the channel producing $\otimes_{i=1}^{N}[\rho]_i$ for all input states). On the other hand, if S is the set of all possible density matrices, decorrelation is forbidden by linearity of quantum mechanics [4]. A stronger conclusion immediately follows [3]: if S contains the states ρ', ρ'' and their convex combination $\lambda\rho' + (1-\lambda)\rho''$, and at least two of their respective one particle reductions are different, then exact decorrelatability of S is impossible. Impossibility of exact decorrelatability of some two-state sets can be proved [4] due to increase in state distinguishability (see also Ref. [8] for some results on disentangling rather than decorrelating states). Notice, however, that nonincrease in distinguishability of states is a necessary, but not a sufficient condition for decorrelatability.

The approximate state-decorrelation problem that we want to address here is the decorrelation of an unknown state while preserving as much as possible the features of the local states. More precisely, with an information-theoretical motivation, as in Ref. [3] we will consider the following problem

Problem: Optimal locally faithful decorrelation of symmetric sets of states. Consider a set of states of the form

$$\mathsf{S} = \{\rho_{\boldsymbol{g}}\}, \quad \rho_{\boldsymbol{g}} := U_{\boldsymbol{g}} \rho_{\boldsymbol{e}} U_{\boldsymbol{g}}^{\dagger}, \quad U_{\boldsymbol{g}} := U_{g_1} \otimes \cdots \otimes U_{g_N}, \tag{3}$$

where $\boldsymbol{G}$ is a group, U_g $g \in \boldsymbol{G}$ are unitary operators acting over the Hilbert space H of the local quantum system $\boldsymbol{g} = (g_1, \ldots, g_N) \in \boldsymbol{G}^N$ and the "seed" state $\rho_{\boldsymbol{e}}$ is an N-partite correlated state. Find a channel $\mathcal{D}$ that decorrelates all states in S, namely,

$$\mathcal{D}(\rho) = [\tilde{\rho}]_1 \otimes \cdots \otimes [\tilde{\rho}]_N, \quad \forall \quad \rho \in \mathsf{S} \tag{4}$$

where $[\tilde{\rho}]_i$ is not necessarily equal to the local state $[\rho]_i$, and is optimally locally faithful, i.e., it maximizes the averaged local fidelity

$$\bar{F}[\rho_{\boldsymbol{e}}, \mathcal{D}] = \frac{1}{N}\sum_{i=1}^{N} \int_{\boldsymbol{G}^N} d\boldsymbol{g} F([\rho_{\boldsymbol{g}}]_i, [\mathcal{D}(\rho_{\boldsymbol{g}})]_i), \tag{5}$$

where $d\boldsymbol{g}$ denotes the Haar measure of the group [9].

As a result of the application of channel $\mathcal{D}$, subsystems become perfectly decorrelated, however, at expense of losing some information about local states. The faithfulness of decorrelation will be judged based on the fidelity between input and output local states, averaged over systems and over the group. The seed state $\rho_{\boldsymbol{e}}$ ($\boldsymbol{e}$ denotes the identity element of $\boldsymbol{G}^N$) plays the role of the noisy carrier on which the "signals" $\boldsymbol{g} \in \boldsymbol{G}^N$ are encoded by the unitary modulation $U_{\boldsymbol{g}} = U_{g_1} \otimes \cdots \otimes U_{g_N}$. The unitary operators, being local, do not affect the correlation of the seed state, whence all states of the set have the same correlation. The problem of decorrelation is now to find a channel $\mathcal{D}$ that decorrelates all states of the form (3) while optimally preserving the signal on local states. The word "signal" may suggest a sequence of pieces of information being transmitted: in our case this will correspond to sequels of preparations of states within the ensemble described by $\rho_{\boldsymbol{g}}$. We emphasize that in the present framework we are not dealing with decorrelation of signals, but rather with decorrelation of states carrying them. Hence, there is no contradiction in performing decorrelation and still claiming, e.g., that the encoded signals are identical, e.g., when $g_1 = \cdots = g_N$.

The figure of merit (5) is a natural choice, in consideration of the special form (3) of the set S to be decorrelated as orbit of the seed state ρ under the group $\boldsymbol{G}^N$. Using the fact that $\sqrt{U_{\boldsymbol{g}} \rho U_{\boldsymbol{g}}^{\dagger}} = U_{\boldsymbol{g}} \sqrt{\rho} U_{\boldsymbol{g}}^{\dagger}$, along with the strong concavity of the Uhlmann fidelity, we obtain the following bound:

$$\bar{F}[\rho_{\boldsymbol{e}}, \mathcal{D}] \le \frac{1}{N}\sum_{i=1}^{N} F\left\{[\rho_{\boldsymbol{e}}]_i, \int_{G^N} d\boldsymbol{g} U_{g_i}^{\dagger} [\mathcal{D}(\rho_{\boldsymbol{g}})]_i U_{g_i}\right\}. \tag{6}$$

From the last inequality it is clear that the group-averaged map

$$\tilde{\mathcal{D}}(\rho) = \int_{G^N} d\boldsymbol{g} U_{\boldsymbol{g}}^{\dagger} \mathcal{D}(U_{\boldsymbol{g}} \rho U_{\boldsymbol{g}}^{\dagger}) U_{\boldsymbol{g}} \tag{7}$$

has always fidelity greater or equal than that achieved by $\mathcal{D}$. The map $\tilde{\mathcal{D}}$ is covariant under the group $\boldsymbol{G}^N$ (shortly $\boldsymbol{G}^N$ covariant), i.e., for all states ρ it satisfies the identity

$$\widetilde{\mathcal{D}}(U_g \rho U_g^\dagger) = U_g \widetilde{\mathcal{D}}(\rho) U_g^\dagger. \quad (8)$$

Since every $\boldsymbol{G}^N$-covariant map is the group-average of itself, we can restrict the search of the optimal map to covariant maps only.

Notice that for a covariant channel $\mathcal{D}$ it is sufficient to decorrelate only one state of $\mathbf{S}$, since then it will automatically decorrelate all states of the set. Therefore, the problem is reduced to find a $\boldsymbol{G}^N$-covariant map that decorrelates only the seed state (notice that, however, this does not trivialize the problem, since the channel that sends all states to the same fixed decorrelated state is not covariant).

If we have additional constraints on the signals (e.g., we know that they are identical) the set $\mathbf{S}$ becomes smaller and the problem of decorrelation easier. We will also consider this special case of tensor representation $U_g = U_g^{\otimes N}$ of the group $\boldsymbol{G}$, i.e., with all identical signals $g_1 = \cdots = g_N$.

In conclusion of this section we want to comment more about the fidelity figure of merit for the case of qubits. Here the fidelity of two states has a simple expression in terms of their Bloch vector. It is not clear, *a priori*, whether it is possible to have a decorrelating covariant map that increases the length of Bloch vectors of local states (thus decreasing the fidelity). However, as a result of maximizing the fidelity it turns out that the Bloch vector is always shrunk, whence the optimal fidelity corresponds to maximum length of the output local Bloch vector. This optimization will be carried out in detail in the next sections.

III. COVARIANCE CONSTRAINTS

For the same reason that led us to consider only covariant decorrelation channels, we can take the channel as permutationally covariant, namely, for every N party state ρ we have

$$\mathcal{D}(\Pi \rho \Pi^\dagger) = \Pi \mathcal{D}(\rho) \Pi^\dagger, \quad (9)$$

where Π is an arbitrary permutation of subsystems. In the particular case in which $g_1 = \cdots = g_N$, all the signals are equal and we will consider permutationally invariant input states ρ. Correspondingly, we can impose a stronger permutational symmetry on the map, namely, permutational invariance both at the input and at the output, namely,

$$\mathcal{D}(\Pi \rho \Pi^\dagger) = \mathcal{D}(\rho), \quad \Pi \mathcal{D}(\rho) \Pi^\dagger = \mathcal{D}(\rho). \quad (10)$$

A. Structure of covariant channels

Covariance constraints are conveniently expressed using Choi-Jamiołkowski isomorphism. Under this isomorphism a completely positive map $\mathcal{D}$ from $\mathrm{Lin}(\mathbf{H}^{\mathrm{in}})$ to $\mathrm{Lin}(\mathbf{H}^{\mathrm{out}})$ is mapped in a one-to-one way to the positive operator $R_{\mathcal{D}} \in \mathrm{Lin}(\mathbf{H}^{\mathrm{out}} \otimes \mathbf{H}^{\mathrm{in}})$:

$$R_{\mathcal{D}} = \mathrm{Choi}(\mathcal{D}) = \mathcal{D} \otimes \mathcal{I}(|\Psi\rangle\langle\Psi|), \quad (11)$$

where $|\Psi\rangle = \Sigma_i |i\rangle \otimes |i\rangle$ is a maximally entangled vector in $\mathbf{H}^{\mathrm{in}} \otimes \mathbf{H}^{\mathrm{in}}$. The trace-preserving condition of $\mathcal{D}$ implies that

$$\mathrm{Tr}_{\mathrm{out}}(R_{\mathcal{D}}) = \mathbb{1}_{\mathrm{in}}. \quad (12)$$

One can express the state transformation using operator $R_{\mathcal{D}}$ with

$$\mathcal{D}(\rho) = \mathrm{Tr}_{\mathrm{in}}[R_{\mathcal{D}}(\mathbb{1}_{\mathrm{out}} \otimes \rho^T)]. \quad (13)$$

The general covariance condition

$$\mathcal{D}(V_g \rho V_g^\dagger) = W_g \mathcal{D}(\rho) W_g^\dagger, \quad (14)$$

with V_g and W_g unitary representations of a group, translates to the commutation condition for $R_{\mathcal{D}}$

$$[R_{\mathcal{D}}, W_g \otimes V_g] = 0. \quad (15)$$

B. Different signals

Let us consider a covariant operation $\mathcal{D}$ acting on N qubit states fulfilling the covariance condition (8), where $g_i \in \mathrm{SU}(2)$, U_g is the defining representation of SU(2) and we do not impose any additional constraints on g_i. The covariance condition (15) applied to this case reads

$$[R_{\mathscr{D}}, \underbrace{U_{g_1} \otimes \cdots \otimes U_{g_N}}_{\mathcal{H}^{\mathrm{out}}} \otimes \underbrace{U_{g_1}^* \otimes \cdots \otimes U_{g_N}^*}_{\mathcal{H}^{\mathrm{in}}}] = 0. \quad (16)$$

Since for SU(2) group the conjugated representation U_g^* is equivalent to U_g, we may simplify the above condition by introducing the new operator

$$\bar{R}_{\mathcal{D}} = (\mathbb{1}_{\mathcal{H}^{\mathrm{out}}} \otimes \sigma_y^{\otimes N}) R_{\mathcal{D}} (\mathbb{1}_{\mathcal{H}^{\mathrm{out}}} \otimes \sigma_y^{\otimes N}). \quad (17)$$

For this operator the covariance condition no longer involves conjugated representations

$$[\bar{R}_{\mathscr{D}}, \underbrace{U_{g_1} \otimes \cdots \otimes U_{g_N}}_{\mathcal{H}^{\mathrm{out}}} \otimes \underbrace{U_{g_1} \otimes \cdots \otimes U_{g_N}}_{\mathcal{H}^{\mathrm{in}}}] = 0. \quad (18)$$

Evolution of the state can be expressed using $\bar{R}_{\mathcal{D}}$ as follows

$$\mathcal{D}(\rho) = \mathrm{Tr}_{\mathcal{H}^{\mathrm{in}}}[\bar{R}_{\mathcal{D}}(\mathbb{1}_{\mathcal{H}^{\mathrm{out}}} \otimes \bar{\rho})], \quad (19)$$

where $\bar{\rho} = \sigma_y^{\otimes N} \rho^T \sigma_y^{\otimes N}$. We will write the operator $\bar{R}_{\mathcal{D}}$ by changing the order of the Hilbert spaces, such that input and output spaces of the ith qubit stand next to each other, namely,

$$\mathcal{H}_1^{\mathrm{out}} \otimes \cdots \otimes \mathcal{H}_N^{\mathrm{out}} \otimes \mathcal{H}_1^{\mathrm{in}} \otimes \cdots \otimes \mathcal{H}_N^{\mathrm{in}} \rightarrow \mathcal{H}_1^{\mathrm{out}} \otimes \mathcal{H}_1^{\mathrm{in}} \otimes \cdots \otimes \mathcal{H}_N^{\mathrm{out}} \otimes \mathcal{H}_N^{\mathrm{in}}. \quad (20)$$

After this rearrangement the covariance condition takes the form

$$[\bar{R}_{\mathcal{D}}, U_{g_1} \otimes U_{g_1} \otimes \cdots \otimes U_{g_N} \otimes U_{g_N}] = 0, \quad (21)$$

which implies that $\bar{R}_{\mathcal{D}}$ can be expressed in a simple way using projections on two-qubit singlet ($P^{(0)}$) and triplet ($P^{(1)}$) subspaces

$$\bar{R}_{\mathcal{D}} = \sum_{i_1,\ldots,i_N=0}^{1} a_{i_1,\ldots,i_N} P^{(i_1)} \otimes \cdots \otimes P^{(i_N)}, \quad (22)$$

where $a_{i_1,\ldots,i_N}$ are positive coefficients. Additionally, in order to assure permutational covariance of $\mathcal{D}$, coefficients $a_{i_1,\ldots,i_N}$ cannot depend on the order of indices. Then, we can introduce a smaller number of coefficients $q_n := a_{i_1,\ldots,i_N}$, where n

is the number of indices i_k equal to 1. The most general covariant map is thus characterized by $N+1$ non-negative coefficients q_n. Equation (22) becomes then

$$\bar{R}_{\mathcal{D}} = \sum_{n=0}^{N} q_n \left\{ \sum_{\pi \in D_n} \pi (P^{(1)\otimes n} \otimes P^{(0)\otimes N-n}) \pi \right\}, \qquad (23)$$

where D_n is the set of permutation operators π of the N qubits that do not leave $P^{(1)\otimes n} \otimes P^{(0)\otimes N-n}$ invariant. Clearly, the cardinality of D_n is $\binom{N}{n}$. Since one has $\mathrm{Tr}_{\mathcal{H}_i^{\mathrm{out}}}[P^{(0)}]=\frac{1}{2}\mathbb{1}$ and $\mathrm{Tr}_{\mathcal{H}_i^{\mathrm{out}}}[P^{(1)}]=\frac{3}{2}\mathbb{1}$ for $1 \le i \le N$, the trace-preserving condition (12) leads then to the following constraint on the coefficients q_n

$$\sum_{n=0}^{N} \frac{3^n}{2^N} \binom{N}{n} q_n = 1. \qquad (24)$$

Eventually, we have N independent coefficients characterizing covariant transformations. This is the freedom that we have when attempting to decorrelate set of states (3) in a covariant way in the case of different SU(2) signals being encoded. Notice that the above characterization may be simply generalized from qubits to arbitrary d-dimensional systems, by encoding signals via SU(d) defining representation (we do not use the equivalence of U and U*, and $P^{(0)} = \frac{1}{d}|\Psi\rangle\langle\Psi|$ and $P^{(1)}=\mathbb{1}-P^{(0)}$).

C. Identical signals

We now characterize covariant operations in the case of identical signals $g_1 = \cdots = g_N$. This is an especially interesting case due to its relevance for quantum cloning, broadcasting and state estimation problems. In this case, the information about the quantum state (playing the role of the signal) is distributed to many subsystems. The covariance condition (18) for the N qubit transformation in the case of identical signals has form

$$[\overline{R}_{\mathscr{D}}, \underbrace{U_g^{\otimes N}}_{\mathcal{H}^{\mathrm{out}}} \otimes \underbrace{U_g^{\otimes N}}_{\mathcal{H}^{\mathrm{in}}}] = 0. \qquad (25)$$

This is a much weaker condition than Eq. (18), and hence the structure of covariant operations will be significantly richer. Recall that an N-fold tensor product of two-dimensional Hilbert spaces can be decomposed with respect to the action of $U^{\otimes N}$ in the following way:

$$\mathcal{H}^{\otimes N} = \bigoplus_{j=s_N}^{N/2} \mathcal{H}_j \otimes \mathbb{C}^{\kappa_j}, \qquad (26)$$

where $s_N=(N \bmod 2)/2$, $\mathcal{H}_j$ carries an irreducible representation of SU(2) corresponding to the total angular momentum j, and

$$\kappa_j = \frac{2j+1}{N/2+j+1} \binom{N}{N/2+j} \qquad (27)$$

denotes the multiplicity of this representation. To evaluate the operator $\bar{R}_{\mathcal{D}}$ we will decompose the output and input subspaces as follows:

$$\underbrace{\mathcal{H}^{\otimes N}}_{\mathcal{H}^{\mathrm{out}}} \otimes \underbrace{\mathcal{H}^{\otimes N}}_{\mathcal{H}^{\mathrm{in}}} = \bigoplus_{j,l=s_N}^{N/2} \mathcal{H}_j^{\mathrm{out}} \otimes \mathbb{C}_{\mathrm{out}}^{\kappa_j} \otimes \mathcal{H}_l^{\mathrm{in}} \otimes \mathbb{C}_{\mathrm{in}}^{\kappa_l}. \qquad (28)$$

Conveniently, we change the notation order, so that the subspaces are ordered as $\mathcal{H}_j^{\mathrm{out}} \otimes \mathcal{H}_l^{\mathrm{in}} \otimes \mathbb{C}_{\mathrm{out}}^{\kappa_j} \otimes \mathbb{C}_{\mathrm{in}}^{\kappa_l}$, and we have $\mathcal{H}_j^{\mathrm{out}} \otimes \mathcal{H}_l^{\mathrm{in}} = \oplus_{J=|j-l|}^{j+l} \mathcal{H}_J$. We will focus now attention to the simple case of permutationally invariant seed state, and hence permutationally invariant output state. Therefore, without loss of generality, we can limit the optimization to maps with permutationally invariant input and output. It turns out that the irreducible spaces for the permutations of N systems are exactly the multiplicity spaces $\mathbb{C}^{\kappa_j}$ for the irreducible representations of $U^{\otimes N}$. This implies that permutational invariance selects maps of the form

$$\bar{R}_{\mathcal{D}} = \bigoplus_{j,l=s_N}^{N/2} R^{(jl)} \otimes \mathbb{1}_{\kappa_j} \otimes \mathbb{1}_{\kappa_l}. \qquad (29)$$

Finally, it can be easily shown that the covariance condition above together with the permutational invariance leads to the following structure of the operator $\bar{R}_{\mathcal{D}}$ [5]:

$$\bar{R}_{\mathcal{D}} = \bigoplus_{j,l=s_N}^{N/2} \bigoplus_{J=|j-l|}^{j+l} s_{j,l}^{J} P_{j,l}^{(J)} \otimes \mathbb{1}_{\kappa_j} \otimes \mathbb{1}_{\kappa_l}, \qquad (30)$$

where $s_{j,l}^{J}$ are nonnegative coefficients and $P_{j,l}^{(J)} \in \mathrm{Lin}(\mathcal{H}_j^{\mathrm{out}} \otimes \mathcal{H}_l^{\mathrm{in}})$ is a projector on the subspace $\mathcal{H}_J$ with total angular momentum J. The trace-preserving condition is given by [5]

$$\sum_{j=s_N}^{N/2} \sum_{J=|j-l|}^{j+l} \frac{2J+1}{2l+1} \kappa_j s_{j,l}^{J} = 1, \quad \forall \quad l: s_N \le l \le \frac{N}{2}. \qquad (31)$$

Up to the leading order in N, the number of independent parameters $s_{j,l}^{J}$ scales as $N^3/6$, which reflects the fact that covariance condition in the case of identical signals is much weaker than in the case of different ones, where the leading order of the scaling is N.

IV. DECORRELATABILITY OF QUBITS

The problem of decorrelatability of N qubit states can now be stated in a simple way. Without loss of generality we may assume that the single qubit reduced density matrices of the seed state ρ are diagonal in the σ_z eigenbasis, i.e., have the form $\rho_i = \frac{1}{2}(\mathbb{1} + \eta \sigma_z)$, where η is the length of the Bloch vector. Indeed, one has complete freedom in choosing the seed state in the set S, and by local rotations one can always obtain a seed state whose reduced local states are diagonal on the σ_z basis.

Indeed, such a state is always contained in the set S, being obtained from the seed through local unitary transformations. The set of N qubit states is nontrivially decorrelatable in the different [identical] signal scenario if there exist positive parameters q_n [$s_{j,l}^{J}$] satisfying the trace-preserving constraints in Eq. (24) [Eq. (31)], such that the corresponding map generates a product state from the seed ρ, namely,

$$\mathrm{Tr}_{\mathcal{H}^{\mathrm{in}}}[\bar{R}_{\mathcal{D}}(\mathbb{1}_{\mathcal{H}^{\mathrm{out}}} \otimes \bar{\rho})] = \left[\frac{1}{2}(\mathbb{1} + \tilde{\eta}\sigma_z)\right]^{\otimes N}, \quad (32)$$

with $\tilde{\eta} > 0$ ($\tilde{\eta}=0$ would mean a complete loss of information). The maximum achievable $\tilde{\eta}$ is a measure of quality of decorrelation process. The interesting question is now for which kind of seed states decorrelation is possible and for which kind of seed states it is not.

We now present the full solution for the simplest case of two qubits. Consider a couple of qubits A and B. Permutational invariance of the seed state ρ_{AB}, along with the condition that local states are diagonal in the σ_z eigenbasis implies that ρ_{AB} has the form

$$\rho_{AB} = \frac{1}{4}\left(\mathbb{1} + \eta(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) - \sum_{i,j=x,y,z} \lambda_{ij}\sigma_i \otimes \sigma_j\right), \quad (33)$$

with $\lambda_{ij}=\lambda_{ji}$.

A. Different signals

Applying the general results of Sec. III B, we find that a covariant operation $\mathcal{D}$ is parametrized with three parameters q_0, q_1, q_2 [see Eq. (22)] satisfying the trace-preserving condition

$$q_0 + 6q_1 + 9q_2 = 4 \quad (34)$$

and one has

$$\bar{R}_{\mathcal{D}} = q_0 P^{(0)} \otimes P^{(0)} + q_1(P^{(0)} \otimes P^{(1)} + P^{(1)} \otimes P^{(0)}) + q_2 P^{(1)} \otimes P^{(1)}. \quad (35)$$

In order to get a better intuition, we write explicitly the map as follows:

$$\mathcal{D}(\rho_{AB}) = \frac{q_0}{4}\mathcal{D}_0(\rho_{AB}) + \frac{3q_1}{2}\mathcal{D}_1(\rho_{AB}) + \frac{9q_2}{4}\mathcal{D}_2(\rho_{AB}), \quad (36)$$

where $\mathcal{D}_i$ are the trace-preserving maps

$$\mathcal{D}_0(\rho_{AB}) = \rho_{AB}, \quad (37)$$

$$\mathcal{D}_1(\rho_{AB}) = \frac{1}{3}(\rho_A \otimes \mathbb{1} + \mathbb{1} \otimes \rho_B - \rho_{AB}), \quad (38)$$

$$\mathcal{D}_2(\rho_{AB}) = \frac{1}{9}(4\mathbb{1} \otimes \mathbb{1} - 2\rho_A \otimes \mathbb{1} - 2\mathbb{1} \otimes \rho_B + \rho_{AB}). \quad (39)$$

Using the decorrelatability condition (32) and the expression of ρ_{AB} in Eq. (33) we obtain that decorrelation is possible when $\lambda_{ij}=0$ apart from $\lambda_{zz} := \lambda$. Decorrelation then corresponds to the following conditions:

$$q_0 = \frac{1}{4}\left(1 + \frac{6\tilde{\eta}}{\eta} - \frac{9\tilde{\eta}^2}{\lambda}\right), \quad (40)$$

$$q_1 = \frac{1}{4}\left(1 + \frac{2\tilde{\eta}}{\eta} + \frac{3\tilde{\eta}^2}{\lambda}\right), \quad (41)$$

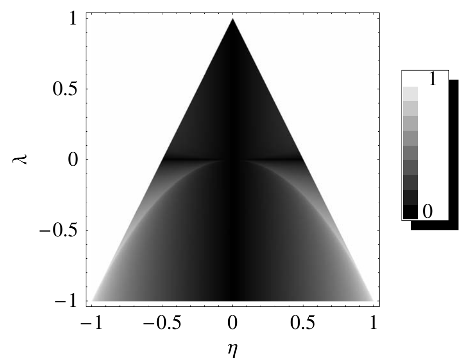

FIG. 1. Length $\tilde{\eta}$ of the Bloch vectors of the decorrelated states of two qubits starting from the joint state in Eq. (43). The plot depicts the maximal achievable $\tilde{\eta}$ in gray scale versus the parameters η and λ of the input state.

$$q_2 = \frac{1}{4}\left(1 - \frac{2\tilde{\eta}}{\eta} - \frac{\tilde{\eta}^2}{\lambda}\right). \quad (42)$$

Analysis of the above equations [together with the trace preserving condition (34)] leads to the following conclusions. Equations are always satisfied for arbitrary seed state ρ for $q_0=\frac{1}{4}$, $q_1=\frac{1}{4}$, $q_2=\frac{1}{4}$. This case is, however, not of much interest since it corresponds to a completely mixing channel resulting in $\tilde{\eta}=0$, and hence destroying all encoded information. We can now write decorrelatable states as in Eq. (33)

$$\rho_{AB} = \frac{1}{4}[\mathbb{1} \otimes \mathbb{1} + \eta(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) - \lambda\sigma_z \otimes \sigma_z], \quad (43)$$

where positivity corresponds to the following conditions:

$$|\eta| \leq 1, \quad \lambda \leq 1 - |\eta|. \quad (44)$$

Notice that all states of the form (43) are separable, by just using the PPT criterion [10]. Finally, to get the optimal decorrelation quality to find which states are decorrelatable we find solutions of Eqs. (40)–(42) with the maximally achievable $\tilde{\eta}$, which is

$$|\tilde{\eta}| = \frac{-\lambda - \sqrt{\eta^2\lambda + \lambda^2}}{|\eta|}, \quad -1 \leq \lambda \leq -\eta^2, \quad (45)$$

$$|\tilde{\eta}| = \frac{-\lambda + \sqrt{\lambda^2 - 3\eta^2\lambda}}{3|\eta|}, \quad -\eta^2 \leq \lambda \leq 0, \quad (46)$$

$$|\tilde{\eta}| = \frac{\lambda + \sqrt{\eta^2\lambda + \lambda^2}}{3|\eta|}, \quad 0 \leq \lambda \leq \eta^2/3, \quad (47)$$

$$|\tilde{\eta}| = \frac{-\lambda + \sqrt{\eta^2\lambda + \lambda^2}}{|\eta|}, \quad \eta^2/3 \leq \lambda \leq 1. \quad (48)$$

This solution is plotted in Fig. 1. The visible parabola in the picture corresponds to the initial states which are already in the product form, i.e.,

$$\rho_{AB} = \left[\frac{1}{2}(1+\eta\sigma_z)\right]^{\otimes 2}$$
$$= \frac{1}{4}[\mathbb{1}\otimes\mathbb{1} + \eta(\sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z) + \eta^2\sigma_z\otimes\sigma_z]. \quad (49)$$

Clearly, such states are trivially decorrelatable, as they are already decorrelated, and $\eta'=\eta$. These states correspond to the case $\lambda=-\eta^2$, and this explains the parabolic structure in the figure.

B. Identical signals

We introduce the following notation to denote bipartite vectors:

$$|A\rangle\rangle := \sum_{m,n=0}^{1} A_{mn}|m\rangle\otimes|n\rangle, \quad (50)$$

where A_{mn} are the matrix elements on the basis $\{|i\rangle\}$ of the operator A. The useful properties of this notation are the following:

$$A\otimes C\,|B\rangle\rangle = |ABC^T\rangle\rangle, \quad \langle\langle A|B\rangle\rangle = \mathrm{Tr}[A^\dagger B]. \quad (51)$$

Consider the situation in which signals encoded on the two qubits are equal. In particular, this is the situation after performing $1\to 2$ universal cloning of qubits, starting form an unknown input state $|\psi\rangle=U|0\rangle$. The optimal cloning operation produces two clones in the state $U\otimes U\rho_{AB}U^\dagger\otimes U^\dagger$, where $\rho_{AB}=\frac{2}{3}|00\rangle\langle 00|+\frac{1}{3}|\psi^+\rangle\langle\psi^+|$ and $|\psi^+\rangle=(1/\sqrt{2})\,|\sigma_x\rangle\rangle$. Notice that this is a correlated state.

We want to know which two-qubit states are decorrelatable and what is the maximal attainable length of the output Bloch vector $\tilde{\eta}$. Since we now impose a weaker covariance condition, we expect decorrelation to succeed for a larger class of states than in the case of independent signals.

Using the general covariance conditions described in Sec. III C, we get a parametrization of covariant operations using six parameters $s^J_{j,l}$, that for convenience we relabel as: $q_0=s^0_{0,0}$, $q_1=s^1_{1,0}$, $q_2=s^1_{0,1}$, $q_3=s^0_{1,1}$, $q_4=s^1_{1,1}$, $q_5=s^2_{1,1}$. The trace-preserving conditions (31) are rewritten as follows:

$$q_0+3q_1=1, \quad q_2+\frac{1}{3}q_3+q_4+\frac{5}{3}q_5=1. \quad (52)$$

The projections $P^J_{j,l}$ can be written as follows using the notation of Eq. (50):

$$P^{(0)}_{00} = \frac{1}{4}\,|\sigma_y\rangle\rangle\langle\langle\sigma_y| \otimes |\sigma_y\rangle\rangle\langle\langle\sigma_y|, \quad (53)$$

$$P^{(1)}_{10} = \frac{1}{2}\left(\mathbb{1}-\frac{1}{2}|\sigma_y\rangle\rangle\langle\langle\sigma_y|\right)\otimes|\sigma_y\rangle\rangle\langle\langle\sigma_y|, \quad (54)$$

$$P^{(1)}_{01} = \frac{1}{2}|\sigma_y\rangle\rangle\langle\langle\sigma_y| \otimes \left(\mathbb{1}-\frac{1}{2}|\sigma_y\rangle\rangle\langle\langle\sigma_y|\right), \quad (55)$$

$$P^{(0)}_{11} = \frac{1}{12}\sum_{i=0,x,z} s(i)\,|\sigma_i\rangle\rangle\,|\sigma_i\rangle\rangle \sum_{j=0,x,z} s(j)\langle\langle\sigma_j|\,\langle\langle\sigma_j|, \quad (56)$$

$$P^{(1)}_{11} = \frac{1}{16}\sum_{i,j=0,x,z}(|\sigma_i\rangle\rangle\,|\sigma_j\rangle\rangle - |\sigma_j\rangle\rangle\,|\sigma_i\rangle\rangle) \times(\langle\langle\sigma_i|\,\langle\langle\sigma_j| - \langle\langle\sigma_j|\,\langle\langle\sigma_i|), \quad (57)$$

$$P^{(2)}_{11} = \frac{1}{16}\sum_{i,j=0,x,z}(|\sigma_i\rangle\rangle\,|\sigma_j\rangle\rangle + |\sigma_j\rangle\rangle\,|\sigma_i\rangle\rangle) \times(\langle\langle\sigma_i|\,\langle\langle\sigma_j| + \langle\langle\sigma_j|\,\langle\langle\sigma_i|) - P^0_{11}, \quad (58)$$

where $\sigma_0:=\mathbb{1}$ and $s(0)=1$ $s(x)=s(z)=-1$. The action on the states ρ_{AB} of Eq. (33) of the (normalized) maps corresponding to each of the operators above is the following:

$$\mathcal{D}^{(0)}_{00}(\rho_{AB}) = P^{(0)}\,\mathrm{Tr}[P^{(0)}\rho_{AB}], \quad (59)$$

$$\mathcal{D}^{(1)}_{10}(\rho_{AB}) = \frac{1}{3}P^{(1)}\,\mathrm{Tr}[P^{(0)}\rho_{AB}], \quad (60)$$

$$\mathcal{D}^{(1)}_{01}(\rho_{AB}) = P^{(0)}\,\mathrm{Tr}[P^{(1)}\rho_{AB}], \quad (61)$$

$$\mathcal{D}^{(0)}_{11}(\rho_{AB}) = P^{(1)}\rho_{AB}P^{(1)}, \quad (62)$$

$$\mathcal{D}^{(1)}_{11}(\rho_{AB}) = \frac{1}{2}(P^{(1)}\,\mathrm{Tr}[P^{(1)}\rho_{AB}] - P^{(1)}\bar{\rho}_{AB}P^{(1)}), \quad (63)$$

$$\mathcal{D}^{(2)}_{11}(\rho_{AB}) = \frac{3}{10}(P^{(1)}\,\mathrm{Tr}[P^{(1)}\rho_{AB}] + P^{(1)}\bar{\rho}_{AB}P^{(1)}) - \frac{1}{5}P^{(1)}\rho_{AB}P^{(1)}, \quad (64)$$

The most general covariant and permutation invariant map is then of the form

$$\mathcal{D}(\rho_{AB}) = q_0\mathcal{D}^0_{00}(\rho_{AB}) + 3q_1\mathcal{D}^1_{10}(\rho_{AB}) + q_2\mathcal{D}^1_{01}(\rho_{AB}) + \frac{q_3}{3}\mathcal{D}^0_{11}(\rho_{AB}) + q_4\mathcal{D}^1_{11}(\rho_{AB}) + \frac{5q_5}{3}\mathcal{D}^2_{11}(\rho_{AB}). \quad (65)$$

We can now write the output state as follows:

$$\mathcal{D}(\rho_{AB}) = \left(\frac{q_3}{3}-\frac{q_4}{2}+\frac{q_5}{6}\right)\rho_{AB} + (q_4-q_5)\left(\frac{\eta}{4}(\sigma_z\otimes\mathbb{1}+\mathbb{1}\otimes\sigma_z)\right) + \frac{1}{4}\left(q_0-\frac{q_3}{3}+\frac{q_4}{2}-\frac{q_5}{6}\right)(1+\Lambda)P^{(0)} + \frac{1}{4}\left(\frac{q_4+q_5}{2}\right)(3-\Lambda)P^{(1)} + \frac{q_2}{4}(3-\Lambda)P^{(0)} + \frac{q_1}{4}(1+\Lambda)P^{(1)}, \quad (66)$$

where $\Lambda=\lambda_{xx}+\lambda_{yy}+\lambda_{zz}$. If we consider the terms in $\sigma_i\otimes\sigma_j$ with $i\neq j$, it is clear that either $\frac{q_3}{3}-\frac{q_4}{2}+\frac{q_5}{6}=0$, or it is impossible to decorrelate the input state. However, the condition $\frac{q_3}{3}-\frac{q_4}{2}+\frac{q_5}{6}=0$ would lead to trivial decorrelation, with total loss of information. We must then have $\lambda_{ij}=0$ for $i\neq j$ at the input state. Moreover, considering that

$$P^{(0)} = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z), \quad (67)$$

and $P^{(1)}=\mathbb{1}-P^{(0)}$, the only term in Eq. (66) containing $\sigma_i^{\otimes 2}$ with possibly different weights is the first one, in order to have $\mathcal{D}(\rho_{AB})$ without terms in $\sigma_x^{\otimes 2}$ or in $\sigma_y^{\otimes 2}$, we must have $\lambda_{xx}=\lambda_{yy}$, namely, decorrelatable states are of the form

$$\rho_{AB} = \frac{1}{4}\left[\mathbb{1} \otimes \mathbb{1} + \eta(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) - \frac{\Lambda - \lambda}{2}(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) - \lambda \sigma_z \otimes \sigma_z\right]. \quad (68)$$

1. Symmetric input state

Let us first restrict to seed states supported on symmetric subspace, i.e., $\mathrm{Tr}[P^{(0)}\rho_{AB}]=\frac{1}{4}(1+\Lambda)=0$ (this set of states contains the states produced by $1\rightarrow 2$ optimal universal cloning machine). The relevant variables in this case are q_3, q_4, and q_5, since q_0 and q_1 do not enter the equations and q_2 is automatically determined by $q_2=(1-\tilde{\eta}^2)/4$.

In terms of the variables η (length of the initial Bloch vector of reduced density matrix) and λ, we can write symmetric decorrelatable states states using Pauli matrices as

$$\rho_{AB}^{\mathrm{sym}} = \frac{1}{4}\{\mathbb{1} \otimes \mathbb{1} + \eta(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + (1+\lambda)/2(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) - \lambda \sigma_z \otimes \sigma_z\}. \quad (69)$$

Starting from Eqs. (32) and (33), we find that a nontrivial solution to the decorrelation problem exists provided that $\eta \neq 0$ and $\lambda \neq -\frac{1}{3}$, and one has

$$q_3 = \frac{1}{12}\left[3 + \tilde{\eta}\left(\tilde{\eta} - \frac{40\tilde{\eta}}{1+3\lambda} + \frac{12}{\eta}\right)\right], \quad (70)$$

$$q_4 = \frac{1}{12}\left[3 + \tilde{\eta}\left(\tilde{\eta} + \frac{20\tilde{\eta}}{1+3\lambda} + \frac{6}{\eta}\right)\right], \quad (71)$$

$$q_5 = \frac{1}{12}\left[3 + \tilde{\eta}\left(\tilde{\eta} - \frac{4\tilde{\eta}}{1+3\lambda} - \frac{6}{\eta}\right)\right]. \quad (72)$$

Looking for the maximal $\tilde{\eta}$ that keeps q_i nonnegative we obtain

$$\tilde{\eta} = \frac{-(1+3\lambda) - \sqrt{(1+3\lambda)^2 + \eta^2[1+(2-3\lambda)\lambda]}}{|\eta|(1-\lambda)}, \quad -1 \le \lambda \le \lambda_1, \quad (73)$$

$$\tilde{\eta} = \frac{-(1+3\lambda) + \sqrt{(1+3\lambda)[1+3\lambda - \eta^2(7+\lambda)]}}{|\eta|(7+\lambda)}, \quad \lambda_1 \le \lambda \le -\frac{1}{3}, \quad (74)$$

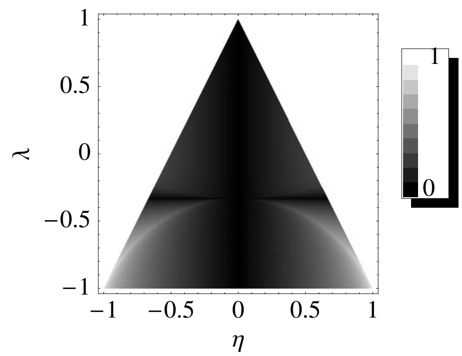

FIG. 2. Length $\tilde{\eta}$ of the Bloch vectors of the decorrelated states of two qubits starting from a seed state supported on the symmetric subspace parametrized as in Eq. (72). The plot depicts the maximal achievable $\tilde{\eta}$ versus the parameters η and λ of the input state.

$$\tilde{\eta} = \frac{2(1+3\lambda) + \sqrt{(1+3\lambda)[\eta^2(13-\lambda) + 4(1+3\lambda)]}}{|\eta|(13-\lambda)}, \quad -\frac{1}{3} \le \lambda \le \lambda_2, \quad (75)$$

$$\tilde{\eta} = \frac{-(1+3)\lambda + \sqrt{(1+3\lambda)^2 + \eta^2[1+(2-3\lambda)\lambda]}}{|\eta|(1-\lambda)}, \quad \lambda_2 \le \lambda \le \lambda_1, \quad (76)$$

where

$$\lambda_1 = \frac{1}{3}(2\sqrt{4-3\eta^2} - 5), \quad \lambda_2 = \frac{1}{3}(7 - 2\sqrt{16-3\eta^2}). \quad (77)$$

See Fig. 2 for visualization of these results. It is worth observing that undecorrelatable states corresponding to $\lambda=-\frac{1}{3}$ are exactly those that can be obtained by a 1-to-2 universal cloning machine. This is a manifestation of a general theorem of no-cloning without correlations proved in Sec. V.

2. Permutationally invariant input state

A general two-qubit state containing also a singlet fraction can be written as

$$\rho_{AB} = p|\Psi^-\rangle\langle\Psi^-| + (1-p)\rho_{AB}^{\mathrm{sym}}. \quad (78)$$

Without writing and analyzing equations which is a bit tedious we just summarize the final results. If either $p=1$, $\lambda=-\frac{1}{3}$, or $\eta=0$, then nontrivial decorrelation is impossible (notice that λ and η are calculated from the symmetric fraction of the state: ρ_{AB}^{sym} in the same way as in the previous subsection). Otherwise, two situations may occur. (i) if $\tilde{\eta}$ evaluated by Eqs. (73)–(76) fulfills the condition $1-\tilde{\eta}^2-4p\geq 0$, then this is a valid maximal achievable length of the output Bloch vector also in the case when the state contains a singlet fraction; (ii) otherwise $\tilde{\eta}$ should be calculated as follows. For $-1\le\lambda\le\lambda_1'$ or $\lambda_2'\le\lambda\le 1$

$$\tilde{\eta} = \frac{\sqrt{\alpha}}{2|\eta|\sqrt{2}}\sqrt{9\alpha + 8\eta^2(1-p) - 3\sqrt{\alpha[9\alpha + 16\eta^2(1-p)]}};$$ (79)

for $\lambda_1' \leq \lambda \leq -\frac{1}{3}$

$$\tilde{\eta} = \frac{\sqrt{\alpha}}{10|\eta|\sqrt{2}}\sqrt{9\alpha - 40\eta^2(1-p) + 3\sqrt{\alpha[9\alpha - 80\eta^2(1-p)]}};$$ (80)

for $-\frac{1}{3} \leq \lambda \leq \lambda_2'$

$$\tilde{\eta} = \frac{\sqrt{\alpha}}{10|\eta|\sqrt{2}}\sqrt{9\alpha + 20\eta^2(1-p) + 3\sqrt{\alpha[9\alpha + 40\eta^2(1-p)]}};$$ (81)

where $\alpha = 1 + 3\lambda$ and

$$\lambda_1' = -\frac{1}{3}[1 + 2\eta^2(1-p)], \quad \lambda_2' = -\frac{1}{3}[1 - \eta^2(1-p)].$$ (82)

One can summarize this by observing (which may not be evident from the above equations) that adding a singlet fraction decreases the achievable $\tilde{\eta}$, but otherwise does not qualitatively change the decorrelatability of states. In particular, the completely nondecorrelatable states are still those that have $\lambda = -1/3$ or $\eta = 0$ in their symmetric fraction.

V. NO APPROXIMATE CLONING WITHOUT CORRELATIONS FOR QUDIT CONTINUOUS SETS OF STATES

In Sec. IV B we noticed that two-qubit states obtained via universal $1 \to 2$ cloning of a single qubit cannot be decorrelated. The same statement holds for clones obtained via phase-covariant $1 \to 2$ cloning. More generally, here we will show that there does not exist an approximate N-to-M cloning channel of d-dimensional systems (qudits) such that the obtained clones are decorrelated, if the cloning channel is to work at least for a phase-set of states. By a phase-set we mean a set containing states of the form

$$|\phi\rangle := \sqrt{p}|0\rangle + \sqrt{1-p}e^{i\phi}|1\rangle,$$ (83)

for some finite continuous range of phases ϕ, where $|0\rangle$, $|1\rangle$ are some orthogonal vectors and p is a real number $0 < p < 1$. Of course, this implies that clones obtained from any cloning machines working for a phase-set of states (such as, e.g., universal, phase covariant, etc.) cannot be decorrelated.

In order to assure the full generality of the proof, we allow cloning to be both asymmetric, and not necessarily covariant. Consider a channel Λ, which acting on N copies of a qudit state produces M ($M > N$) approximate, possibly different clones which are required to be uncorrelated:

$$\Lambda(|\phi\rangle\langle\phi|^{\otimes N}) = \bigotimes_{k=1}^{M} \rho_k^{\phi}.$$ (84)

We will show that such a transformation is impossible, if one requires that every clone ρ_k^{ϕ} carries some (possibly infinitesimally small) information on the identity of the input state $|\phi\rangle$ and additionally that the channel works at least for all states from some phase-set.

Since the channel should work for states from a phase-set, let us consider its action on states $|\phi\rangle = \sqrt{p}|0\rangle + \sqrt{1-p}e^{i\phi}|1\rangle$. Notice that the input product state $|\phi\rangle\langle\phi|^{\otimes N}$ depends on the phase ϕ via linear functions of $e^{in\phi}$, where $n \in \{-N, \ldots, N\}$. Thanks to linearity of Λ, the dependence of the output state $\Lambda(|\phi\rangle\langle\phi|^{\otimes N})$ on ϕ has the same character.

Consider now a map Λ_k which is obtained from the map Λ [Eq. (84)] by tracing out all output qudits except the qudit number k. Its action clearly reads

$$\Lambda_k(|\phi\rangle\langle\phi|^{\otimes N}) = \rho_k^{\phi}.$$ (85)

Since Λ_k is again a channel it follows that ρ_k^{ϕ} may depend on ϕ only via linear functions of $e^{in\phi}$, where again $n \in \{-N, \ldots, N\}$. Notice that since cloning is to preserve some information on the input state, the output state of each clone ρ_k^{ϕ} has to depend on ϕ. Since the matrix of each clone ρ_k^{ϕ} include at least terms $e^{\pm i\phi_j}$ (or possibly higher powers of these), then it follows that $\otimes_{k=1}^{M}\rho_k^{\phi}$ contains entries that depend on ϕ via terms $e^{\pm i\bar{M}\phi}$ where $\bar{M} \geq M > N$.

This leads to a contradiction, since for decorrelation to be successful we would need the equality of a polynomial in $e^{in\phi}$, where $-N < n < N$, with a polynomial containing higher powers (at least M) of $e^{\pm i\phi}$, and this is impossible to hold for a continuous range of parameters ϕ. Hence, approximate cloning with decorrelated clones is impossible for any set of pure states which contains a finite arch of states of the form (83). This no-go theorem clearly can be extended to any set of mixed states containing an arch of the form

$$\rho_{\phi} := U_{\phi}\rho U_{\phi}^{\dagger}.$$ (86)

In fact, an arch of mixed states ρ_{ϕ} can be obtained as $\rho_{\phi} = \mathcal{N}(|\phi\rangle\langle\phi|)$ with $\mathcal{N}$ amplitude-damping channel $\mathcal{N}(\rho) = \alpha\rho + \beta\sigma_z\rho\sigma_z$. Therefore, if a map $\mathcal{D}$ is able to clone an arch of ρ_{ϕ} without correlations, then the map $\mathcal{D} \circ \mathcal{N}$ would do the same for an arch of pure states, which contradicts our previous result. We have then proved that in finite dimension any set of mixed states containing an arch of states of the form (86) cannot be cloned without correlations in any approximate and asymmetric way. This is clearly true, as a special case, for covariant universal cloning, or any other covariant cloning of symmetric sets of input states, for groups containing U(1) as a subgroup. Notice that in our derivation we have used only linearity of the transformation and we have not used the trace preserving condition. This implies that cloning without correlations is impossible also probabilistically.

The present no-cloning-without-correlation result is already quite general, however, it is likely to be of even larger validity. We conjecture that it holds more generally for linearly dependent sets of states. Such conjecture is supported by the fact that linearly independent states can be probabilistically perfectly cloned [11], so if we consider, e.g., N copies of an unknown qubit state, nothing forbids cloning without correlations for $N+1$ different qubit states, since $|\phi\rangle\langle\phi|^{\otimes N}$ will be linear independent states.

VI. DECORRELATION FOR CONTINUOUS VARIABLES

We consider now the case of decorrelation for continuous variables. For a couple of continuous variables in a joint seed state ρ_{AB} the information (α,β) (with α and β complex) is encoded as follows:

$$D(\alpha)\otimes D(\beta)\rho_{AB}D(\alpha)^{\dagger}\otimes D(\beta)^{\dagger}, \tag{87}$$

$D(z)=\exp(za^{\dagger}-z^{*}a)$ for $z\in\mathbb{C}$ denoting a single-mode displacement operator, a and $a^{\dagger}$ being the annihilation and creation operators of the mode. Here we show that it is always possible to decorrelate any joint state of the form (87), with ρ_{AB} representing a two-mode Gaussian state, namely,

$$\rho_{AB}=\frac{1}{\pi^{2}}\int d^{4}\boldsymbol{q}e^{-1/2\boldsymbol{q}^{T}\boldsymbol{M}\boldsymbol{q}}D(\boldsymbol{q}), \tag{88}$$

where $\boldsymbol{q}=(q_1,q_2,q_3,q_4)$, $D(\boldsymbol{q})=D(q_1+iq_2)\otimes D(q_3+iq_4)$, and $\boldsymbol{M}$ is the 4×4 (real, symmetric, and positive) correlation matrix of the state, that satisfies the Heisenberg uncertainty relation [12] $\boldsymbol{M}+\frac{i}{4}\boldsymbol{\Omega}\geq0$, with $\boldsymbol{\Omega}=\oplus_{k=1}^{2}\boldsymbol{\omega}$ and $\boldsymbol{\omega}=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$.

A Gaussian decorrelation channel covariant under $D(\alpha)\otimes D(\beta)$ is given by

$$\mathcal{D}(\rho)=\frac{\sqrt{\det\boldsymbol{G}}}{(2\pi)^{2}}\int d^{4}\boldsymbol{x}e^{-1/2\boldsymbol{x}^{T}\boldsymbol{G}\boldsymbol{x}}D(\boldsymbol{x})\rho D^{\dagger}(\boldsymbol{x}), \tag{89}$$

with positive matrix $\boldsymbol{G}$. For suitable $\boldsymbol{G}$, the resulting state $\mathcal{D}(\rho_{AB})$ is still Gaussian, with a new block-diagonal covariance matrix $\tilde{\boldsymbol{M}}$, thus corresponding to a decorrelated state. In fact, it is easily seen that the map $\mathcal{D}$ is covariant. Using the relation

$$D(\boldsymbol{x})D(\boldsymbol{q})D(\boldsymbol{x})=e^{2i(q_1x_2-q_2x_1+q_3x_4-q_4x_3)}D(\boldsymbol{q}), \tag{90}$$

explicitly one has

$$\mathcal{D}(\rho_{AB})=\frac{\sqrt{\det\boldsymbol{G}}}{(2\pi)^{2}\pi^{2}}\int d^{4}qe^{-1/2\boldsymbol{q}^{T}\boldsymbol{M}\boldsymbol{q}}D(\boldsymbol{q})\times\int d^{4}xe^{-1/2(\boldsymbol{q}\oplus\boldsymbol{x})^{T}\boldsymbol{G}'(\boldsymbol{q}\oplus\boldsymbol{x})}, \tag{91}$$

where $\boldsymbol{G}'$ is the 8×8 block matrix

$$\boldsymbol{G}'=\begin{pmatrix}0&\boldsymbol{\Sigma}^{T}\\\boldsymbol{\Sigma}&\boldsymbol{G}\end{pmatrix}, \tag{92}$$

with

$$\boldsymbol{\Sigma}=\begin{pmatrix}\sigma_y&0\\0&-\sigma_y\end{pmatrix} \tag{93}$$

and σ_y denoting the usual Pauli matrix $\sigma_y=\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)$. Notice also that $\boldsymbol{\Sigma}^{T}=-\boldsymbol{\Sigma}$.

The integral on x in Eq. (91) can be performed, and one obtains

$$\mathcal{D}(\rho_{AB})=\frac{1}{\pi^{2}}\int d^{4}qe^{-1/2\boldsymbol{q}^{T}(\boldsymbol{M}+\boldsymbol{U})\boldsymbol{q}}D(\boldsymbol{q}), \tag{94}$$

where $\boldsymbol{U}=\boldsymbol{\Sigma}\boldsymbol{G}^{-1}\boldsymbol{\Sigma}$. Then, by writing the correlation matrix $\boldsymbol{M}$ of the input seed state in block form, namely,

$$\boldsymbol{M}=\begin{pmatrix}\boldsymbol{A}&\boldsymbol{C}\\\boldsymbol{C}^{T}&\boldsymbol{B}\end{pmatrix} \tag{95}$$

and writing $\boldsymbol{G}^{-1}$ as

$$\boldsymbol{G}^{-1}=\begin{pmatrix}\boldsymbol{W}&\boldsymbol{V}\\\boldsymbol{V}^{T}&\boldsymbol{Z}\end{pmatrix} \tag{96}$$

a decorrelation map is obtained just by taking

$$\boldsymbol{V}=\sigma_y\boldsymbol{C}\sigma_y. \tag{97}$$

Since for physical maps one must have $\boldsymbol{G}^{-1}>0$, then $\boldsymbol{W}$ and $\boldsymbol{Z}$ are subject to constraints. Typically, one will take $\boldsymbol{W}$ and $\boldsymbol{Z}$ such that $\boldsymbol{G}^{-1}>0$ and the added noise is minimal. Since the channel in Eq. (89) is covariant also for $D(\alpha)^{\otimes2}$, notice that the above derivation holds for the case of encoding with the same unitary on both continuous variables as well. In the following we will give two relevant examples of decorrelation maps for Gaussian states.

(1) *Decorrelating twin-beam states*. A special example of Gaussian state of two continuous variables is the so-called "twin beam," which is an entangled state that can be generated in a quantum optical laboratory by parametric down-conversion of vacuum. On the Fock basis $\{|n\rangle\}$, this state can be written as

$$|\psi\rangle=\sqrt{1-\lambda^{2}}\sum_{n=0}^{\infty}\lambda^{n}|n\rangle\otimes|n\rangle, \tag{98}$$

with $0\leq\lambda<1$, and the correlation matrix $\boldsymbol{M}$ for $\rho_{AB}=|\psi\rangle\langle\psi|$ is given by

$$\boldsymbol{M}=\frac{1+\lambda^{2}}{1-\lambda^{2}}\mathbb{1}-\frac{2\lambda}{1-\lambda^{2}}\begin{pmatrix}0&\sigma_z\\\sigma_z&0\end{pmatrix}. \tag{99}$$

For any state in the set (87), the covariant map (89) with

$$\boldsymbol{G}^{-1}=\frac{2\lambda}{1-\lambda^{2}}\left[(1+\epsilon)\mathbb{1}+\begin{pmatrix}0&\sigma_z\\\sigma_z&0\end{pmatrix}\right], \tag{100}$$

and arbitrary $\epsilon>0$, provides two decorrelated states, independently of the signal (α,β). The covariance matrix of the decorrelated seed state is $\tilde{\boldsymbol{M}}=(\frac{1+\lambda}{1-\lambda}+\epsilon')\mathbb{1}$, with $\epsilon'=\frac{2\lambda\epsilon}{1-\lambda^{2}}$, which correspond to two thermal states with mean photon number $\bar{n}=\frac{\lambda}{1-\lambda}+\frac{\epsilon'}{2}$ each.

(2) *Decorrelating classically correlated coherent states*. Coherent states that are classically correlated via a Gaussian function are given by the set (87), where the seed state is written as

$$\rho_{AB}=\int\frac{d^{2}\gamma}{\pi\delta^{2}}e^{-|\gamma|^{2}/\delta^{2}}|\gamma\rangle\langle\gamma|^{\otimes2}, \tag{101}$$

and $|\gamma\rangle$ are coherent states. This seed state can be easily obtained by mixing a thermal state with mean photon number $\bar{n}=2\delta^{2}$ with the vacuum in a 50:50 beam splitter. The corresponding correlation matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M}=(1+2\delta^{2})\mathbb{1}+2\delta^{2}\begin{pmatrix}0&\sigma_z\\\sigma_z&0\end{pmatrix}. \tag{102}$$

A decorrelating map is obtained from Eq. (89) with

$$G^{-1} = 2\delta^2\left[(1+\epsilon)\mathbb{1} - \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}\right], \qquad (103)$$

and arbitrary $\epsilon > 0$. For any state in the set (87), such a covariant map provides two decorrelated states, independently of the signal (α,β). The covariance matrix of the decorrelated seed state is $\tilde{M} = (1+4\delta^2+\epsilon')\mathbb{1}$, with $\epsilon' = 2\delta^2\epsilon$, which correspond to two factorized thermal states with mean photon number $\bar{n} = 2\delta^2 + \frac{\epsilon'}{2}$ each.

Relation with cloning of continuous variables. The striking difference between the qubit and the continuous variables cases is that for qubits only few states can be decorrelated, whereas for continuous variables any joint Gaussian state can be decorrelated. This is due to the fact that the covariance group for qubits comprises all local unitary transformations, whereas for continuous variables includes only local displacements, which is a very small subset of all possible local unitary transformations in infinite dimension. In particular, unlike the case of qudits, it can be shown that states obtained via Gaussian cloning of continuous variables can be decorrelated and the no-go proof valid for finite-dimensional cases does not apply here.

Cloning for continuous variables with minimal added noise can be obtained from N to M copies both for coherent states [13] and mixed states [6] as follows: (1) use a N-port beam splitter which concentrates the signal in one mode and discards the other $N-1$ modes; (2) amplify the signal by a phase-insensitive amplifier with power gain $G = \frac{M}{N}$; (3) distribute the amplified mode by mixing it in an M-port beam splitter with $M-1$ vacuum modes. The noise in each mode a_i is evaluated by the sum of variances $\Delta x_i^2 + \Delta y_i^2$ of conjugated quadratures $x_i = \frac{a_i + a_i^\dagger}{2}$ and $y_i = \frac{a_i - a_i^\dagger}{2i}$. Notice that for Heisenberg relations necessarily one has $\Delta x_i^2 + \Delta y_i^2 \geq \frac{1}{2}$. In the concentration stage the N modes with amplitude $\langle a_i \rangle = \alpha$ and noise $\Delta x_i^2 + \Delta y_i^2 = \gamma_i$ are reduced to a single mode with amplitude $\sqrt{N}\alpha$ and noise $\gamma = \frac{1}{N}\sum_{i=0}^{N-1}\gamma_i$. The amplification stage gives a mode with amplitude $\sqrt{M}\alpha$ and noise $\gamma' = \frac{M}{N}\gamma + \frac{M}{2N} - \frac{1}{2}$. Finally, the distribution stage gives M modes, with amplitude α and noise $\Gamma = \frac{1}{M}(\gamma' + \frac{M-1}{2})$ each. The distribution stage produces highly correlated copies. The correlated clones of coherent states and displaced thermal states can be simply decorrelated as follows. First, apply the inverse transformation of the distribution stage, retaining just the copy with amplitude $\sqrt{M}\alpha$, and then (4) distribute by mixing in an M-port beam splitter with $M-1$ modes in thermal states with noise γ' (corresponding to mean photon number $\bar{n} = \gamma' - \frac{1}{2}$). In such a way, continuous variables clones will be decorrelated. Clearly, the concatenation of stages (1), (2), and (4) gives directly a N-to-M continuous variables covariant cloning without correlation for coherent states and displaced thermal states.

VII. CONCLUSIONS

We addressed the problem of removing correlation from sets of states while preserving as much local quantum information as possible. We reviewed the problem of decorrelation for two qubits and provided sets of decorrelatable states and the minimum amount of noise to be added for decorrelation. In continuous variables, we showed that an arbitrary set of bipartite Gaussian state can be decorrelated in a covariant way with respect to the group of displacement operators, i.e., independently of the coherent signal. The striking difference between the qubit and the continuous variables cases is that for qubits only few states can be decorrelated, whereas for continuous variables any joint Gaussian state can be decorrelated. This is due to the fact that the covariance group for qubits comprises all local unitary transformations, whereas for continuous variables includes only local displacements, which is a very small subset of all possible local unitary transformations in infinite dimension. Indeed, for the same reason decorrelation becomes much easier when considering covariance with respect to unitary transformations of the form $U \otimes U$ (i.e., with the same information encoded on the quantum systems, e.g., the qubit Bloch vectors have the same direction, or the continuous variables are displaced in the same direction), which is actually the case when considering broadcasted states. Covariant decorrelation of this kind for multiple copies gives insight into the problem of how much individual information can be preserved, while all correlations between copies are removed. As a rule of thumb, for covariant sets of states we can say that only a small subset of states can be decorrelated if the set is too large.

We proved that states obtained from universal cloning can only be decorrelated at the expense of a complete erasure of local information (i.e., information about the copied state). More generally, we proved that cloning without correlations among the copies is impossible for sets of qudits that contain phase-set of states. In infinite dimension, on the contrary, we showed that it is possible to realize continuous variable cloning without correlation between the copies, by slightly modifying the setup of the customary cloning of coherent states. Among the open problems for future work, we notice that we did not provide any experimental scheme for covariant decorrelation, even for two qubits. Moreover, in the case of continuous variables, we just gave a covariant channel for decorrelation, without facing the problem of minimizing the noise added to the output decorrelated states. Finally, it would be interesting to prove or disprove our conjecture about discrete set of states, namely, that cloning without correlations is impossible for a linear-dependent set of states.

The problem of removing correlations from sets of states while preserving local information can be seen as the simplest version of a quantum cocktail-party problem [14]. In general, such a problem can be formulated as follows. Assume we have a bipartite quantum system (e.g., two qubits, two quantum modes of electromagnetic field, etc.) initially in a state $|0\rangle \otimes |0\rangle$ (or more generally in some mixed state ρ_{AB}). The signal is encoded using unitary operations $U_A(t)$, $U_B(t)$ acting locally at time t on subsystems A and B, respectively. The communication of quantum signals will amount to sending the states $[U_A(t) \otimes U_B(t)]|0\rangle \otimes |0\rangle$ at different times t, each time rotated by a different pair of unitary matrices $U_A(t)$ and $U_B(t)$, depending on the quantum message intended to be transmitted. After this encoding, the systems pass through the environment which causes the two signals to be mixed in analogy to classical mixing of signals in microphones. This mixing can be represented by a unitary operation V that entangles both systems with the environment state $|E\rangle$ as follows:

$$|\psi(t)\rangle_{ABE} = V(U_A(t) \otimes U_B(t) \otimes I)|0\rangle \otimes |0\rangle \otimes |E\rangle. \tag{104}$$

The analog of the classical cocktail-party problem [15] would be now to determine the "signals" $U_A(t)$ and $U_B(t)$—or the state $[U_A(t) \otimes U_B(t)]|0\rangle \otimes |0\rangle$—from the output state of AB only, without even knowing the interaction with the environment V: this would be a strict quantum analog of blind independent component separation. In this sense we would decorrelate the signals $U_A(t)$ and $U_B(t)$. This quantum version of the cocktail-party problem is much harder than its classical counterpart, for many reasons, including the no-cloning theorem, which forbids one to determine the output state from a single copy: an approximate solution, if possible, would need at least some additional assumptions about the time self-correlation of each separate signal, along with the aid of a quantum memory to store the whole time sequence of output states of AB and a full joint measurement on the whole sequence. We posed in this paper a simpler, but a closely related problem of decorrelating two quantum signals, in the scenario where the signals U_A and U_B are encoded on a correlated state ρ_{AB} as: $U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger$, but no additional mixing operation V is applied. We wanted to decorrelate the received state, and the desired result is two completely uncorrelated systems A and B, each one in a state that carries information about the signals U_A and U_B, respectively.

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