Robustness of entangled states that are positive under partial transposition

Somshubhro Bandyopadhyay*
DIRO, Université de Montréal, Case Postale 6128, Succursale Centre-Ville, Montréal, Québec, Canada H3C 3J7

Sibasish Ghosh†
The Institute of Mathematical Sciences, C. I. T Campus, Taramani, Chennai 600113, India

Vwani Roychowdhury‡
Electrical Engineering Department, UCLA, Los Angeles, California 90095, USA

(Received 19 September 2007; revised manuscript received 7 December 2007; published 12 March 2008)

We study robustness of bipartite entangled states that are positive under partial transposition (PPT). It is shown that almost all PPT entangled states are unconditionally robust, in the sense, both inseparability and positivity are preserved under sufficiently small perturbations in its immediate neighborhood. Such unconditionally robust PPT entangled states lie inside an open PPT entangled ball. We construct examples of such balls whose radii are shown to be finite and can be explicitly calculated. This provides a lower bound on the volume of all PPT entangled states. Multipartite generalization of our constructions is also outlined.

DOI: 10.1103/PhysRevA.77.032318 PACS number(s): 03.67.Mn, 03.65.Ud

I. INTRODUCTION

Robustness of an entangled quantum state quantifies its ability to remain inseparable or entangled in the presence of decoherence—that is, how much noise can be added before the entangled state becomes separable [1–7]. Recently, it was shown that weakly entangled states are dense and robust, and in particular, bound entangled states constructed from an unextendible product basis (UPB) [8] are *conditionally* robust, in the sense that sufficiently small perturbations along certain directions preserve both inseparability and the positivity under partial transposition (PPT) properties [9]. While this is a significant result, robustness of generic bound entangled states [10] (bound entangled states are assumed to be PPT unless otherwise stated), i.e., preservation of their (a) inseparability and (b) positivity under partial transposition, in their immediate neighborhood under sufficiently small perturbation, is not well understood. Consider a bipartite quantum system AB, described by the joint Hilbert space $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, an inseparable PPT density matrix $\rho\in\mathcal{H}$, an arbitrary perturbation of ρ as follows:

$$\rho' = \frac{1}{1+\epsilon}(\rho+\epsilon\sigma), \tag{1}$$

where σ is any other density matrix and $\epsilon>0$ is an infinitesimal noise parameter. We say that ρ is unconditionally robust if and only if it is always inside a PPT ball, that is, for any sufficiently small perturbation along an arbitrary direction the state remains PPT, and inseparable.

The question, whether a given bound entangled state is unconditional robust, is a nontrivial one. If we choose σ in the above equation to be a PPT state, then although PPT property is surely preserved for any choice of ϵ, it does not guarantee that the perturbed state remains inseparable. On the other hand, if σ is chosen to be an entangled state with a nonpositive spectrum under partial transposition (NPT), then it is possible that the perturbed state becomes distillable for any choice of ϵ, thereby losing the PPT property. In fact, such examples have been found, although in a different context [11].

We prove that any PPT entangled state is either inside or on the surface of a closed PPT entangled ball. Thus, almost all PPT entangled states are unconditionally robust, and those on the surface of such balls are conditionally robust. The radius of such a PPT entangled ball may be suitably defined using an appropriate distance measure (trace norm, Bures norm, or Hilbert-Schmidt norm) between the center-of-the-ball state and the states that are on the surface of the ball. A corollary of the above result is that almost all PPT states are unconditionally robust.

We provide examples where the radius of PPT entangled balls, constructed in the neighborhood of bound entangled states from an unextendible product basis [8] (such bound entangled states are denoted by BE-UPB), are shown to be finite and can be explicitly calculated.

Moreover, we show that bound entangled states can also be maximally robust in certain directions. That is, one can mix a bound entangled state with certain product states, such that the mixture remains bound entangled as long as the proportion of the bound entangled state is nonzero.

Finally, we prove that for every BE-UPB state (i.e., an edge BE state [12]), there is a region such that a mixture (the coefficients of such a mixture are bounded) of a BE-UPB state with any separable state is bound entangled inside the region. This may be considered as dual to the result—every PPT entangled state can be expressed as a mixture of a separable state with an edge PPT entangled state—obtained in Ref. [12].

II. BACKGROUND

Consider a bipartite quantum system AB, described by the joint Hilbert space $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$, where dimensions of

*bandyo@iro.umontreal.ca
†sibasish@imsc.res.in
‡vwani@ee.ucla.edu

$\mathcal{H}_A, \mathcal{H}_B$, are d_1, d_2, respectively. Let $\mathcal{D}$ be the set of density matrices of the system AB, and $\mathcal{B}$ be the set of linear operators on $\mathcal{H}$. Thus $\mathcal{D}$ is a convex subset of the $(d_1 d_2)^2$-dimensional space $\mathcal{B}$. Let $\mathcal{S}$ be the set of all separable states. Thus $\mathcal{S}$ is a convex as well as a compact (with respect to the usual metrics such as trace norm or Hilbert-Schmidt norm, etc.) subset of $\mathcal{D}$.

Let $\{|i\rangle_A: i=1,2,\ldots,d_1\}$, $\{|j\rangle_B: j=1,2,\ldots,d_2\}$ be the standard orthonormal basis of $\mathcal{H}_A$, $\mathcal{H}_B$, respectively. The partial transpose ρ^{T_B} of any $\rho \in \mathcal{D}$ (defined with respect to the standard orthonormal product basis $\{|i\rangle_A \otimes |j\rangle_B: i=1,2,\ldots,d_1;\ j=1,2,\ldots,d_2\}$ of $\mathcal{H}$), is given by

$$_B\langle j| \otimes _B\langle i|\rho^{T_B}|i'\rangle_A \otimes |j'\rangle_B \equiv {}_B\langle j'| \otimes _B\langle i|\rho|i'\rangle_A \otimes |j\rangle_B \qquad (2)$$

for all $i, i' \in \{1,2,\ldots,d_1\}$ and for all $j, j' \in \{1,2,\ldots,d_2\}$. Let $\mathcal{P}$ be the set of all elements ρ of $\mathcal{D}$, such that $\rho^{T_B} \geq 0$. Thus $\mathcal{S}$ is a proper subset of $\mathcal{P}$ whenever $d_1 d_2 \geq 8$.

Throughout this paper we will extensively use the theory of entanglement witness. Here we provide a brief review of the pertinent results. We begin with the definition of entanglement witness [13–15] and discuss some of its properties.

Definition 1 (Entanglement witness). An entanglement witness W is a member of $\mathcal{B}$ such that

(i) $W = W^\dagger$;

(ii) $\mathrm{Tr}(W\sigma) \geq 0$ for all $\sigma \in \mathcal{S}$;

(iii) there exists at least one entangled state ρ of AB such that $\mathrm{Tr}(W\rho) < 0$; and

(iv) $\mathrm{Tr}(W) = 1$ [16].

If W is an entanglement witness and ρ is an entangled state such that $\mathrm{Tr}(W\rho) < 0$, then we say W *witnesses* (or *detects*) the entanglement in ρ. For each entanglement witness W, one can write the spectral decomposition as

$$W = \sum_{i=1}^{p} \lambda_i^+ |e_i^+\rangle\langle e_i^+| - \sum_{j=1}^{n} \lambda_j^- |e_j^-\rangle\langle e_j^-|, \qquad (3)$$

where λ_i^+'s are positive eigenvalues of W with corresponding eigenvectors $|e_i^+\rangle$ for $i=1,2,\ldots,p$ (p a positive integer) and $-\lambda_j^-$'s are negative eigenvalues of W with corresponding eigenvectors $|e_j^-\rangle$ for $j=1,2,\ldots,n$ (n a positive integer). Thus $W = W^+ - W^-$ and

$$\mathrm{Tr}(W) = \mathrm{Tr}(W^+) - \mathrm{Tr}(W^-) = \sum_{i=1}^{p} \lambda_i^+ - \sum_{j=1}^{n} \lambda_j^- = 1. \qquad (4)$$

For all density matrices $\pi \in \mathcal{D}$,

$$-\mathrm{Tr}(W^-) \leq \mathrm{Tr}(W\pi) \leq \mathrm{Tr}(W^+). \qquad (5)$$

W^+ is therefore called the *positive* part of W and W^- is called the *negative* part of W. Note that both $p, n \geq 1$ and the n-dimensional subspace spanned by the eigenvectors $|e_j^-\rangle$ for $j=1,2,\ldots,n$, contains no product state.

Lemma 1 [12,14]. Let ρ be any given entangled state in $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\dim \mathcal{H}_A = d_1$, $\dim \mathcal{H}_B = d_2$, and $d_1 d_2 \geq 8$. There exists an entanglement witness W_ρ such that

(i) $\mathrm{Tr}(W_\rho \rho) < 0$, and

(ii) there also exists a separable state σ_ρ such that $\mathrm{Tr}(W_\rho \sigma_\rho) = 0$.

Let ρ be any state of AB, taken from $(\mathcal{P} - \mathcal{S})$. Let $\mathcal{W}_\rho$ be the collection of all entanglement witnesses such that $\mathrm{Tr}(W\rho) < 0$, $W \in \mathcal{W}_\rho$. $\mathcal{W}_\rho$ is a nonempty subset of $\mathcal{B}$ as for each entangled state $\rho \in \mathcal{D}$ there exists at least one entanglement witness [17]. For each $W \in \mathcal{W}_\rho$, let $\mathcal{D}_W$ be the set of all entangled density matrices of AB, whose inseparability is witnessed by W. For two entanglement witnesses $W_1, W_2 \in \mathcal{W}$, W_2 is said to be *finer* than W_1 if D_{W_1} is a subset of D_{W_2}. An element $W_\rho \in \mathcal{W}_\rho$ is an *optimal* entanglement witness [12] for ρ if there is no $W \in \mathcal{W}_\rho$ which is finer than W_ρ.

Definition 2 (Edge state). An element $\delta \in (\mathcal{P} - \mathcal{S})$ is said to be an edge state if there is no product state $|\psi\rangle_A\langle\psi| \otimes |\phi\rangle_B\langle\phi| \in \mathcal{S}$ and there is a positive number ϵ such that $\delta - \epsilon|\psi\rangle_A\langle\psi| \otimes |\phi\rangle_B\langle\phi|$ is a positive operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ or is positive under partial transposition or both [12,13].

It was shown in [12] that for any $\rho \in (\mathcal{P} - \mathcal{S})$, there exists an element $\sigma \in \mathcal{S}$, an edge state $\delta \in (\mathcal{P} - \mathcal{S})$, and a number $\Lambda \in [0,1]$ such that $\rho = \Lambda\sigma + (1-\Lambda)\delta$, and for fixed δ, this representation is optimal (in the sense that one cannot increase Λ by subtracting a nonzero factor of the projector of a product state from δ). Thus by choosing the nearest [18] separable state σ_ρ (of ρ), one can expect to select an edge state $\delta_\rho \in (\mathcal{P} - \mathcal{S})$ such that $\rho = \Lambda_\rho \sigma_\rho + (1-\Lambda_\rho)\delta_\rho$, where Λ_ρ is the largest achievable value. Note that the BE-UPB states [8] are the edge states.

III. RESULTS

This section is arranged as follows: We first introduce the necessary definitions and then prove the results on unconditional robustness of PPT entangled states.

Definition 3 (*Nonempty ball around a density matrix*). For any $\rho \in \mathcal{D}$ and any $\lambda \in (0,1]$, a nonempty ball $B(\rho;\lambda)$ of radius λ around ρ is defined as $B(\rho;\lambda) = \{\mu\rho' + (1-\mu)\rho: \rho' \in \mathcal{D}$ and $0 \leq \mu < \lambda\}$.

Definition 4 (*Neighborhood robustness*). A PPT entangled state $\rho \in \mathcal{D}$ is

(i) *maximally robust* if there exists a member $\sigma \in \mathcal{D}$ such that $x\sigma + (1-x)\rho$ is a PPT entangled state for all $x \in [0,1[$;

(ii) *robust relative* [1] *to* T if there exists a nonempty subset T of $\mathcal{D}$ and an element $z_0 \in]0,1[$ such that the states $z\sigma + (1-z)\rho$ are PPT bound entangled for all $\sigma \in T$ and for all $z \in [0, z_0[$; and

(iii) *unconditionally robust* if there exists a nonempty ball $B(\rho;\lambda)$ containing only PPT entangled states.

Lemma 2. $B(I/D;\ 1/(D-1))$ is a separable ball.

Proof. In Ref. [19] it was shown that $\rho \in \mathcal{D}$ is separable if its purity, i.e., $\mathrm{Tr}(\rho^2)$, is less than $\frac{1}{D-1}$, where $D = d_1 d_2$. Applying this to an arbitrary element $\rho_\mu \equiv \mu\rho' + (1-\mu)\frac{I}{D}$ of $B(I/D;\lambda)$, it follows that ρ_μ is separable if

$$\mathrm{Tr}(\rho_\mu^2) = \frac{1}{D} + \mu^2\left[\mathrm{Tr}(\rho'^2) - \frac{1}{D}\right] < \frac{1}{D-1} \qquad (6)$$

for all elements ρ' of $\mathcal{D}$. Thus $\frac{1}{D} + \mu^2(1 - \frac{1}{D})$ must be less than $\frac{1}{D-1}$, i.e., $\mu < \frac{1}{D-1}$. Hence, every element of the ball $B(I/D;\ 1/(D-1))$ is separable [20]. ■

Definition 5 (*Cut cone*). Let $\rho\in(\mathcal{P}-\mathcal{S})$. Consider the cone $K_S\equiv\{\mu\sigma+(1-\mu)\rho\colon\ 0\leq\mu\leq 1 \text{ and } \sigma\in\mathcal{S}\}$. Let $\lambda\in]0,1]$. Then the set $K_S\cap B(\rho;\lambda)$ is called the cut cone of height λ, with vertex at ρ and is denoted by $K_S(\rho;\lambda)$.

A. Construction of a class of PPT entangled states

Let ρ be any given element of $(\mathcal{P}-\mathcal{S})$. Then from part (i) of Lemma 1, there exists an entanglement witness W_ρ such that $\mathrm{Tr}(W_\rho\rho)=-\lambda_\rho$. Define the following family of states:

$$\mathcal{F}_\rho=\left\{\rho_x\in\mathcal{D}\colon\ \rho_x=x\rho+(1-x)\frac{I}{D}\ \text{ and }\ 0\leq x\leq 1\right\},\tag{7}$$

subset of $\mathcal{P}$. Now,

$$\mathrm{Tr}(W_\rho\rho_x)=\frac{1}{D}-x\left(\frac{1}{D}+\lambda_\rho\right)<0,\quad\forall\ x\in\left]\frac{1}{(1+D\lambda_\rho)},1\right].\tag{8}$$

Thus,

$$\rho_x\in(\mathcal{P}-\mathcal{S})\quad\forall\ x\in\left]\frac{1}{1+D\lambda_\rho},1\right].\tag{9}$$

Consider the following subfamily of $\mathcal{F}_\rho$:

$$\mathcal{F}_\rho^{1/(1+D\lambda_\rho)}=\left\{\rho_x\in\mathcal{F}_\rho\colon\ \frac{1}{1+D\lambda_\rho}<x\leq 1\right\}.\tag{10}$$

Thus all elements of $\mathcal{F}_\rho^{1/(1+D\lambda_\rho)}$ are PPT entangled states.

B. Unconditional robustness

We now select an arbitrary element $\rho_x\in\mathcal{F}_\rho^{1/(1+D\lambda_\rho)}$ and construct the following family of density matrices:

$$\begin{aligned}\mathcal{G}_{\rho,1/(1+D\lambda_\rho)}=\{&\tau(\rho,\sigma,x,y)\in\mathcal{D}\colon\ \tau(\rho,\sigma,x,y)\\&=y\sigma+(1-y)\rho_x,\\&\text{for }\ \rho_x\in\mathcal{F}_\rho^{1/(1+D\lambda_\rho)},\ \ \sigma\in\mathcal{D},\ \ y\in[0,1)\}.\end{aligned}\tag{11}$$

For any $\tau(\rho,\sigma,x,y)\in\mathcal{G}_{\rho,1/(1+D\lambda_\rho)}$, we have

$$\tau(\rho,\sigma,x,y)=[1-s(x,y)]\left\{t(x,y)\sigma+[1-t(x,y)]\frac{I}{D}\right\}+s(x,y)\rho,\tag{12}$$

where

$$s(x,y)=1-x(1-y),\tag{13}$$

$$t(x,y)=y/s(x,y),\tag{14}$$

for $(x,y)\in]1/(1+D\lambda_\rho),1][0,1[$.

The function $t(x,y)$ is well defined only for $(x,y)\in]1/(1+D\lambda_\rho),1[\times[0,1[$, and in that case, the range of $t(x,y)$ is $[0,1[$. Also the range of $s(x,y)$ is $]0,1[$ whenever $(x,y)\in]1/(1+D\lambda_\rho),1[\times[0,1[$. From a result in [19], it follows that the density matrix $t(x,y)\sigma+[1-t(x,y)]\frac{I}{D}$ is separable for all $\sigma\in\mathcal{D}$ provided $t(x,y)<1/(D-1)$, i.e., $y\in[0,(1-x)/(D-1-x)[$ whenever $x\in]1/(1+D\lambda_\rho),1[$. Thus $\tau(\rho,\sigma,x,y)$, given in Eq. (12), is PPT for all $\sigma\in\mathcal{D}$ such that $y\in[0,(1-x)/(D-1-x)[$ whenever $x\in]1/(1+D\lambda_\rho),1[$. Now

$$\begin{aligned}\mathrm{Tr}(W_\rho\tau(\rho,\sigma,x,y))&=y\left\{\mathrm{Tr}(W_\rho\sigma)+\frac{x(1+D\lambda_\rho)-1}{D}\right\}\\&\quad-\frac{x(1+D\lambda_\rho)-1}{D}\\&=y\ \mathrm{Tr}(W_\rho\sigma)-(1-y)\frac{x(1+D\lambda_\rho)-1}{D}.\end{aligned}\tag{15}$$

We have

$$\mathrm{Tr}(W_\rho\sigma)+\frac{x(1+D\lambda_\rho)-1}{D}\leq\frac{1}{p(W_\rho)}\mathrm{Tr}(W_\rho^+)+\frac{x(1+D\lambda_\rho)-1}{D},\tag{16}$$

for all $\sigma\in\mathcal{D}$, where W_ρ^+ is the positive part of W_ρ. Thus, for all $\sigma\in\mathcal{D}$, $\tau(\rho,\sigma,x,y)$ is a PPT entangled state provided $x\in]1/(1+D\lambda_\rho),1[$ and $y\in[0,y_0(x)[$, where

$$y_0(x)=\min\left\{\frac{1-x}{D-1-x},\frac{p(W_\rho)\{x(1+D\lambda_\rho)-1\}}{D\ \mathrm{Tr}(W_\rho^+)+p(W_\rho)\{x(1+D\lambda_\rho)-1\}}\right\}.\tag{17}$$

We can therefore state,

Theorem 1 (*Unconditional robustness*). For any PPT bound entangled state ρ and for each $x\in]1/(1+D\lambda_\rho),1[$, the ball $B(\rho_x;y_0(x))$ contains only PPT entangled states, where $y_0(x)$ is given in Eq. (17); λ_ρ is a positive number where $\mathrm{Tr}(W_\rho\rho)=-\lambda_\rho$.

Remark 2. Each member of the set

$$\tilde{\mathcal{F}}_\rho^{1/(1+D\lambda_\rho)}=(\mathcal{F}_\rho^{1/(1+D\lambda_\rho)}-\{\rho\})\tag{18}$$

is an unconditionally robust PPT entangled state. Also $y_0(x)$ [given in Eq. (17)] provides a lower bound on the maximum size of the ball (containing only PPT bound entangled states) around ρ_x for each $x\in]1/(1+D\lambda_\rho),1[$. The largest range of x can be obtained by taking the maximum possible value of λ_ρ (for example, as given in Lemma 1). However, x cannot be arbitrarily close to 0, as for all such x, ρ_x must be separable [19]. Indeed $\frac{1}{1+D\lambda_\rho}\geq\frac{1}{D-1}$, i.e., $\lambda_\rho\leq\left(1-\frac{2}{D}\right)$. Let us also note that the above result is consistent with the argument presented in [21] that the set of PPT entangled states includes a nonempty ball.

Theorem 2. For every PPT entangled state ρ, there is always a nonempty PPT entangled ball of finite radius in its neighborhood. Thus almost all PPT entangled states are unconditionally robust.

Denoting the ball $B(\rho_x;y_0(x))$ in Theorem 1 as $B(\rho_x;y_{\rm opt}(x))$, where we have assumed that the entanglement witness considered in deriving the value of $y_0(x)$ is an optimal entanglement witness $W_{\rm opt}$ and $\lambda_{\rm opt}=-{\rm Tr}(W_{\rm opt}\rho)$, consider the following nonempty subset of $(\mathcal{P}-\mathcal{S})$:

$$\mathcal{N}_{\rm PPTBE}=\cup_{\rho\in(\mathcal{P}-\mathcal{S})}\cup_{x\in 1/(1+D\lambda_{\rm opt}),1[}B(\rho_x;y_{\rm opt}(x)). \quad (19)$$

It seems that $\mathcal{N}_{\rm PPTBE}$ is a proper subset of $(\mathcal{P}-\mathcal{S})$ as it appears that (in particular) the edge states of $(\mathcal{P}-\mathcal{S})$ should not have unconditional robustness properties.

IV. NEIGHBORHOOD ROBUSTNESS OF BOUND ENTANGLED STATES FROM AN UNEXTENDIBLE PRODUCT BASIS

In what follows we illustrate all the above-mentioned properties considering only bound entangled states generated from an unextendible product basis (UPB) [8], construct a PPT entangled ball whose radius can be explicitly found, and use these results to obtain a lower bound on the volume of PPT entangled states. We further note that entanglement.

A. Bound entangled states from an UPB and entanglement witness [8,15]

We begin with the definition of bound entangled states constructed from an UPB.

Let H be a finite dimensional Hilbert space of the form $H_A\otimes H_B$. For simplicity we assume that $\dim H_A=\dim H_B=d$. Let $S=\{|\omega_i\rangle=|\psi_i^A\rangle\otimes|\varphi_i^B\rangle\}_{i=1}^n$ be an UPB with cardinality $|S|=n$. Let the projector on H_S (the subspace spanned by the UPB), be denoted by $P_S=\Sigma_{i=1}^n|\omega_i\rangle\langle\omega_i|$.

Lemma 3 [8]. Let $P_S^\perp$ be the projector on $H_S^\perp$ (the subspace orthogonal to H_S). Then, the state

$$\Omega=\frac{1}{d^2-n}(I-P_S)=\frac{P_S^\perp}{d^2-n}, \quad (20)$$

where $D=d^2$, is PPT entangled.

The state Ω is the bound entangled state generated from UPB and will be referred to as the BE-UPB state. In [15], the following result was proved.

Lemma 4. Let $S=\{|\omega_i\rangle=|\psi_i^A\rangle\otimes|\varphi_i^B\rangle\}_{i=1}^n$ be an UPB. Then

$$\lambda=\min\sum_{i=1}^n\langle\phi_A\phi_B|\omega_i\rangle\langle\omega_i|\phi_A\phi_B\rangle$$
$$=\min\sum_{i=1}^n|\langle\phi_A|\psi_i^A\rangle|^2|\langle\phi_B|\varphi_i^B\rangle|^2 \quad (21)$$

over all pure states $|\phi_A\rangle\in H_A,|\phi_B\rangle\in H_B$ exists and is strictly larger than 0.

It was also shown in [15] that in many cases where UPB states have considerable symmetry, λ can be explicitly calculated.

One can accordingly define the entanglement witness operator unnormalized, that detects UPB-BE states as follows:

$$W=P_S-\lambda I. \quad (22)$$

First of all note that the operator is Hermitian. Next, for any product state $|\phi_A,\phi_B\rangle\in H$, $\langle\phi_A,\phi_B|W|\phi_A,\phi_B\rangle\geq 0$, where the equality is achieved by the product state for which $\langle\phi_A,\phi_B|P_S|\phi_A,\phi_B\rangle=\lambda$ and from Lemma 4 we know such a product state exists. So, for any convex combination of projectors on these later product states (let σ_Ω be one such convex combination), we have ${\rm Tr}(W\sigma_\Omega)=0$ and for all separable states σ, ${\rm Tr}(W\sigma)\geq 0$. One can trivially check that ${\rm Tr}(W\Omega)=-\lambda<0$. Note that ${\rm Tr}(W)=n-\lambda d^2$, and hence, we must have $\lambda<n/d^2$.

B. PPT entangled balls whose radii can be explicitly calculated and a lower bound on the volume of PPT entangled states

From now on, we shall consider the normalized entanglement witness

$$W_\Omega=\frac{W}{n-\lambda d^2}. \quad (23)$$

The witness operator W_Ω can also detect a large class of other bound entangled states constructed from UPBs and, in particular, the bound entangled states that satisfy the range criterion besides having less than full rank [6].

Notationwise,

$$\lambda_\Omega\equiv-{\rm Tr}(W_\Omega\Omega)=\lambda/(n-\lambda d^2), \quad (24)$$

$$p(W_\Omega)=n, \quad (25)$$

$$\frac{1}{1+D\lambda_\rho}=1-\frac{\lambda d^2}{n}, \quad (26)$$

$${\rm Tr}(W_\Omega^+)=\frac{n(1-\lambda)}{n-\lambda d^2}. \quad (27)$$

Thus all the states

$$\Omega_x=x\Omega+(1-x)(I/d^2)\in\mathcal{F}_\Omega^{(1-\lambda d^2/n)} \quad (28)$$

[see Eq. (10)] are PPT entangled for $x\in\left(1-\frac{\lambda d^2}{n},1\right]$. Now for the following family of states [see Eq. (11)]

$$\tau(\Omega,\sigma,x,y)\equiv y\sigma+(1-y)\Omega_x, \quad (29)$$

we have [using Eq. (17)]

$$y_0(x)=\min\left\{\frac{1-x}{d^2-1-x},1-\frac{(1-\lambda)d^2}{nx+d^2-n}\right\},$$
$$\text{where } x\in\left(1-\frac{\lambda d^2}{n},1\right]. \quad (30)$$

Thus we see that for each PPT-BE state $\Omega_x\in\tilde{\mathcal{F}}_\Omega^{(1-\lambda d^2/n)}$ [see Eq. (18)], there exists a ball $B(\Omega_x;y_0(x))$ that contains only PPT-BE states, where

$$y_0(x) = \begin{cases} \dfrac{1-x}{d^2-1-x} & \text{for all } x \in \left]1-\dfrac{\lambda d^2}{n}, x_0\right[\\ \dfrac{nx-n+\lambda d^2}{nx-n+d^2} & \text{for all } x \in [x_0, 1[\end{cases}, \quad (31)$$

where

$$x_0 = \frac{n(d^2-2)+d^2\{1-\lambda(d^2-1)\}}{n(d^2-2)d^2(1-\lambda)}. \quad (32)$$

Thus $y_0(x)$ in Eq. (31) can be explicitly calculated for those cases of UPB-BE states Ω where λ can be explicitly obtained [15]. We therefore have the following result:

Theorem 3. For any PPT-BE state Ω corresponding to the UPB $S=\{|\omega_i\rangle = |\psi_i^A\rangle \otimes |\varphi_i^B\rangle\}_{i=1}^n$ in $d\otimes d$, the PPT-BE states

$$\Omega_x = x\Omega + (1-x)(I/d^2), \quad (33)$$

where $1-\frac{\lambda d^2}{n} < x < 1$, are unconditionally robust.

Given any nonempty subset $\mathcal{T}$ of $\mathcal{D}$, the *volume* $|\mathcal{T}|$ of $\mathcal{T}$ is defined as the probability of randomly selecting an element of $\mathcal{D}$ from $\mathcal{T}$. From Theorem 5, one can have the following result regarding lower bounds on the volume of PPT-BE states.

Corollary 2. $|(\mathcal{P}-\mathcal{S})| \geq |\mathcal{N}_{\text{PPTBE}}| \geq \max\left\{\left|B(\Omega_x; y_0(x))\right| : 1-\frac{\lambda d^2}{n} < x \leq 1\right\}$, where $y_0(x)$ is given in Eq. (31) and $\mathcal{N}_{\text{PPTBE}}$ is given in Eq. (19).

Remark 3. As a special case of Theorem 2, for every $x \in]1-\frac{\lambda d^2}{n}, 1]$, the PPT-BE state $\Omega_x = x\Omega + (1-x)(I/d^2)$ is maximally robust. In fact, in this scenario, the corresponding separable state σ_Ω is taken as any convex combination of all the product states $|\chi\rangle$ such that $\langle\chi|P_S|\chi\rangle = \lambda$.

Remark 4. As a special case of Theorem 1, the BE-UPB state Ω is robust with respect to $\mathcal{S}$. Since every BE-UPB state is an edge state, this is simply the converse of the fact [12] that every PPT-BE state can be expressed as a mixture of a separable state with an edge PPT BE state.

Theorem 4. For every BE-UPB state, there is an adjacent PPT-BE ball of finite radius, obtained by mixing the BE-UPB state with all possible separable states.

Proof. We focus our attention on the class of states obtained by mixing an UPB-BE state Ω with *any separable state* σ,

$$\sigma_{z,\Omega} = z\sigma + (1-z)\Omega. \quad (34)$$

The state in Eq. (34) is PPT by construction, and is inseparable in the domain $z \in [0, \lambda[$ because $\lambda_\Omega p(W_\Omega)/[\text{Tr}(W_\Omega^+) + \lambda_\Omega p(W_\Omega)] = \lambda$. ■

Remark 6. Robustness of the BE-UPB state Ω, that appears in Theorem 6, can also be extended with respect to the set $\mathcal{S}_\Omega$ of all elements σ of $\mathcal{P}$, where $\text{Tr}(W_\Omega \sigma) \geq 0$. Therefore, the state $z\sigma + (1-z)\Omega$ is a PPT-BE state for all $z \in \left[0, \frac{\lambda_\Omega}{\lambda_\Omega + z_1}\right[$, where $z_1 = \inf\{\text{Tr}(W_\Omega \sigma) : \sigma \in \mathcal{S}_\Omega\}$. Orús and Tarrach [9] have recently shown that for sufficiently small perturbation of any BE-UPB state Ω in $d_1 \otimes d_2$ by a density matrix σ, $\sigma^{T_B} > 0$ on the subspace spanned by the kernel of Ω^{T_B}, the resulting state is PPT.

Remark 7. Numerical methods have already been implemented to obtain entanglement witnesses for other classes of PPT entangled states [22–24]. It is quite possible those witnesses, and the pertinent class of bound entangled states may be used to obtain lower bounds on the volume of the PPT entangled class, and a comparison with our result would be worth studying. However, this is beyond the scope of this work and will be taken up in the future.

V. MULTIPARTITE GENERALIZATION

It is easy to generalize the above results to the case of multipartite entangled states that are PPT across every bipartition. One may consider the set $\mathcal{P}_n$ corresponding to all states ρ of an n-partite system in the Hilbert space $d_1 \otimes d_2 \otimes \cdots \otimes d_n$, where ρ is PPT across every bipartition. Let $\mathcal{S}_n$ be the subset of $\mathcal{P}_n$, where each element of $\mathcal{S}_n$ is fully separable. Thus every $\rho \in (\mathcal{P}_n - \mathcal{S}_n)$ has genuine m-partite entanglement, where $2 \leq m \leq n$. The set $\mathcal{S}_n$ is convex and compact (with respect to some suitable metric). Applying the Hahn-Banach theorem, for each $\rho \in (\mathcal{P}_n - \mathcal{S}_n)$, one can obtain a Hermitian operator W_ρ (acting on $d_1 \otimes d_2 \otimes \cdots \otimes d_n$) such that

(i) $\text{Tr}(W_\rho \sigma) \geq 0$ for all $\sigma \in \mathcal{S}_n$,

(ii) $\text{Tr}(W_\rho \rho) < 0$,

(iii) $\text{Tr}(W_\rho) = 1$, and

(iv) there exists at least one element $\sigma_\rho \in \mathcal{S}_n$ where $\text{Tr}(W_\rho \sigma_\rho) = 0$.

Thus a result analogous to Theorem 3 holds because there exists a separable ball $B(I/(d_1 d_2, \ldots, d_n); \lambda)$ of finite radius $\lambda > 0$, centered around the maximally mixed state $I/(d_1 d_2, \ldots, d_n)$ [25]. The maximal robustness of $\rho \in (\mathcal{P}_n - \mathcal{S}_n)$, in the direction of $\sigma_\rho \in \mathcal{S}_n$ can then be proved in a straightforward manner. Similarly, robustness of ρ with respect to $\mathcal{S}_n$ can also be proved analogous to Theorem 1. The results similar to Lemma 4, Theorem 5, Corollary 2, and Theorem 6 also hold because all completely product pure states in $\mathcal{S}_n$ form a compact set. In this case the quantity $\inf\{\langle\phi|P_S|\phi\rangle : |\phi\rangle\langle\phi| \in \mathcal{S}_n\}$, where P_S is the projector on the subspace spanned by the UPB S, is positive and is attained for some pure state in $\mathcal{S}_n$.

ACKNOWLEDGMENTS

Many thanks to Anne Broadbent and Andrew Scott for careful reading of the manuscript and helpful suggestions. S.G. thanks S. L. Braunstein for pointing out the possibility of generalizing the results of the present paper to multipartite case. S.B. was supported by iCORE, MITACS, CIFAR, and General Dynamics Canada. Most of this work was completed while S.B. was at the Institute for Quantum Information

Science, University of Calgary. Part of this work was completed while S.G. was visiting the Institute for Quantum Information Science at the University of Calgary. Research of S.G. was funded in part by EPSRC Grant No. GR/87406. V.R. was sponsored in part by the Defense Advanced Research Projects Agency (DARPA) project No. MDA972-99-1-0017, and in part by the U.S. Army Research Office/DARPA under Contract or Grant No. DAAD 19-00-1-0172.

[1] G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141 (1999).
[2] M. A. Nielsen and J. Kempe, Phys. Rev. Lett. **86**, 5184 (2001).
[3] A. W. Harrow and M. A. Nielsen, Phys. Rev. A **68**, 012308 (2003).
[4] D. Cavalcanti, Phys. Rev. A **73**, 044302 (2006).
[5] J. F. Du, M. J. Shi, X. Y. Zhou, and R. D. Han, Phys. Lett. A **267**, 244 (2000).
[6] S. Bandyopadhyay, S. Ghosh, and V. Roychowdhury, Phys. Rev. A **71**, 012316 (2005).
[7] M. Steiner, Phys. Rev. A **67**, 054305 (2003); F. G. S. L. Brandao, *ibid.* **76**, 030301(R) (2007); L. Clarisse, e-print arXiv:quant-ph/0612072; G. Adesso, A. Serafini, and F. Illuminati, New J. Phys. **9**, 60 (2007); M. A. Jafarizadeh, M. Mirzaee, and M. Rezaee, Int. J. Quantum Inf. **3**, 511 (2005); D. Cavalcanti and M. O. T. Cunha, Appl. Phys. Lett. **89**, 084102 (2006).
[8] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. **82**, 5385 (1999).
[9] R. Orús and R. Tarrach, Phys. Rev. A **70**, 050101(R) (2004).
[10] P. Horodecki, Phys. Lett. A **232**, 333 (1997); M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **78**, 574 (1997); **80**, 5239 (1998); **82**, 1056 (1999); M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
[11] G. Vidal and J. I. Cirac, Phys. Rev. A **65**, 012323 (2001).
[12] M. Lewenstein, B. Kraus, P. Horodecki, and J. I. Cirac, Phys. Rev. A **63**, 044304 (2001).
[13] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A **62**, 052310 (2000).
[14] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[15] B. M. Terhal, Phys. Lett. A **271**, 319 (2000); Linear Algebr. Appl. **323**, 61 (2001).
[16] We have $\mathrm{Tr}(W)=d_1 d_2\,\mathrm{Tr}\big(W\frac{I}{d_1 d_2}\big)\geq 0$, by condition (i). If $\mathrm{Tr}(W)=0$, we then see that the hyperplane in $\mathcal{B}$, orthogonal to the vector W, *touches* the compact and convex set $\mathcal{S}$ at its point $I/(d_1 d_2)$. Thus, $I/(d_1 d_2)$ turns out to be a point on the boundary of $\mathcal{S}$, and hence, each open neighborhood of $I/(d_1 d_2)$ will always contain at least one inseparable state of AB, which is not true, whatever be the suitable metric we choose (see [26]).
[17] D. Bruß, J. I. Cirac, P. Horodecki, F. Hulpke, B. Kraus, M. Lewenstein, and A. Sanpera, J. Mod. Opt. **49**, 1399 (2002); K. Ekert, O. Gühne, F. Hulpke, P. Hyllus, J. Korbicz, J. Mompart, D. Bruß, M. Lewenstein, and A. Sanpera, in *Quantum Information Processing*, edited by G. Leuchs and T. Beth (Wiley-VCH, Verlag GmbH and Co. KGaA, Weinheim 2003).
[18] The state $\sigma_\rho \in \mathcal{S}$ is "nearest" to the state $\rho \in (\mathcal{P}-\mathcal{B})$ in the sense that the distance $d(\rho,\sigma)$ of ρ from any element σ of $\mathcal{S}$ achieves its minimum at the point σ_ρ.
[19] L. Gurvits and H. Barnum, Phys. Rev. A **66**, 062311 (2002).
[20] Note that this is not true, in general, for an arbitrary element of $\mathcal{S}$. For example, if $|\Psi\rangle\langle\Psi|$ is an entangled state in $\mathcal{D}$, then the (unnormalized) state $\rho=|\psi\rangle_A\langle\psi|\otimes|\psi^*\rangle_B\langle\psi^*|+\eta|\Psi\rangle\langle\Psi|$ is always entangled for all $\eta>0$ (where we have assumed that $d_1\leq d_2$, and for $|\psi\rangle_A=\Sigma_{i=1}^{d_1}\lambda_i|i\rangle_A$, we have $|\psi^*\rangle_B=\Sigma_{i=1}^{d_1}\lambda_i^*|i\rangle_B$). This is because ρ^{T_B} (the partial transposition of ρ with respect to the subsystem B) is not positive semidefinite when $\eta>0$. Hence there does not exist $\lambda>0$ such that the ball $B(|\psi\rangle_A\langle\psi|\otimes|\psi^*\rangle_B\langle\psi^*|;\lambda)$ contains only separable states.
[21] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A **58**, 883 (1998).
[22] F. G. S. L. Brandão, Phys. Rev. A **72**, 022310 (2005).
[23] F. G. S. L. Brandão and R. O. Vianna, Phys. Rev. Lett. **93**, 220503 (2004).
[24] J. Eisert, P. Hyllus, O. Gühne, and M. Curty, Phys. Rev. A **70**, 062317 (2004).
[25] L. Gurvits and H. Barnum, Phys. Rev. A **68**, 042312 (2003).
[26] S. L. Braunstein, C. M. Caves, R. Jozsa, N. Linden, S. Popescu, and R. Schack, Phys. Rev. Lett. **83**, 1054 (1999).