Coulomb corrections to the Delbrück scattering amplitude at low energies

G. G. Kirilin* and I. S. Terekhov†
Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia
(Received 18 October 2007; published 27 March 2008)

In this paper we study the Coulomb corrections to the Delbrück scattering amplitude. We consider the limit when the energy of the photon is much less than the electron mass. The calculations are carried out in the coordinate representation using the exact relativistic Green function of an electron in a Coulomb field. The resulting relative corrections are of the order of a few percent for a large charge of the nucleus. We compare the corrections with the corresponding ones calculated through the dispersion integral of the pair production cross section and also with the magnetic loop contribution to the g factor of a bound electron. The last one is in a good agreement with our results but the corrections calculated through the dispersion relation are not.

DOI: 10.1103/PhysRevA.77.032118 PACS number(s): 12.20.Ds, 32.10.Dk

I. INTRODUCTION

The elastic scattering of photons by an external Coulomb field (so-called Delbrück scattering [1]) is one of the nontrivial predictions of quantum electrodynamics. In the perturbation theory, the Delbrück scattering amplitude starts from the second order in $Z\alpha$ ($Z|e|$ is the charge of the nucleus, $\alpha = e^2 \approx 1/137$ is the fine-structure constant, we set $c=1$, $\hbar = 1$). Significant efforts have been made to calculate this amplitude for the arbitrary scattering angles and energies even in the lowest-order Born approximation. The results of these calculations and the detailed bibliography can be found in Ref. [2].

To calculate the Delbrück scattering amplitude for $Z \gg 1$ it is necessary to take into account Coulomb field exactly. The analytical expression for the amplitude exact in $Z\alpha$ has been derived in Ref. [3] without any additional assumptions, but numerical results have not yet been obtained because this expression is fairly cumbersome. Considerable progress in the calculation of the Coulomb corrections to the lowest-order Born approximation has been achieved for the case of the photon energy ω much larger than the electron mass m_e and small scattering angles $\Delta/\omega \ll 1$ ($\Delta = |\mathbf{k}_1 - \mathbf{k}_2|$, where $\mathbf{k}_1$ and $\mathbf{k}_2$ are the momenta of the photon in the initial and final states, correspondingly) [4–8], or large momentum transfer $\Delta/m_e \gg 1$ [9,10]. It turns out, that the Coulomb corrections strongly decrease the Delbrück amplitude in comparison with the lowest-order Born approximation (the theoretical results and the corresponding experimental data are reviewed in detail in [11,12]). At the moment, the minimal photon energy at which Delbrück scattering is experimentally observed is $\omega = 889$ keV (see Refs. [13–17]), the corresponding energy for the Coulomb corrections is $\omega = 2754$ keV (see Ref. [18]).

In the present paper we have calculated Coulomb corrections to the Delbrück scattering amplitude at low photon energy $\omega \ll m_e$, using integral representation for the electron Green function in a Coulomb field obtained in Ref. [19]. These corrections have not yet been investigated neither experimentally nor theoretically. Nevertheless, they are closely connected with the Coulomb corrections to the pair production cross section due to the dispersion relation [20,21] and with the magnetic-loop contribution to the g factor of a bound electron [22]. We use the Delbrück scattering amplitude at low energy to estimate the Coulomb corrections for both phenomena.

The structure of this paper is as follows: In Sec. II we present the general parametrization of the Delbrück scattering amplitude. In Sec. III, we show that the calculation in the coordinate representation reproduces the result of the lowest-order Born approximation derived in the momentum representation. The results for the Coulomb corrections are given in Sec. IV. We also provide the simple parametrization of their dependence on Z. In Sec. V we compare our results with those obtained via the dispersion relation. The estimated value for the magnetic-loop contribution to the g factor of a bound electron is given in Sec. VI.

II. DELBRÜCK SCATTERING AMPLITUDE

We parametrize the Delbrück scattering amplitude as follows:

$$A = \epsilon_\mu^{(1)} \epsilon_\nu^{\star(2)} \Pi^{\mu\nu}(\omega, \mathbf{k}_1, \mathbf{k}_2, Z), \tag{1}$$

$$\Pi^{\mu\nu}(\omega,\mathbf{k}_1,\mathbf{k}_2,Z) = \frac{\alpha(Z\alpha)^2}{m_e^3}\{f_1(\omega,\mathbf{k}_1,\mathbf{k}_2,Z)(g^{\mu\nu}k_1\cdot k_2 - k_2^\mu k_1^\nu) + f_2(\omega,\mathbf{k}_1,\mathbf{k}_2,Z)[\omega^2 g^{\mu\nu} - \omega(n^\mu k_1^\nu + n^\nu k_2^\mu) + n^\mu n^\nu k_1\cdot k_2]\}, \tag{2}$$

where $k_1 = (\omega, \mathbf{k}_1)$, $k_2 = (\omega, \mathbf{k}_2)$ are the 4-momenta of the photon in the initial and final states, correspondingly, $\epsilon^{(1,2)}$ are the photon polarization vectors, the 4-vector n is defined as $k_1 \cdot n = k_2 \cdot n = \omega$, f_1 and f_2 are the form factors. In order to obtain tensor $\Pi^{\mu\nu}$ we should find two scalar functions $f_{(1,2)}$. Furthermore to find $\Pi^{\mu\nu}$ in the case $\omega, |\mathbf{k}_1|, |\mathbf{k}_2| \ll m_e$ we can perform all calculations for the nonphysical photons $k_1 = (0, \boldsymbol{k}_1)$, $k_2 = (0, \boldsymbol{k}_2)$, and also neglect the dependence of functions $f_{(1,2)}$ from k_1 and k_2. In a pointlike charge approximation (Coulomb field), the polarization tensor $\Pi^{\mu\nu}$ has the following form:

*g.g.kirilin@inp.nsk.su
†i.s.terekhov@inp.nsk.su

$$\Pi^{\mu\nu}(\cdots,Z)=\tilde{\Pi}^{\mu\nu}(\cdots,Z)-\tilde{\Pi}^{\mu\nu}(\cdots,0), \quad (3)$$

$$\tilde{\Pi}^{\mu\nu}(\omega,\mathbf{k}_1,\mathbf{k}_2,Z)=i\alpha\int d^3r_1d^3r_2 \times\exp(i\mathbf{k}_1\cdot\mathbf{r}_1-i\mathbf{k}_2\cdot\mathbf{r}_2)\int_C\frac{d\epsilon}{2\pi} \times\mathrm{Sp}[\gamma^\mu\hat{G}(\mathbf{r}_1,\mathbf{r}_2|\epsilon)\gamma^\nu\hat{G}(\mathbf{r}_2,\mathbf{r}_1|\epsilon-\omega)], \quad (4)$$

where $\hat{G}(\mathbf{r}_1,\mathbf{r}_2|\epsilon)$ is the Green function of an electron in a Coulomb field. The contour of integration over ϵ in the expression (4) goes from $-\infty$ to ∞ so it is below the real axis on the left half-plane and above the real axis on the right half-plane. The Green function has the following form (see Ref. [19]):

$$\hat{G}(\mathbf{r}_1,\mathbf{r}_2|\epsilon)=\sum_{l=1}^{\infty}\int_0^\infty ds\hat{K}(l,s,\mathbf{r}_1,\mathbf{r}_2,\epsilon,Z\alpha), \quad (5)$$

$$\hat{K}(l,s,\mathbf{r}_1,\mathbf{r}_2,\epsilon,Z\alpha)=\frac{-i}{4\pi r_1r_2}\exp[2iZ\alpha s\epsilon+ip(r_1+r_2)\cot(ps) -i\pi\nu]\left[\left(R_+\frac{y}{2}J'_{2\nu}(y)B_l+R_-J_{2\nu}(y)lA_l\right) \times(\gamma_0\epsilon+m)+iZ\alpha\gamma^0[m(\hat{\mathbf{n}}_1+\hat{\mathbf{n}}_2) -pR_+\cot(ps)]J_{2\nu}(y)B_l +\left(ip^2\frac{r_1-r_2}{2\sin^2 ps}(\hat{\mathbf{n}}_1+\hat{\mathbf{n}}_2)B_l-p\cot(ps) \times(\hat{\mathbf{n}}_1-\hat{\mathbf{n}}_2)lA_l\right)J_{2\nu}(y)\right], \quad (6)$$

where

$$R_\pm=1\pm\mathbf{n}_1\cdot\mathbf{n}_2\pm i\boldsymbol{\Sigma}(\mathbf{n}_1\times\mathbf{n}_2),$$

$$\Sigma^k=i\epsilon^{ijk}[\gamma^i,\gamma^j]/4,$$

$$\mathbf{n}_{(1,2)}=\mathbf{r}_{(1,2)}/r_{(1,2)},$$

$$\hat{\mathbf{n}}_{(1,2)}=\boldsymbol{\gamma}\cdot\mathbf{n}_{(1,2)},$$

$$A_l=\frac{d}{dx}[P_l(x)+P_{l-1}(x)],$$

$$B_l=\frac{d}{dx}[P_l(x)-P_{l-1}(x)],$$

$$x=\mathbf{n}_1\cdot\mathbf{n}_2,$$

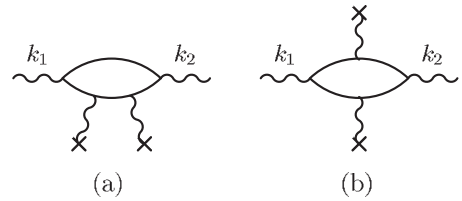

FIG. 1. The lowest-order Born approximation.

$$\nu=\sqrt{l^2-(Z\alpha)^2},\quad y=2p\sqrt{r_1r_2}/\sin ps,\quad p=\sqrt{m^2-\epsilon^2}. \quad (7)$$

Using Eq. (6) we express the polarization tensor (4) as the ninefold integral and twofold sum:

$$\tilde{\Pi}^{\mu\nu}(\omega,\mathbf{k}_1,\mathbf{k}_2,Z)=i\alpha\sum_{l_1=1}^{\infty}\sum_{l_2=1}^{\infty}\int d^3r_1d^3r_2\int_C\frac{d\epsilon}{2\pi} \times\int_0^\infty ds_1ds_2 \times\exp(i\mathbf{k}_1\cdot\mathbf{r}_1-i\mathbf{k}_2\cdot\mathbf{r}_2) \times\mathrm{Sp}[\gamma^\mu\hat{K}(l_1,s_1,\mathbf{r}_1,\mathbf{r}_2,\epsilon,Z\alpha)\gamma^\nu \times\hat{K}(l_2,s_2,\mathbf{r}_2,\mathbf{r}_1,\epsilon-\omega,Z\alpha)]. \quad (8)$$

For the sake of convenience, we calculate the time-time component of the polarization tensor and the trace of the spatial components separately,

$$\Pi^{(ii)}(\mathbf{k},Z)=(n_\mu n_\nu-g_{\mu\nu})\Pi^{\mu\nu}(0,\mathbf{k},\mathbf{k},Z), \quad (9)$$

$$\Pi^{(00)}(\mathbf{k},Z)=n_\mu n_\nu\Pi^{\mu\nu}(0,\mathbf{k},\mathbf{k},Z). \quad (10)$$

Substituting the parametrization of $\Pi^{\mu\nu}$ (See Eq. (2)) on the right-hand side of Eqs. (9) and (10) yields the relation between $\Pi^{(ii)},\Pi^{(00)}$ and the form factors $f_{(1,2)}$,

$$\Pi^{(ii)}(\mathbf{k},Z)=2\frac{\alpha(Z\alpha)^2}{m_e^3}\mathbf{k}^2f_1(0,\mathbf{k},\mathbf{k},Z), \quad (11)$$

$$\Pi^{(00)}(\mathbf{k},Z)=-\frac{\alpha(Z\alpha)^2}{m_e^3}\mathbf{k}^2[f_1(0,\mathbf{k},\mathbf{k},Z)+f_2(0,\mathbf{k},\mathbf{k},Z)]. \quad (12)$$

Let us find the functions $f_{(1,2)}(0,0,0,Z)$ in the lowest-order Born approximation.

III. LOWEST-ORDER BORN APPROXIMATION

The diagrams of the second order of the perturbation theory in $Z\alpha$ are depicted in Fig. 1. Their contribution were calculated in Refs. [23,24]. We aim here to demonstrate that the calculation of these diagrams in the coordinate representation reproduces the result of the lowest-order Born approximation. We shall also restrict our consideration to the low-energy limit of the scattering amplitude. To obtain functions $f_{(1,2)}(0,0,0,Z)$ we expand the exponent function in the expression (4) up to the second order in $|\mathbf{k}_1|=|\mathbf{k}_2|=|\mathbf{k}|$. It is convenient to turn the contour of the integration over ϵ along the imaginary axis. In this case, the contribution of the lowest-order Born approximation takes the form

$$2\Pi^{\mu\nu}_{(a)}+\Pi^{\mu\nu}_{(b)}=\frac{\alpha\mathbf{k}^2}{6}\int d^3r_1d^3r_2|\mathbf{r}_1-\mathbf{r}_2|^2\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\mathrm{Sp}[2\gamma^\mu \times G^{(0)}(\mathbf{r}_1,\mathbf{r}_2|i\epsilon)\gamma^\nu G^{(2)}(\mathbf{r}_2,\mathbf{r}_1|i\epsilon) + \gamma^\mu G^{(1)}(\mathbf{r}_1,\mathbf{r}_2|i\epsilon)\gamma^\nu G^{(1)}(\mathbf{r}_2,\mathbf{r}_1|i\epsilon)], \quad (13)$$

where $G^{(n)}$ is the contribution to the Green function (6) of the nth order in $Z\alpha$,

$$G^{(0)}(\mathbf{r}_1,\mathbf{r}_2|i\epsilon)=-\left((\gamma_0 i\epsilon+m)+i\frac{\hat{\mathbf{r}}_1-\hat{\mathbf{r}}_2}{|\mathbf{r}_1-\mathbf{r}_2|}\frac{\partial}{\partial|\mathbf{r}_1-\mathbf{r}_2|}\right)\frac{\exp(-|\mathbf{r}_1-\mathbf{r}_2|)}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}, \quad (14)$$

$$G^{(1)}(\mathbf{r}_1,\mathbf{r}_2|i\epsilon)=-\frac{Z\alpha}{16\pi}\frac{1-t^2}{r_1r_2}\left[2i\frac{\epsilon\rho}{p^2}\left(\gamma_0 i\epsilon+m+\frac{ip^2}{\rho^2}(\hat{\mathbf{r}}_1-\hat{\mathbf{r}}_2)\frac{\partial}{\sigma\,\partial\sigma}+\frac{ip}{2}(\hat{\mathbf{n}}_1-\hat{\mathbf{n}}_2)\frac{\partial}{\partial\rho}\right)F(\rho,\sigma)+\frac{1}{1-\sigma^2}\gamma^0\left(\hat{R}_+-i\frac{m}{p}(\hat{\mathbf{n}}_1+\hat{\mathbf{n}}_2)\frac{\partial}{\partial\rho}\right)\times\left(\frac{1}{\sigma}e^{-\rho\sigma}-e^{-\rho}\right)\right], \quad (15)$$

$$G^{(2)}(\mathbf{r}_1,\mathbf{r}_2|i\epsilon)=\frac{(Z\alpha)^2}{4\pi r_1r_2}\int_0^\infty ds\,\exp(ip(r_1+r_2)\coth s)\left\{\sum_{l=1}^{\infty}\left[\left(\hat{R}_+B_l\frac{y\,\partial}{2\,\partial y}+\hat{R}_-lA_l\right)(\gamma_0 i\epsilon/p+m/p)-i\frac{p(r_1-r_2)}{2\sinh^2 s}(\hat{\mathbf{n}}_1+\hat{\mathbf{n}}_2)B_l - i\coth s(\hat{\mathbf{n}}_1-\hat{\mathbf{n}}_2)lA_l\right]\left.\frac{\partial}{4l\,\partial\nu}I_{2\nu}(y)\right|_{\nu=l}+\frac{\epsilon s}{2p}\left[\frac{\epsilon s}{2p}\hat{T}+\gamma^0\left(\frac{m}{p}(\hat{\mathbf{n}}_1+\hat{\mathbf{n}}_2)-\hat{R}_+ i\coth s\right)\frac{y}{\cos\frac{\phi}{2}}I_1\left(y\cos\frac{\phi}{2}\right)\right]\right\}, \quad (16)$$

where

$$F(\rho,\sigma)=\frac{1}{2\rho\sigma}\left[e^{-\rho\sigma}\ln\frac{1+\sigma}{1-\sigma}+e^{\rho\sigma}\Gamma(0,\rho(1+\sigma))-e^{-\rho\sigma}\Gamma(0,\rho(1-\sigma))\right], \quad (17)$$

$$\hat{T}=\left(2y^2(\gamma_0 i\epsilon/p+m/p)-iy^2\coth s(\hat{\mathbf{n}}_1-\hat{\mathbf{n}}_2)-i\frac{p(r_1-r_2)}{\sinh^2 s}\frac{(\hat{\mathbf{n}}_1+\hat{\mathbf{n}}_2)}{\cos^2\frac{\phi}{2}}\frac{y\,\partial}{\partial y}\right)I_0\left(y\cos\frac{\phi}{2}\right), \quad (18)$$

where $\Gamma(a,z)=\int_z^\infty t^{a-1}\exp(-t)dt$ is the incomplete Γ function and

$$\rho=p(r_1+r_2),\quad \sigma=\frac{|\mathbf{r}_1-\mathbf{r}_2|}{r_1+r_2}, \quad (19)$$

$$t=\frac{r_1-r_2}{r_1+r_2},\quad \cos\frac{\phi}{2}=\left(\frac{1+x}{2}\right)^{1/2}. \quad (20)$$

The contribution to the Green function $G^{(2)}$ consists of two parts, which are separated by the square brackets in Eq. (16). The first one, which is proportional to $\partial_\nu I_{2\nu}(y)$, arises from the $Z\alpha$ expansion of index of the Bessel functions in Eq. (6). The second one contains $\hat{T}$ and I_1 and corresponds to the zero-order expansion of the Bessel function in $Z\alpha$, i.e., the Bessel function indices 2ν have been replaced to $2l$. After the replacement, the summation over l can be performed analytically. This part can be called conditionally quasiclassical contribution because it corresponds to the contribution of the large angular momenta $l\gg Z\alpha$.

Let us note, that the contribution of each diagram in Fig. 1 is infrared divergent, i.e., it diverges at large distances. Since the divergence is cancelled between the contributions of the diagrams in Figs. 1(a) and 1(b), the contribution of each diagram depends on the regularization of this divergence. However, the contribution of separated terms (for example, the conditionally quasiclassical contribution or the contribution proportional to $\partial_\nu I_{2\nu}$) and even the presence of the divergence depends on the order of the iterated integration over the spatial variables $\mathbf{r}_i$ and the inner variables of the Green functions—"proper times" s_i. An iterated improper integral, the value of which depends on the order of integration, we call as conditionally convergent iterated integral. The example of such integral, that appears during the calculation of the diagram in Fig. 1(b), is given in the Appendix. To avoid the complications due to an explicit regularization and difficulties during the calculation of separated terms, one should fix the order of integration for all diagrams and sum the contribution of each one before the integration with respect to the last variable. It is convenient to change integration variables in Eq. (13) as follows: $(r_1,r_2,x)\to(t,\sigma,\rho)$, the latter are defined in Eqs. (19) and (20). The integration of an arbitrary function $f(\mathbf{r}_1,\mathbf{r}_2)$, such that $f(\mathbf{r}_1,\mathbf{r}_2)=f(\mathbf{r}_2,\mathbf{r}_1)=f(\hat{U}\mathbf{r}_1,\hat{U}\mathbf{r}_2)$, $\hat{U}\in\mathrm{SO}(3)$, takes the form

$$\int d^3r_1 \int d^3r_2 f(\mathbf{r}_1,\mathbf{r}_2) = \frac{2\pi^2}{p^6}\int_0^1 (1-t^2)dt \times \int_t^1 \sigma d\sigma \int_0^\infty \rho^5 d\rho f(\mathbf{r}_1,\mathbf{r}_2). \tag{21}$$

We choose the variable t as the last integration variable. In this case, the contributions of the diagrams (Fig. 1) have the following form:

$$\Pi_{(a)}^{(ii)} = \frac{\alpha(Z\alpha)^2}{m_e^3}\mathbf{k}^2\left(-\frac{5}{2304} - \frac{5}{128}\int_0^1 \frac{dt}{t^2}\right), \tag{22}$$

$$\Pi_{(b)}^{(ii)} = \frac{\alpha(Z\alpha)^2}{m_e^3}\mathbf{k}^2\left(\frac{19}{1152} + \frac{5}{64}\int_0^1 \frac{dt}{t^2}\right), \tag{23}$$

$$\Pi_{\text{Born}}^{(ii)} = 2\Pi_{(a)}^{(ii)} + \Pi_{(b)}^{(ii)} = \frac{\alpha(Z\alpha)^2}{m_e^3}\mathbf{k}^2\frac{7}{576}. \tag{24}$$

The time-time component is derived in a similar manner,

$$\Pi_{\text{Born}}^{(00)} = \frac{\alpha(Z\alpha)^2}{m_e^3}\mathbf{k}^2\frac{59}{2304}. \tag{25}$$

Substitution of the expressions (24) and (25) in (11) and (12) leads to the form factors in the lowest-order Born approximation

$$f_{1B} = \frac{7}{16\cdot 72}, \quad f_{2B} = -\frac{73}{32\cdot 72}. \tag{26}$$

As noticed above, they coincide with the results derived in Refs. [23,24].

IV. COULOMB CORRECTIONS

Analytical derivation of the Coulomb corrections to the lowest-order Born approximation (26) is a rather complicated problem. We have calculated these corrections mostly numerically. To increase the accuracy of the numerical calculations we have subtracted the lowest-order Born approximation (13) from the general expression (4) before any transformations,

$$\Pi_C^{\mu\nu} = \Pi^{\mu\nu} - \Pi_{\text{Born}}^{\mu\nu}. \tag{27}$$

Similar to the lowest-order Born approximation, we split the amplitude into two parts,

$$\Pi_C^{\mu\nu} = \Pi_{\text{quasicl}}^{\mu\nu} + \Pi_{\text{res}}^{\mu\nu}. \tag{28}$$

The conditionally quasiclassical part $\Pi_{\text{quasicl}}^{\mu\nu}$ contains I_{2l} instead of $I_{2\nu}$. The residual part $\Pi_{\text{res}}^{\mu\nu}$ contains all other terms, i.e., the terms of the integrand that contains one of the following expressions:

$$(Z\alpha)^2 I_{2l_1}(I_{2\nu_2} - I_{2l_2}) = O(Z^4\alpha^4), \tag{29}$$

$$(Z\alpha)^2 (I_{2\nu_1} - I_{2l_1}) I_{2l_2} = O(Z^4\alpha^4), \tag{30}$$

$$(I_{2\nu_1} - I_{2l_1})(I_{2\nu_2} - I_{2l_2}) = O(Z^4\alpha^4), \tag{31}$$

$$I_{2l_1}\left(I_{2\nu_2} - I_{2l_2} - \frac{(Z\alpha)^2}{2l_2}\frac{\partial I_{2\nu}}{\partial\nu}\bigg|_{\nu=l_2}\right) = O(Z^4\alpha^4), \tag{32}$$

$$\left(I_{2\nu_1} - I_{2l_1} - \frac{(Z\alpha)^2}{2l_1}\frac{\partial I_{2\nu}}{\partial\nu}\bigg|_{\nu=l_1}\right) I_{2l_2} = O(Z^4\alpha^4). \tag{33}$$

After the subtraction, it is convenient to change variables as follows [see Eq. (8)]: $(r_1,r_2)\to(\eta=\sqrt{r_1r_2}, t'=r_1/r_2)$. Then we integrate analytically over ϵ, x, t' one by one. After that we also perform the analytical summation over l_1. Thus, the expression (4) can be reduced to the sum over l_2 and the iterated integral over s_1, s_2, and η. The explicit expression for the integrand is omitted here as bulky. Further analytical integration is only possible for separate terms. The conditionally quasiclassical contribution can be represented as a onefold integral or as an infinite series over $Z\alpha$. For example, the corresponding contribution to $\Pi^{(00)}$ is the following:

$$\Pi_{\text{quasicl}}^{(00)} = \frac{\alpha(Z\alpha)^2}{m_e^3}\mathbf{k}^2\left(\frac{287\pi^2-39}{18\,432}(Z\alpha)^2 - \frac{\pi^2(49\pi^2-15)}{69120}(Z\alpha)^4 + \frac{\pi^4(158\pi^2-63)}{3\,096\,576}(Z\alpha)^6 - \frac{\pi^6(21\pi^2-10)}{5\,160\,960}(Z\alpha)^8 + \frac{\pi^8(83\,290\pi^2-46\,431)}{245\,248\,819\,200}(Z\alpha)^{10} + \cdots\right). \tag{34}$$

The first two terms in Eq. (34) are dominant and give 97%–96% of the conditionally quasiclassical contribution in spite of the fact that we calculate the complete series in $Z\alpha$.

This contribution is finite, i.e., the infrared divergence is absent in Eq. (34). Nevertheless, the residual part must be integrated in the same order as that used to derive Eq. (34). We examine explicitly that the contributions containing the subtraction from the single Bessel function, i.e., which are proportional to Eq. (32) or Eq. (33), are conditionally convergent iterated integrals.

The complete results of our numerical calculations, i.e., the sum of conditionally quasiclassical contribution and the residual part, are presented in Figs. 2 and 3, and Table I.

It should be noted that the conditionally quasiclassical contribution is of the order of 60% of the lowest-order Born approximation for $Z=80$, but the residual part and the conditionally quasiclassical contribution have opposite signs and almost cancel each other, so that the complete results for the Coulomb corrections are of the order of a few percent of the lowest-order Born approximation. This cancellation adversely affects the accuracy of the calculation. We estimate the accuracy of the results (Table I) to be of the order of 1%.

Now we consider the dependence of the Coulomb corrections on Z. As noted above, the first two terms in the $Z\alpha$ expansion (34) are dominant in the conditionally quasiclassical contribution. It comes as a surprise that the complete results (Table I), which are much smaller than the condition-

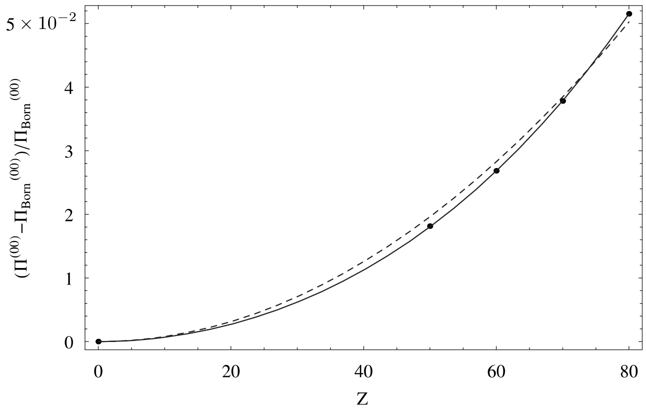

FIG. 2. The relative Coulomb corrections to the trace of the time-time component of the polarization tensor. The dashed curve corresponds to the fit $a(Z\alpha)^2$, the solid line corresponds to $a(Z\alpha)^2+b(Z\alpha)^4$.

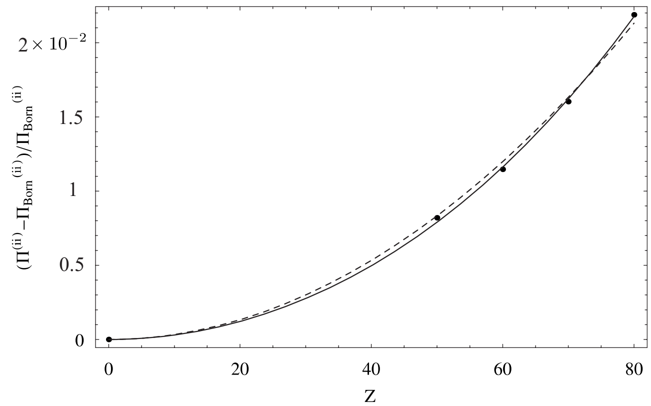

FIG. 3. The relative Coulomb corrections to the trace of the spatial components of the polarization tensor. The dashed curve corresponds to the fit $a(Z\alpha)^2$, the solid line corresponds to $a(Z\alpha)^2+b(Z\alpha)^4$.

ally quasiclassical contribution, can be adequately fitted by a biquadratic polynomial in $Z\alpha$ without a free term,

$$\Pi_C^{(00)}=\frac{\alpha(Z\alpha)^2}{m_e^3}\mathbf{k}^2[(3.22\pm0.01)\times10^{-3}(Z\alpha)^2+(1.90\pm0.02)\times10^{-3}(Z\alpha)^4], \quad (35)$$

$$\Pi_C^{(ii)}=\frac{\alpha(Z\alpha)^2}{m_e^3}\mathbf{k}^2[(6.69\pm0.17)\times10^{-4}(Z\alpha)^2+(3.18\pm0.54)\times10^{-4}(Z\alpha)^4]. \quad (36)$$

The results of the fitting with a quadratic function $a(Z\alpha)^2$ and also the functions in Eqs. (35) and (36) are shown in Figs. 2 and 3. One further comment is in order. The coefficients at $(Z\alpha)^2$ in Eqs. (35) and (36) have a magnitude one or two orders less than those at $(Z\alpha)^0$ in the lowest-order Born approximation, Eqs. (24) and (25). If one assumes the same hierarchy between the coefficients at $(Z\alpha)^2$ and $(Z\alpha)^4$, then the coefficients of $(Z\alpha)^4$ could not be distinguished from zero with our accuracy. In this case, the maximal difference between the dashed and solid curves in Figs. 2 and 3 shows the actual accuracy of our calculations. Substituting Eqs. (35) and (36) in the relations (11) and (12), we obtain the Coulomb corrections to the form factors $f_{(1,2)}$,

$$f_{1C}=3.35\times10^{-4}(Z\alpha)^2+1.6\times10^{-4}(Z\alpha)^4, \quad (37)$$

$$f_{2C}=-3.36\times10^{-3}(Z\alpha)^2-2.1\times10^{-3}(Z\alpha)^4. \quad (38)$$

TABLE I. Relative Coulomb corrections.

Z	$\Pi_C^{(00)}/\Pi_{\rm Born}^{(00)}$	$\Pi_C^{(ii)}/\Pi_{\rm Born}^{(ii)}$
50	1.82×10^{-2}	8.20×10^{-3}
60	2.69×10^{-2}	1.15×10^{-2}
70	3.78×10^{-2}	1.60×10^{-2}
80	5.15×10^{-2}	2.19×10^{-2}

V. PAIR PRODUCTION CROSS SECTION AND DELBRÜCK SCATTERING AMPLITUDE

In Refs. [21,20], Gluckstern and Rohrlich have derived the relation between the pair production cross section in a Coulomb field and the Delbrück amplitude averaged over the polarizations

$$A(\omega)=\frac{\omega^2}{2\pi^2}\int_{2m}^{\infty}\frac{\sigma_{\gamma\to e^+e^-}(\omega')}{\omega'^2-\omega^2+i0}d\omega'. \quad (39)$$

The amplitude (1) averaged over the polarizations of the photon has the form

$$A=\frac{1}{2}\left(\delta^{ij}-\frac{k^ik^j}{\omega^2}\right)\Pi^{ij}=-\frac{\alpha(Z\alpha)^2}{m_e^3}f_2\omega^2. \quad (40)$$

One can find the relation between the Coulomb corrections to the form factor f_2 and the pair production cross section in a Coulomb field by using the expressions (39) and (40) (we set $m_e=1$ in this section),

$$f_2=-\frac{1}{2\pi^2\alpha(Z\alpha)^2}\int_2^{\infty}\frac{\sigma(\omega')}{\omega'^2}d\omega'. \quad (41)$$

Let us check the formula (41) in the Born approximation. Substituting $Z=82$ (lead) yields

$$\frac{1}{2\pi^2\alpha(Z\alpha)^2f_{2B}}\int_2^{\infty}\frac{-\sigma_B(\omega')}{\omega'^2}d\omega'=1+4\times10^{-5}, \quad (42)$$

where σ_B is replaced by the asymptotical formulas derived by Maximon in Ref. [25] for $\omega<2.1$,

$$\sigma_B(\omega)=\alpha(Z\alpha)^2\frac{2\pi}{3}\left(\frac{\omega-2}{\omega}\right)^3\left(1+\frac{\epsilon}{2}+\frac{23}{40}\epsilon^2+\frac{11}{60}\epsilon^3+\frac{29}{960}\epsilon^4+O(\epsilon^5)\right), \quad (43)$$

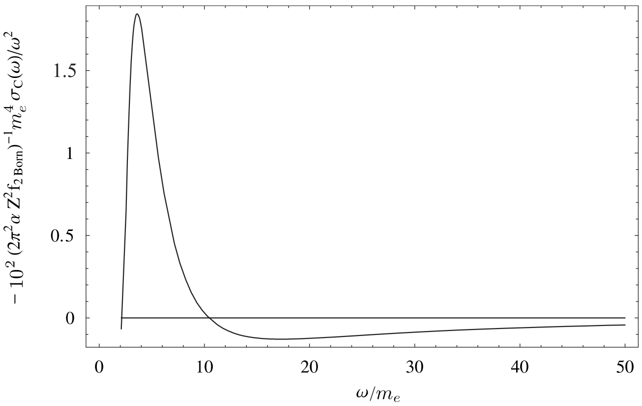

FIG. 4. Coulomb corrections to the pair production cross section (Z=82).

$$\epsilon = \frac{2\omega - 4}{2 + \omega + 2(2\omega)^{1/2}}, \tag{44}$$

for $\omega > 2.1$,

$$\sigma_B(\omega) = \alpha(Z\alpha)^2\Bigg[\frac{28}{9}\ln 2\omega - \frac{218}{27} + \left(\frac{2}{\omega}\right)^2\Bigg(6\ln 2\omega - \frac{7}{2} + \frac{2}{3}\ln^3 2\omega - \ln^2 2\omega - \frac{\pi^2}{3}\ln 2\omega + \frac{\pi^2}{6} + 2\zeta(3)\Bigg) - \left(\frac{2}{\omega}\right)^4\left(\frac{3}{16}\ln 2\omega + \frac{1}{8}\right) - \left(\frac{2}{\omega}\right)^6\left(\frac{29\ln 2\omega}{9\cdot 256} - \frac{77}{27\cdot 512}\right) + O\left(\frac{2^8}{\omega^8}\right)\Bigg]. \tag{45}$$

Now we discuss the Coulomb corrections to the form factors. Using Eq. (38) we have obtained the relative correction to the form factor in the Born approximation (here and below all the calculations are carried out for Z=82),

$$\frac{f_2 - f_{2B}}{f_{2B}} = 4.9 \times 10^{-2}. \tag{46}$$

However, if we use the Coulomb corrections to the pair production cross section $\sigma_C(\omega) = \sigma(\omega) - \sigma_B(\omega)$ derived in Ref. [26] for the photon energy $\omega < 10$ and the interpolation equation derived in Ref. [27] for $\omega > 10$, then the relative correction to the form factor in the Born approximation is

$$-\frac{1}{2\pi^2\alpha(Z\alpha)^2 f_{2B}}\int_{2.01}^{\infty}\frac{\sigma_C(\omega)}{\omega^2}d\omega = 2.7 \times 10^{-3}. \tag{47}$$

This result is 20 times less than that in Eq. (46). The integrand (47) as a function of ω is shown in Fig. 4. It varies mainly in the region $2 < \omega < 30$ but there is a long negative "tail" for $\omega \to \infty$. The total integral is a result of the almost complete cancellation between the positive contribution for $\omega \lesssim 10$ and the negative one for $\omega \gtrsim 10$. The following ratio shows it clearly:

TABLE II. Integration of the $1/\omega$ corrections over the negative "tail" [see Fig. 4 and Eq. (49)].

Contribution in σ_C when $\omega \to \infty$	$-\int_{\omega_0}^{\infty} d\omega\,\sigma_C(\omega)/\omega^2 2\pi^2\alpha(Z\alpha)^2 f_{2B}$
$O(1)$	−0.184
$O(1)+O(1/\omega)$	0.068
$O(1)+O(1/\omega)+O(1/\omega^2)$	−0.062

$$\frac{\int_2^{\infty}\sigma_C(\omega)/\omega^2 d\omega}{\int_2^{\infty}|\sigma_C(\omega)/\omega^2| d\omega} = 3.9 \times 10^{-2}. \tag{48}$$

For the integral (47) to be calculated with the sufficient accuracy it is necessary to derive the Coulomb corrections to the cross section with an accuracy better than a few percent. It is quite possible that this cancellation causes the discrepancy due to the lack of precision in the calculations of the positive part of the integrand in Ref. [26].

The cause of the discrepancy could also be the interpolation equation derived in Ref. [27] (the region $10 \lesssim \omega \lesssim 30$). Another interpolation formula for the Coulomb corrections to the pair production process is derived in Refs. [28,29] up to terms which are of the order of $1/\omega$ and $(1/\omega^2)\ln \omega/2$,

$$\sigma_C(\omega \gg 2) = \alpha(\alpha Z)^2\left[-\frac{28}{9}f(Z\alpha) + \frac{1}{\omega}\left(-\frac{\pi^4}{2}\mathrm{Im}\,g(Z\alpha) - 4\pi(Z\alpha)^3 f_1(Z\alpha)\right) + \frac{b}{\omega^2}\ln\frac{\omega}{2}\right], \tag{49}$$

where the functions f, g, and f_1 are derived analytically but the coefficient b is obtained by an interpolation procedure from the experimental data for $\omega \gtrsim 30$ [30,31]. The absolute value of the approximation formula (49) is always less than the corresponding corrections of Ref. [27] for $\omega > 25$. The expression (49) is zero when ω_0=8.95 (see Fig. 4, the corresponding value for the approximation formula of Ref. [27] is ω_0=10.45). In order to estimate the accuracy of the integral over the negative part, let us calculate the integral $-\int_{\omega_0}^{\infty} d\omega\sigma_C(\omega)/[\omega^2 2\pi^2\alpha(Z\alpha)^2 f_{2B}]$ so that the terms of higher orders in $1/\omega$ are accounted for in $\sigma_C(\omega)$ one after another. The results are presented in Table II. One can see that the successive terms from Eq. (49) thus integrated give the contributions of the same order, i.e., the process does not converge to a certain value of the integral.

It is also quite possible that, in order to resolve the contradiction between the results (46) and (47), the pair production in bound-free states should be taken into account because of the strong cancellation, Eq.(48), of the contribution of free-free states.

The expression (47) coincides with that calculated in Ref. [32] (more precisely, $-D_1/f_{2B}$ in the notations of Ref. [32]). The comparison of our results, i.e., $[f_2(Z) - f_{2B}]/f_{2B}$, and the results of Ref. [32], $-D_1/f_{2B}$, is made in Fig. 5.

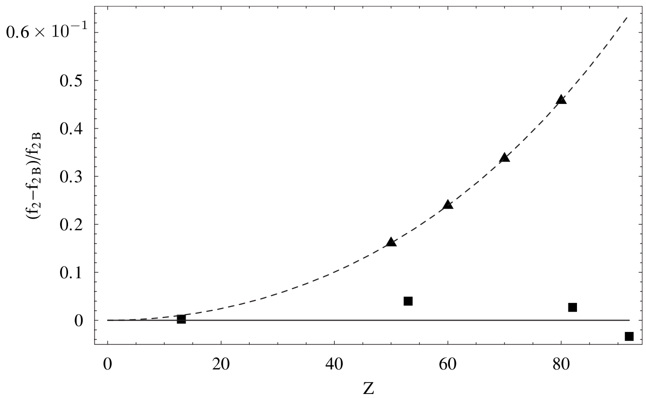

FIG. 5. Our results (triangles) for $[f_2(Z)-f_{2B}]/f_{2B}$ and the approximation formula (38) (dashed line) in comparison with the results of Ref. [32] (squares).

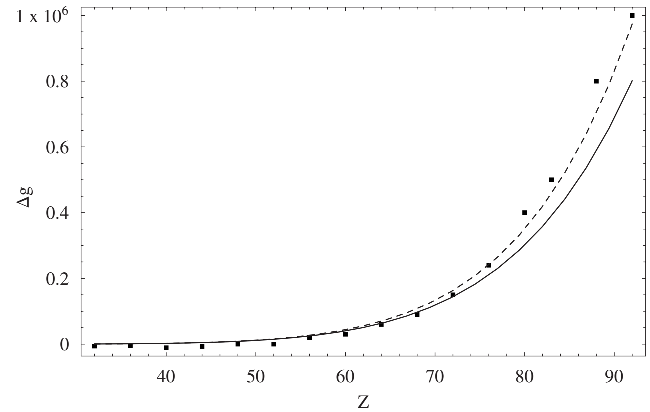

FIG. 6. The squares represent the difference of the g factor corrections obtained in Refs. [22,23]. The solid line corresponds to the function $(16/3)\alpha(Z\alpha)^5[3.35\times10^{-4}(Z\alpha)^2]$, the dashed line is the function $(16/3)\alpha(Z\alpha)^5[3.35\times10^{-4}(Z\alpha)^2+1.6\times10^{-4}(Z\alpha)^4]$, corresponding to the Coulomb corrections to the form factor f_1 in Eq. (37).

It should be noted that our results and those of Ref. [32] are essentially different because the last one has a nonmonotonic dependence on Z.

VI. g FACTOR OF A BOUND ELECTRON

The amplitude of virtual light-by-light scattering is known to be a part of the so-called magnetic loop contribution to the g factor of a bound electron [34].

For the $1S_{1/2}$ electron state, this contribution reads as (see Ref. [33])

$$\Delta g = -\frac{32}{3}\frac{\alpha(Z\alpha)^2}{\pi m^2}\int_0^\infty dq f_1(q/m)\int_0^\infty dr r\tilde{f}_1(r)\tilde{f}_2(r)\left(\frac{\sin qr}{qr} - \cos qr\right), \tag{50}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are the radial parts of the electron wave function in a Coulomb field,

$$\psi(\mathbf{r}) = \begin{pmatrix} \tilde{f}_1\Omega \\ -i\tilde{f}_2(\sigma\cdot\mathbf{n})\Omega \end{pmatrix}, \tag{51}$$

where Ω is the spherical spinor (see, for example, Ref. [35]). Using the lowest-order Born approximation for the form factor f_1 and the nonrelativistic expressions for the components of the wave function $\tilde{f}_1(r)=2\exp(-r/a_B)$ and $\tilde{f}_2=\tilde{f}_1'(r)/2m$ yields the leading correction to the g factor of a bound electron in $1S_{1/2}$ state [34],

$$\Delta g = \frac{7}{216}\alpha(Z\alpha)^5. \tag{52}$$

In the case of small $Z\alpha$, one can expand the form factor f_1 in power series of $Z\alpha$,

$$f_1(0,0,\mathbf{q},Z\alpha) = F\left(\frac{q}{m}\right) + (Z\alpha)^2F_{(1)}\left(\frac{q}{m}\right) + O(Z^4\alpha^4). \tag{53}$$

The contribution of $F(q/m)$ was considered in Ref. [33] in detail. To calculate the correction in $Z\alpha$, it is sufficient to use the functions $\tilde{f}_1$ and $\tilde{f}_2$ in the nonrelativistic limit and the expression (37) $f_1-7/1152$ as $F_{(1)}(0)$. The results of the numerical calculation of the magnetic-loop contribution exact in $Z\alpha$ are presented in Ref. [22].

The comparison of the contribution of the Coulomb corrections to the form factor f_1 in Eq. (37) and the difference of the results obtained in Ref. [22] and Ref. [33] is depicted in Fig. 6. It is surprising that our correction coincides with this difference not only for the small $Z\alpha$, but for $Z\alpha\sim1$ also.

VII. CONCLUSIONS

The Coulomb corrections to the Delbrück scattering amplitude have been considered in this paper. We have calculated these corrections in the low-energy limit but taking into account all orders of the parameter $Z\alpha$. The accuracy of the calculation is of the order of 1% for $Z=50$ and increases with Z. Our result is in good agreement with the corresponding contribution to the g factor of a bound electron calculated previously in Ref. [22]. However, there is a contradiction with the dispersion integral of the Coulomb corrections to the pair production cross section calculated in Ref. [32].

ACKNOWLEDGMENTS

The authors thank A. I. Milstein and R. N. Lee for their helpful comments and discussions.

APPENDIX: EXAMPLE OF REARRANGEMENT OF A CONDITIONALLY CONVERGENT INTEGRAL

While calculating the contribution of the first order in $Z\alpha$, we have expanded the expression (6) on $Z\alpha$ and integrated over s. The equation (23) corresponding to the contribution of the diagram [Fig. 1(b)] has been derived by integration over ρ, σ, and t in order [see Eq. (20)]. However, one can calculate this contribution in an alternative way—by substituting the expansion of Eq. (6) in Eq. (13) and integrating over r_1 and r_2 before the integration over s_1 and s_2 in the Green functions (6). One of the typical expressions appeared as

$$Y(t_1,t_2,z) = \frac{1+t_1t_2}{[1+2z(t_1+t_2)^2]^2(t_1+t_2)}\left(1-\frac{6(1+t_1t_2)}{(t_1+t_2)^2}\right), \tag{A1}$$

where $t_{1,2}=\coth s_{1,2}\in(1,\infty)$ and $z=(1+x)/2\in(0,1)$. The expression (A1) must be integrated over the total variables' domain. One can easily integrate over t_1, z, and t_2 one after another (or z, t_1, and t_2) and obtain a finite result, that is

$$\int_1^\infty\int_0^1\int_1^\infty Y dt_1 dz dt_2 = \int_1^\infty\int_1^\infty\int_0^1 Y dz dt_1 dt_2 = \frac{133}{60} - \frac{13\pi}{4\sqrt{2}} + \frac{119\arctan 2\sqrt{2}}{60\sqrt{2}} + \frac{38}{15}\ln\frac{32}{81}. \tag{A2}$$

However, if one integrates Eq. (A1) over t_1 and t_2 at first then the result is the function of z,

$$\widetilde{Y}(z) = \int_1^\infty\left(\int_1^\infty Y(t_1,t_2,z)dt_1\right)dt_2 = \frac{1}{60z}\left[\frac{1+120z}{2\sqrt{2z}}\left(\arctan(2\sqrt{2z})-\frac{\pi}{2}\right)+16z(5-48z)\ln\left(\frac{1+8z}{8z}\right)+96z-1\right], \tag{A3}$$

which has a nonintegrable singularity at $z=0$,

$$\widetilde{Y}(z) = -\frac{\pi}{240\sqrt{2}z^{3/2}} + O(z^{-1/2}). \tag{A4}$$

[1] L. Meitner, H. Küsters, and M. Delbrück, Z. Phys. **84**, 137 (1933).
[2] P. Papatzacos and K. Mork, Phys. Rep. **21C**, 81 (1975).
[3] A. Scherdin, A. Schafer, W. Greiner, and G. Soff, Phys. Rev. D **45**, 2982 (1992).
[4] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1873 (1969).
[5] H. Cheng and T. T. Wu, Phys. Rev. D **2**, 2444 (1970).
[6] H. Cheng and T. T. Wu, Phys. Rev. D **5**, 3077 (1972).
[7] A. I. Milstein and V. M. Strakhovenko, Sov. Phys. JETP **58**, 8 (1983).
[8] A. I. Mil'shtein and V. M. Strakhovenko, Phys. Lett. A **95**, 135 (1983).
[9] H. Cheng, E.-C. Tsai, and X.-q. Zhu, Phys. Rev. D **26**, 908 (1982).
[10] A. I. Milstein and R. Z. Shaisultanov, J. Phys. A **21**, 2941 (1988).
[11] A. I. Milstein and M. Schumacher, Phys. Rep. **243**, 183 (1994).
[12] H. Falkenberg, A. Hunger, M. Rullhusen, P. Schumacher, A. I. Milstein, and K. Mork, At. Data Nucl. Data Tables **50**, 1 (1992).
[13] M. Schumacher, Phys. Rev. **182**, 7 (1969).
[14] F. Smend, M. Schumacher, and I. Borchert, Nucl. Phys. A **213**, 309 (1973).
[15] M. Schumacher, F. Smend, and I. Borchert, Nucl. Phys. A **206**, 531 (1973).
[16] F. Smend and M. Schumacher, Nucl. Phys. A **223**, 423 (1974).
[17] W. Muckenheim and M. Schumacher, J. Phys. G **6**, 1237 (1980).
[18] P. Rullhusen, F. Smend, M. Schumacher, A. Hanser, and H. Rebel, Z. Phys. A **293**, 287 (1979).
[19] A. I. Mil'shtein and V. M. Strakhovenko, Phys. Lett. A **90**, 447 (1982).
[20] F. Rohrlich and R. L. Gluckstern, Phys. Rev. **86**, 1 (1952).
[21] F. Rohrlich, Phys. Rev. **108**, 169 (1957).
[22] T. Beier, I. Lindgren, H. Persson, S. Salomonson, P. Sunnergren, H. Häffner, and N. Hermanspahn, Phys. Rev. A **62**, 032510 (2000).
[23] V. Costantini, B. De Tollis, and G. Pistoni, Nuovo Cimento Soc. Ital. Fis., A **2**, 733 (1971).
[24] P. Papatzacos and K. Mork, Phys. Rev. D **12**, 206 (1975).
[25] L. Maximon, J. Res. Natl. Bur. Stand., Sect. B **72**, 79 (1968).
[26] I. Øverbø, K. J. Mork, and H. A. Olsen, Phys. Rev. A **8**, 668 (1973).
[27] I. Øverbø, Phys. Lett. B **71**, 412 (1977).
[28] R. N. Lee, A. I. Milstein, and V. M. Strakhovenko, Phys. Rev. A **69**, 022708 (2004).
[29] R. N. Lee, A. I. Milstein, and V. M. Strakhovenko, Phys. Rev. A **70**, 059902 (2004).
[30] E. S. Rosenblum, E. F. Shrader, and R. M. Warner, Phys. Rev. **88**, 612 (1952).
[31] H. Gimm and J. Hubbell, National Bureau of Standards Technical Report No. 968, 1978 (unpublished).
[32] R. Solberg, K. Mork, and I. Øverbø, Phys. Rev. A **51**, 359 (1995).
[33] R. N. Lee, A. I. Milstein, I. S. Terekhov, and S. G. Karshenboim, Phys. Rev. A **71**, 052501 (2005).
[34] S. G. Karshenboim and A. I. Milstein, Phys. Lett. B **549**, 321 (2002).
[35] V. B. Berestetskii, E. Lifshitz, and L. Pitaevskii, *Relativistic Quantum Theory* (Pergamon, Oxford, UK, 1982).